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A

COURSE

OF

MATHEMATICS,

IN TWO VOLUMES:

19825

COMPOSED, AND MORE ESPECIALLY DESIGNED,

FOR THE USE OF THE GENTLEMEN CADETS

IN THE ROYAL MILITARY ACADEMY AT WOOLWICH.

BY

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VOL. I.

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PREFACE.

A SHORT and Eaſy Courſe of the Mathematical Sciences has long been conſidered as a deſideratum for the uſe of Students in the different ſchools of education: one that ſhould hold a middle rank between the more voluminous and bulky collections of this kind, and the mere abſtract and brief common-place forms, of principles and memorandums.

For long experience in all Seminaries of Learning, and particularly in the Royal Military Academy at Woolwich, has ſhewn, that ſuch a work was very much wanted, and would prove a great and general benefit; as, for want of it, recourſe has always been obliged to be had to a number of other books, of different authors, ſelecting a part from one and a part from another, as ſeemed moſt ſuitable to the purpoſe in hand, and rejecting the other parts; a practice which occaſions much expence and trouble, in procuring and keeping ſuch a number of odd volumes, of various modes of compoſition and form; beſides wanting the benefit of uniformity and reference, which are found in a regular ſeries of compoſition.

To remove theſe inconveniences, the Author of the preſent work has been induced, from time to time, to compoſe various parts of this Courſe of Mathematics; which the experience of many years uſe in the Academy has enabled him to adapt and improve to the moſt uſeful form and quantity, for the benefit of inſtruction. And, to render that benefit more eminent and laſting, the Maſter General of the Ordnance has been pleaſed to give it its preſent form, by ordering it to be enlarged and printed.

As

As this work has been compoſed expreſsly with the intention of adapting it to the purpoſes of academical education, it is not deſigned to hold out the expectation of new inventions or diſcoveries: but rather to collect and arrange the moſt uſeful principles in a convenient practical form, demonſtrate them in a plain and conciſe way, and illuſtrate them with ſuitable examples: rejecting whatever ſeem to be matters of mere curioſity; and retaining only ſuch parts and branches, as have a direct tendency and application to ſome uſeful purpoſe in life, eſpecially in the military profeſſion, for which the gentlemen educated at this Academy are intended.

As a work of ſuch a nature muſt neceſſarily conſiſt of matters which have, in a manner, become common property, and in a great meaſure are contained, in ſome ſhape or other, in moſt books of this kind, it will not be imputed to the author, as a crime, that he has availed himſelf of the materials of ſome of the beſt books on theſe ſciences, from whence he may have extracted, or which he may have imitated; whether they be any of his own former publications, or thoſe of other authors.

Nevertheleſs it is expected that ſomething new may be found in many parts of theſe volumes, as well in the matter, as in the arrangement and manner of demonſtration, eſpecially in the geometrical part of this work. And here the author hopes he will not be too ſeverely criticiſed if, through a deſire of rendering this branch more eaſy and ſimple, he has in ſome inſtances deviated a little from the tedious and rigid ſtrictneſs of Euclid, particularly in the doctrine of ratios and proportion, which has always been ſo greatly complained of, eſpecially by young ſtudents in theſe ſciences.

Royal Military Academy,
July 30th, 1798.

ARITHMETIC,

OR,

PART I. OF VOL. I.

CONTENTS OF PART I.

VOL. I.

Page

GENERAL Preliminary Principles - - - 1

ARITHMETIC.

Notation and Numeration - - - - 4
Roman Notation - - - - - 7
Addition - - - - - - - 8
Subtraction - - - - - - - 11
Multiplication - - - - - - 13
Division - - - - - - - 18
Reduction - - - - - - - 23
Compound Addition - - - - - - 32
——— *Subtraction* - - - - - 36
——— *Multiplication* - - - - - 38
——— *Division* - - - - - 41
Golden Rule, or Rule-of-Three - - - 44
Compound Proportion - - - - - 49
Vulgar Fractions - - - - - - 51
Reduction of Vulgar Fractions - - - - 52
Addition of Vulgar Fractions - - - - 61
Subtraction of Vulgar Fractions - - - - 62
Multiplication of Vulgar Fractions - - - 63
Division of Vulgar Fractions - - - - 64
Rule-of-Three in Vulgar Fractions - - - 65
Decimal Fractions - - - - - 66
Addition of Decimals - - - - - 67
Subtraction of Decimals - - - - - 68
Multiplication of Decimals - - - - - ib.

	Page
Diviſion of Decimals	71
Reduction of Decimals	73
Rule-of-Three in Decimals	76
Duodecimals	77
Involution	78
Evolution	80
To extract the Square Root	81
To extract the Cube Root	85
To extract any Root whatever	88
Table of Powers and Roots	90
Ratios, Proportions, and Progreſſions	95
Arithmetical Proportion	96
Geometrical Proportion	101
Muſical Proportion	104
Fellowſhip or Partnerſhip	105
Single Fellowſhip	ibid
Double Fellowſhip	108
Simple Intereſt	110
Compound Intereſt	112
Alligation Medial	114
Alligation Alternate	116
Single Poſition	120
Double Poſition	122
Permutation and Combination	125
Practical Queſtions	135

A COURSE

A COURSE OF MATHEMATICS, &c.

GENERAL PRINCIPLES.

1. QUANTITY, or MAGNITUDE, is any thing that will admit of increaſe or decreaſe; or that is capable of any ſort of calculation or menſuration: ſuch as, numbers, lines, ſpace, time, motion, weight.

2. MATHEMATICS is the ſcience which treats of all kinds of quantity whatever, that can be numbered or meaſured.—That part which treats of numbering, is called *Arithmetic*: and that which concerns meaſuring, or figured extenſion, is called *Geometry*.——Theſe two, which are converſant about multitude and magnitude, and are the foundation of all the other parts, are called *Pure* or *Abſtract Mathematics*; becauſe they inveſtigate and demonſtrate the properties of abſtract numbers and magnitudes of all ſorts. And when theſe two parts are applied to particular or practical ſubjects, they conſtitute the branches or parts called *Mixed Mathematics*.—— Mathematics is alſo diſtinguiſhed into *Speculative* and *Practical*; viz. *Speculative*, when it is concerned in diſcovering properties and relations; and *Practical*, when applied to practice and real uſe concerning phyſical objects.

3. In Mathematics are ſeveral general terms or principles; ſuch as, Definitions, Axioms, Propoſitions, Theorems, Problems, Lemmas, Corollaries, Scholiums, &c.

4. *A Definition* is the explication of any term or word in a ſcience; ſhewing the ſenſe and meaning in which the term is employed.—Every Definition ought to be clear, and expreſſed in words that are common and perfectly well underſtood.

5. *A Propoſition* is ſomething propoſed to be proved, or ſomething required to be done; and is accordingly either a Theorem or a Problem.

6. *A Theorem* is a demonſtrative propoſition; in which ſome property is aſſerted, and the truth of it required to be proved. Thus, when it is ſaid that, The ſum of the three angles of any triangle, is equal to two right angles, this is a Theorem, the truth of which is demonſtrated by Geometry. —A ſet or collection of ſuch Theorems conſtitutes a *Theory*.

7. *A Problem* is a propoſition or a queſtion requiring ſomething to be done; either to inveſtigate ſome truth or property, or to perform ſome operation. As, to find out the quantity or ſum of all the three angles of any triangle, or to draw one line perpendicular to another.——*A Limited Problem* is that which has but one anſwer or ſolution. *An Unlimited Problem* is that which has innumerable anſwers. And a *Determinate Problem* is that which has a certain number of anſwers.

8. *Solution* of a Problem, is the reſolution or anſwer given to it. *A Numerical* or *Numeral Solution*, is the anſwer given in numbers. *A Geometrical Solution*, is the anſwer given by the principles of Geometry. And a *Mechanical Solution*, is one which is gained by trials.

9. *A Lemma* is a preparatory propoſition, laid down in order to ſhorten the demonſtration of the main propoſition which follows it.

10. *A Corollary*, or *Conſectary*, is a conſequence drawn immediately from ſome propoſition or other premiſes.

11. *A Scholium*, is a remark or obſervation made on ſome foregoing propoſition or premiſes.

12. *An Axiom*, or *Maxim*, is a ſelf-evident propoſition; requiring no formal demonſtration to prove the truth of it; but is received and aſſented to as ſoon as mentioned. Such as, The whole of any thing is greater than a part of it: or, The whole is equal to all its parts taken together: or, Two quantities that are each of them equal to a third quantity, are equal to each other.

13. *A Poſtu-*

13. *A Poſtulate*, or *Petition*, is ſomething required to be done, which is ſo eaſy and evident that no perſon will heſitate to allow it.

14. *An Hypotheſis* is a ſuppoſition aſſumed to be true, in order to argue from, or to found upon it the reaſoning and demonſtration of ſome propoſition.

15. *Demonſtration*, is the collecting the ſeveral arguments and proofs, and laying them together in proper order, to ſhew the truth of the propoſition under conſideration.

16. *A Direct*, *Poſitive*, or *Affirmative Demonſtration*, is that which concludes with the direct and certain proof of the propoſition in hand.—This kind of Demonſtration is moſt ſatisfactory to the mind; for which reaſon it is called ſometimes an *Oſtenſive Demonſtration*.

17. *An Indirect* or *Negative Demonſtration*, is that which ſhews a propoſition to be true, by proving that ſome abſurdity would neceſſarily follow if the propoſition advanced were falſe. This is alſo ſometimes called *Reductio ad Abſurdum*; becauſe it ſhews the abſurdity and falſehood of all ſuppoſitions contrary to that contained in the propoſition.

18. *Method*, is the art of diſpoſing a train of arguments in a proper order, to inveſtigate either the truth or falſity of a propoſition, or to demonſtrate it to others when it has been found out.——This is either Analytical or Synthetical.

19. *Analyſis*, or the *Analytic Method*, is the art or mode of finding out the truth of a propoſition, by firſt ſuppoſing the thing to be done, and then reaſoning back ſtep by ſtep till we arrive at ſome known truth. This is alſo called the *Method of Invention*, or *Reſolution*; and is that which is commonly uſed in Algebra.

20. *Syntheſis*, or the *Synthetic Method*, is the ſearching out truth, by firſt laying down ſome ſimple and eaſy principles, and purſuing the conſequences flowing from them till we arrive at the concluſion.—This is alſo called the *Method of Compoſition*; and is the reverſe of the Analytic method, as this proceeds from known principles to an unknown concluſion; while the other goes in a retrograde order, from the thing ſought, conſidered as if it were true, to ſome known principle or fact. And therefore, when any truth has been found out by the Analytic method, it may be demonſtrated by a proceſs in the contrary order, by Syntheſis.

ARITHMETIC.

ARITHMETIC is the art or ſcience of numbering; being that branch of Mathematics which treats of the nature and properties of numbers.—When it treats of whole numbers, it is called *Vulgar*, or *Common Arithmetic*; but when of broken numbers, or parts of numbers, it is called *Fractions*.

Unity, or an *Unit*, is that by which every thing is called one; being the beginning of number. As one man, one ball, one gun.

Number is either ſimply one, or a compound of ſeveral units. As one man, three men, ten men.

An *Integer*, or *Whole Number*, is ſome certain preciſe quantity of units; as one, three, ten.—Theſe are ſo called as diſtinguiſhed from *Fractions*, which are broken numbers, or parts of numbers; as one-half, two-thirds, or three-fourths.

NOTATION AND NUMERATION.

NOTATION, or NUMERATION, teaches to denote or expreſs any propoſed number, either by words or characters; or to read and write down any ſum or number.

The numbers in Arithmetic are expreſſed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors, about eight or nine hundred years ſince: viz. 1 one, 2 two, 3 three, 4 four, 5 five, 6 ſix, 7 ſeven, 8 eight, 9 nine, 0 cipher or nothing. Theſe characters or figures were formerly all called by the general name of *Ciphers*; whence it came to paſs, that the art of Arithmetic was then often called *Ciphering*. Alſo the firſt nine are called *Significant Figures*, as diſtinguiſhed from the cipher, which is quite inſignificant of itſelf.

Beſide this value of thoſe figures, they have alſo another, which depends upon the place they ſtand in when joined together; as in the following Table:

Units

&c.	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thouſands	Tens of Thouſands	Thouſands	Hundreds	Tens	Units
&c.	9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3
				9	8	7	6	5	4
					9	8	7	6	5
						9	8	7	6
							9	8	7
								9	8
									9

Here, any figure in the firſt place, reckoning from right to left, denotes only its own ſimple value; but that in the ſecond place, denotes ten times its ſimple value; and that in the third place, a hundred times its ſimple value; and ſo on; the value of any figure, in each ſucceſſive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the firſt place denotes only ſix units, or ſimply ſix; 9 in the ſecond place ſignifies nine tens, or ninety; 7 in the third place, ſeven hundred; and the 1 in the fourth place, one thouſand; ſo that the whole number is read thus, one thouſand ſeven hundred and ninety-ſix.

As to the cipher 0, it ſtands for nothing of itſelf, but being joined on the right hand ſide to other figures, it increaſes their value in the ſame tenfold proportion: thus, 5 ſignifies only five; but 50 denotes 5 tens, or fifty; and 500 is five hundred; and ſo on.

For the more eaſily reading of large numbers, they are divided into periods and half-periods, each half period conſiſting of three figures; the name of the firſt period being units; of the ſecond, millions; of the third, millions of millions, or bi-millions, contracted to billions; of the fourth, millions of millions of millions, or tri-millions, contracted to trillions; and ſo on. Alſo the firſt part of any period is ſo many units of it, and the latter part ſo many thouſands.

The

The following Table contains a ſummary of the whole doctrine:

Periods.	Quadrill.		Trillions;		Billions;		Millions;		Units.	
Half-per.	th.	un.	th.	un.	th.	un.	th.	un.	th.	un.
Figures.	123,	456;	789,	098;	765,	432;	101,	234;	567,	890

NUMERATION is the reading of any number in words that is propoſed or ſet down in figures, which will be eaſily done by help of the following rule, deduced from the foregoing tablets and obſervations, viz.

Divide the figures in the propoſed number, as in the ſummary above, into periods and half-periods; then begin at the left-hand ſide, and read the figures with the names ſet to them in the two foregoing tables.

EXAMPLES.

Expreſs in words the following numbers; viz.

34	15080	13405670
96	72003	47050023
180	109026	309025600
304	483500	4723507689
6134	2500639	274856390000
9028	7523000	6578600307024

NOTATION is the ſetting down in figures any number propoſed in words; which is done by ſetting down the figures inſtead of the words or names belonging to them in the ſummary above; ſupplying the vacant places with ciphers where any words do not occur.

EXAMPLES.

Set down in figures the following numbers:

Fifty ſeven.
Two hundred eighty-ſix
Nine thouſand two hundred and ten.
Twenty-ſeven thouſand five hundred and ninety-four.
Six hundred and forty thouſand, four hundred and eighty-one.
Three millions, two hundred ſixty thouſand, one hundred and ſix.

Four

Four hundred and eight millions, two hundred and fifty-five thouſand, one hundred and ninety-two.

Twenty-ſeven thouſand and eight millions, ninety-ſix thouſand two hundred and four.

Two hundred thouſand and five hundred and fifty millions, one hundred and ten thouſand, and ſixteen.

Twenty-one billions, eight hundred and ten millions, ſixty-four thouſand, one hundred and fifty.

Of the ROMAN NOTATION.

THE Romans, like ſeveral other nations, expreſſed their numbers by certain letters of the alphabet. The Romans uſed only ſeven numeral letters, being the ſeven following capitals: viz. I for *one*; V for *five*; X for *ten*; L for *fifty*; C for an *hundred*; D for *five hundred*; M for a *thouſand*. The other numbers they expreſſed by various repetitions and combinations of theſe, after the following manner:

1	= I	
2	= II	As often as any character is repeated, ſo many times is its value repeated.
3	= III	
4	= IIII or IV	A leſs character before a greater diminiſhes its value.
5	= V	
6	= VI	A leſs character after a greater increaſes its value.
7	= VII	
8	= VIII	
9	= IX	
10	= X	
50	= L	
100	= C	
500	= D or Iↀ	For every ↀ annexed, this becomes 10 times as many.
1000	= M or CIↀ	For every C and ↀ, placed one at each end, it becomes 10 times as much.
2000	= MM	
5000	= $\overline{V}$ or Iↀↀ	A bar over any number, increaſes it 1000 fold.
6000	= $\overline{VI}$	
10000	= $\overline{X}$ or CCIↀↀ	
50000	= $\overline{L}$ or Iↀↀↀ	
60000	= $\overline{LX}$	
100000	= $\overline{C}$ or CCCIↀↀↀ	
1000000	= $\overline{M}$ or CCCCIↀↀↀↀ	
2000000	= $\overline{MM}$	
&c.	&c.	

EXPLA-

Explanation of certain Characters.

There are various characters or marks uſed in Arithmetic, and Algebra, to denote ſeveral of the operations and propoſitions; the chief of which are as follow:

$+$ ſignifies *plus*, or addition.
$-$ - - *minus*, or ſubtraction.
$\times$ - - multiplication.
$\div$ - - diviſion.
$::\ :\ :$ - proportion.
$=$ - - equality.
$\surd$ - - ſquare root.
$\sqrt[3]{}$ - - cube root, &c.

Thus,

$5 + 3$, denotes that 3 is to be added to 5.
$6 - 2$, denotes that 2 is to be taken from 6.
7×3, denotes that 7 is to be multiplied by 3
$8 \div 4$, denotes that 8 is to be divided by 4.
$2 : 3 :: 4 : 6$, ſhews that 2 is to 3 as 4 is to 6.
$6 + 4 = 10$, ſhews that the ſum of 6 and 4 is equal to 10.
$\surd 3$, or $3^{\frac{1}{2}}$, denotes the ſquare root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be ſquared.
8^3, denotes that the number 8 is to be cubed.
&c.

OF ADDITION.

ADDITION is the collecting or putting of ſeveral numbers together, in order to find their *ſum*, or the total amount of the whole. This is done as follows:

Set or place the numbers under each other, ſo that each figure may ſtand exactly under the figures of the ſame value, that

that is, units under units, tens under tens, hundreds under hundreds, &c.; and draw a line under the loweſt number, to ſeparate the given numbers from their ſum, when it is found. —Then add up the figures in the column or row of units, and find how many tens are contained in their ſum.—Set down exactly below, what remains more than thoſe tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next add up the ſecond row, together with the number carried, in the ſame manner as the firſt. And thus proceed till the whole is finiſhed, ſetting down the total amount of the laſt row.

To prove Addition.

Firſt Method.—Begin at the top, and add together all the rows of numbers downwards, in the ſame manner as they were before added upwards; then if the two ſums agree, it may be preſumed the work is right.—This method of proof is only doing the ſame work twice over, a little varied.

Second Method.—Draw a line below the uppermoſt number, and ſuppoſe it cut off.—Then add all the reſt of the numbers together in the uſual way, and ſet their ſum under the number that is to be proved.—Laſtly, add this laſt found number and the uppermoſt line together; then if their ſum be the ſame as that found by the firſt addition, it may be preſumed the work is right.—This method of proof is founded on the plain axiom, that "The whole is equal to all its parts taken together."

Third Method.—Add the figures in the uppermoſt line together, and find how many nines are contained in their ſum.—Reject thoſe nines, and ſet down the remainder towards the right-hand directly even with the figures in the line, as in the annexed example.—Do the ſame with each of the propoſed lines of numbers, ſetting all theſe exceſſes of nines in a column on the right-hand, as here 5, 5, 6. Then, if the exceſs of 9's in this ſum, found as before, be equal to the exceſs of 9's in the total ſum 18304, the work is right.—Thus, the ſum of the right-hand column 5, 5, 6, is 16, the exceſs of which above 9 is 7. Alſo the ſum of the figures in the

EXAMPLE I.

	Exceſs of nines
3497	5
6512	5
8295	6
18304	7

the ſum total 18304 is 16, the exceſs of which above 9 is alſo 7, the ſame as the former *.

OTHER EXAMPLES.

2.	3.	4.
12345	12345	12345
67890	67890	876
98765	9876	9087
43210	543	56
12345	21	234
67890	9	1012
302445	90684	23610
290100	78339	11265
302445	90684	23610

* This method of proof depends on a property of the number 9, which, except the number 3, belongs to no other digit whatever; namely, that "any number divided by 9, will leave the ſame remainder as the ſum of its figures or digits divided by 9;" which may be demonſtrated in this manner.

Demonſtration. Let there be any number propoſed, as 4658. This, ſeparated into its ſeveral parts, becomes $4000 + 600 + 50 + 8$. But $4000 = 4 + 1000 = 4 + (999 + 1) = 4 \times 999 + 4$. In like manner $600 = 6 \times 99 + 6$; and $50 = 5 \times 9 + 5$. Therefore the given number $4658 = 4 \times 999 + 4 + 6 \times 99 + 6 + 5 \times 9 + 5 + 8 = 4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8$; and $4658 \div 9 = (4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8) \div 9$. But $4 \times 999 + 6 \times 99 + 5 \times 9$ is evidently diviſible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the ſame remainder as $4 + 6 + 5 + 8$ divided by 9. And the ſame, it is evident, will hold for any other number whatever.

In like manner, the ſame property may be ſhewn to belong to the number 3; but the preference is uſually given to the number 9, on account of its being more convenient in practice.

Now, from the demonſtration above given, the reaſon of the rule itſelf is evident; for the exceſs of 9's in two or more numbers being taken ſeparately, and the exceſs of 9's taken alſo out of the ſum of the former exceſſes, it is plain that this laſt exceſs muſt be equal to the exceſs of 9's contained in the total ſum of all theſe numbers; all the parts taken together being equal to the whole.

——This rule was firſt given by Dr. Wallis in his Arithmetic, publiſhed in the year 1657.

Ex.

Ex. 5. Add 3426; 9024; 5106; 8390; 1204 together.
Anſ. 27150.

6. Add 509267; 235809; 72910; 8392; 420; 21; and 9 together. Anſ. 826828.

7. Add 2; 19; 817; 4298; 50916: 730205; 9120634 together. Anſ. 9906891.

8. How many days are in the twelve calendar months?
Anſ. 365.

9. How many days are there from the 15th day of April to the 24th day of November, both days included? Anſ. 224.

10. An army conſiſting of 52714 infantry * or foot, 5110 horſe, 6250 dragoons, 3927 light-horſe, 928 artillery or gunners, 1410 pioneers, 250 ſappers, and 406 miners; what is the whole number of men? Anſ. 70995.

OF SUBTRACTION.

SUBTRACTION teaches to find how much one number exceeds another, called their *difference*, or the *remainder*, by taking the leſs from the greater. The method of doing which is as follows:

Place the leſs number under the greater, in the ſame manner as in Addition, that is, units under units, tens under tens, and ſo on; and draw a line below them.—Begin at the right-hand, and take each figure in the lower line or number from the figure above it, ſetting down the remainder below it.—But if the figure in the lower line be greater than that above it, firſt borrow or add 10 to the upper one, and then take the lower figure from that ſum, ſetting down the remainder, and carrying 1, for what was borrowed, to the next lower figure, with which proceed as before, and ſo on till the whole is finiſhed.

* The whole body of foot ſoldiers is denoted by the word *Infantry*; and all thoſe that charge on horſeback, by the word *Cavalry*.—Some authors conjecture, that the term infantry is derived from a certain Infanta of Spain, who, finding that the army commanded by the king her father had been defeated by the Moors, aſſembled a body of the people together on foot, with which ſhe engaged and totally routed the enemy. In honour of this event, and to diſtinguiſh the foot ſoldiers, who were not before held in much eſtimation, they received the name of Infantry, from her own title of Infanta.

To prove Subtraction.

Add the remainder to the leis number, or that which is juſt above it, and if the ſum be equal to the greater or uppermoſt number, the work is right *.

EXAMPLES.

1.	2.	3.
From 5386427	From 5386427	From 1234567
Take 2164315	Take 4258792	Take 702973
Rem. 3222112	Rem. 1127635	Rem. 531594
Proof. 5386427	Proof. 5386427	Proof. 1234567

4. From 5231806 Take 5079918. Anſ. 157888.
5. From 7020914 Take 2766809. Anſ. 4254105.
6. From 8503602 Take 574371. Anſ. 7929231.

7. Sir Iſaac Newton was born in the year 1642, and he died in 1727; how old was he at the time of his deceaſe?
Anſ. 85 years.

8. Homer was born 2530 years ago, and Chriſt 1797 years ago; then how long before Chriſt was the birth of Homer? Anſ. 733 years.

9. Noah's flood happened about the year of the world 1656, and the birth of Chriſt about the year 4000; then how long was the flood before Chriſt? Anſ. 2344 years.

10. The Arabian or Indian method of notation was firſt known in England about the year 1150; then how long is it ſince, to this preſent year 1797? Anſ. 647 years.

11. Gunpowder was invented in the year 1330; then how long was this before the invention of printing, which was in 1441? Anſ. 111 years.

12. The mariner's compaſs was invented in Europe in the year 1302; then how long was that before the diſcovery of America by Columbus, which happened in 1492?
Anſ. 190 years.

* The reaſon of this method of proof is evident: for if the difference of two numbers be added to the leſs, it muſt manifeſtly make up a ſum equal to the greater.

OF

OF MULTIPLICATION

MULTIPLICATION is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times. As 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand*.—The number you multiply by, or number of repetitions, is the *Multiplier*.—And the number found, being the total amount, is called the *Product*.—Alſo, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors*.

Before proceeding to any operations in this rule it is neceſſary to learn off very perfectly the following Table of all the products of the firſt 12. numbers, ſometimes called the Multiplication Table, or Pythagoras's Table, from its inventer.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To multiply any Given Number by a Single Figure, or by any Number not more than 12.

* Set the multiplier under the units figure, or right-hand place, of the multiplicand, and draw a line below it.—Then, beginning at the right hand, multiply every figure in this by the multiplier.—Count how many tens there are in the product of every ſingle figure, and ſet down the remainder directly under the figure that is multiplied; and if nothing remains, ſet down a cipher.—Carry as many units or ones, as there are tens counted, to the product of the next figures; and proceed in the ſame manner till the whole is finiſhed.

EXAMPLE.

Multiply	9876543210	Multiplicand
By	2	Multiplier
	19753086420	Product.

To multiply by a Number conſiſting of Several Figures.

† Set the multiplier below the multiplicand, placing them as in Addition, namely units under units, tens under tens, &c. drawing a line below it.—Multiply the whole of the multiplicand by each figure of the multiplier, as in the laſt article; ſetting

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The reaſon of this rule is the ſame as for the proceſs in Addition, in which 1 is carried for every 10, to the next place, gradually as the ſeveral products are produced, one after another, inſtead of ſetting them all down below each other, as in the annexed Example.

	5678	
	4	
	32 =	8 × 4
	280 =	70 × 4
	2400 =	600 × 4
	20000 =	5000 × 4
	22712 =	5678 × 4

† After having found the produce of the multiplicand by the firſt figure of the multiplier, as in the former caſe, the multiplier is ſuppoſed to be divided into parts, and the product is found for the ſecond figure in the ſame manner: but as this figure ſtands in the place of tens, the product muſt be 10 times its ſimple value; and therefore the firſt figure of this product muſt be ſet in the place of tens; or, which

ſetting down a line of products for each figure in the multiplier, ſo as that the firſt figure of each line may ſtand ſtraight under the figure multiplying by.—Add all the lines of products together, in the order as they ſtand, and their ſum will be the anſwer or whole product required.

To prove Multiplication.

There are three different ways of proving Multiplication, which are as below:

Firſt Method.—Make the multiplicand and multiplier change places, and multiply the latter by the former in the ſame manner as before. Then if the product found in this way be the ſame as the former, the number is right.

Second Method.—* Caſt all the 9's out of the ſum of the figures in each of the two factors, as in Addition, and ſet down the remainders. Multiply theſe two remainders together, and caſt the 9's out of the product, as alſo out of

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which is the ſame thing, directly under the figure multiplied by. And proceeding in this manner ſeparately with all the figures of the multiplier, it is evident that we ſhall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand: therefore theſe ſeveral products being added together, will be equal to the whole required product; as in the example annexed.

1234567	the multiplicand.
4567	
8641969 =	7 times the mult.
7407402 =	60 times ditto.
6172835 =	500 times ditto.
4938268 =	4000 times ditto.
5638267489 =	4567 times ditto.

* This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reaſon for the one may ſerve for that of the other. Another more ample demonſtration of this rule may be as follows:—Let P and Q denote the number of 9's in the factors to be multiplied, and a and b what remain; then $9P + a$ and $9Q + b$ will be the numbers themſelves, and their product is $(9P \times 9Q) + 9P \times b) + (9Q \times a) + (a \times b)$; but the firſt three of theſe products are each a preciſe number of 9's, becauſe their factors are ſo, either one or both: theſe therefore being caſt away, there remains only $a \times b$; and if the 9's be alſo caſt out of this, the exceſs is the exceſs of 9's in the total product: but a and b are the exceſſes in the factors themſelves, and $a \times b$ is their product; therefore the rule is true.

the

the whole product or answer of the question, reserving the remainders of these last two, which remainders must be equal when the work is right.—*Note*, It is common to set the four remainders within the four angular spaces of a cross, as in the example below.

Third Method.—Multiplication is also very naturally proved by Division; for the product divided by either of the factors, will evidently give the other. But this cannot be practised till the rule of Division is learned.

EXAMPLES.

Mult. 3542 by 6196	Proof.	or Mult. 6196 by 3542
21252	2	12392
31878	5 × 4	24784
3542	2	30980
21252		18588
21946232 Product.		21946232 Proof.

OTHER EXAMPLES.

Multiply 123456789	by 3.	Anſ. 370370367.
Multiply 123456789	by 4.	Anſ. 493827156.
Multiply 123456789	by 5.	Anſ. 617283945.
Multiply 123456789	by 6.	Anſ. 740740734.
Multiply 123456789	by 7.	Anſ. 864197523.
Multiply 123456789	by 8.	Anſ. 987654312.
Multiply 123456789	by 9.	Anſ. 1111111101.
Multiply 123456789	by 11.	Anſ. 1358024679.
Multiply 123456789	by 12.	Anſ. 1481481468.
Multiply 302914603	by 16.	Anſ. 4846633648.
Multiply 273580961	by 23.	Anſ. 6292362103.
Multiply 402097316	by 195.	Anſ. 78408976620.
Multiply 82164973	by 3027.	Anſ. 248713373271.
Multiply 7564900	by 579.	Anſ. 4380077100.
Multiply 8496427	by 874359.	Anſ. 7428927415293.
Multiply 2760325	by 37072.	Anſ. 102330768400.

CONTRAC

CONTRACTIONS IN MULTIPLICATION.

I. *When there are Ciphers in the Factors.*

IF the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the product, as are in both the factors.—And when the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the firſt figure of every line of products exactly under the figure multiplying with.

EXAMPLES.

1.		2.
Mult. 9001635		Mult. 390720400
by - 70100		by - 406000
9001635		23443224
63011445		15628816
631014613500	Prod.	158632482400000

3. Multiply 81503600 by 7030. Anſ. 572970308000.
4. Multiply 9030100 by 2100. Anſ. 18963210000.
5. Multiply 8057069 by 70050. Anſ. 564397683450.

II. *When the Multiplier is the Product of two or more Numbers in the Table; then*

* Multiply by each of thoſe parts ſeparately, inſtead of the whole number at once.

EXAMPLES.

1. Multiply 51307298 by 56, or 7 times 8.

51307298
7
359151086
8
2873208688

* The reaſon of this rule is obvious enough; for any number multiplied by the component parts of another, muſt give the ſame product as if it were multiplied by that number at once. Thus, in the 1ſt example, 7 times the product of 8 by the given number, makes 56 times the ſame number, as plainly as 7 times 8 makes 56.

2. Multiply 31704592 by 36. Anſ. 1141365312.
3. Multiply 29753804 by 72. Anſ. 2142273888.
4. Multiply 7128368 by 96. Anſ. 684323328.
5. Multiply 160430800 by 108. Anſ. 17326526400.
6. Multiply 61835720 by 1320. Anſ. 81623150400.

7. There was an army compoſed of 104 *battalions, each conſiſting of 500 men; what was the number of men contained in the whole? Anſ. 52000.

8. A convoy of ammunition †bread, conſiſting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy; what is the number of loaves loſt? Anſ. 80000.

OF DIVISION.

DIVISION is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken out of it, which is the ſame thing.

The number to be divided, is called the *Dividend*.—The number to divide by, is the *Diviſor*.—And the number of times the dividend contains the diviſor, is called the *Quotient*.—Sometimes there is a *Remainder* left, after the diviſion is finiſhed.

The uſual manner of placing the terms, is, the dividend in the middle, having the diviſor on the left-hand, and the quotient on the right, each ſeparated by a curve line; as, to divide 12 by 4, the quotient is 3,

Dividend
Diviſor 4) 12 (3 Quotient;
ſhewing that the number 4 is 3 times contained in 12, or may be three times ſubtracted out of it, as in the margin.

‡ *Rule*.—Having placed the diviſor before the dividend, as above directed, find how often the diviſor is contained in as many figures of the dividend as are juſt neceſſary, and place the number on the right in the quotient.

12	
4	ſubtr.
8	
4	ſubtr.
4	
4	ſubtr.
0	

Mul-

* A battalion is a body of foot, conſiſting of 500, or 600, or 700 men, more or leſs.

† The ammunition-bread is that which is provided for, and diſtributed to the ſoldiers; the uſual allowance being a loaf of 6 pounds to every ſoldier, once in 4 days.

‡ In this way we reſolve the dividend into parts, and find by trial how

Multiply the diviſor by this number, and ſet the product under the figures of the dividend before-mentioned.—Subtract this product from that part of the dividend under which it ſtands, and bring down the next figure of the dividend, or more if neceſſary, to join on the right of the remainder.—Divide this number, ſo increaſed, in the ſame manner as before; and ſo on till all the figures are brought down and uſed.

N. B. If it be neceſſary to bring down more figures than one to any remainder, in order to make it as large as the diviſor, or larger, a cipher muſt be ſet in the quotient for every figure ſo brought down more than one.

To prove Division.

* Multiply the quotient by the diviſor; to this product add the remainder, if there be any; then the ſum will be equal to the dividend when the work is right.

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how often the diviſor is contained in each of thoſe parts, one after another, and arranging the ſeveral figures of the quotient one after another, into one number.

When there is no remainder to a diviſion, the quotient is the whole and perfect anſwer to the queſtion. But when there is a remainder, it goes ſo much towards another time as it approaches to the diviſor: ſo, if the remainder be half the diviſor, it will go the half of a time more; if the 4th part of the diviſor, it will go one fourth of a time more; and ſo on. Therefore, to complete the quotient, ſet the remainder at the end of it, above a ſmall line, and the diviſor below it, thus forming a fractional part of the whole quotient.

* This method of proof is plain enough: for ſince the quotient is the number of times the dividend contains the diviſor, the quotient multiplied by the diviſor muſt evidently be equal to the dividend.

There are alſo ſeveral other methods ſometimes uſed for proving Diviſion, ſome of the moſt uſeful of which are as follow:

Second Method.—Subtract the remainder from the dividend, and divide what is left by the quotient; ſo ſhall the new quotient from this laſt diviſion be equal to the former diviſor, when the work is right.

Third Method.—Add together the remainder and all the products of the ſeveral quotient figures by the diviſor, according to the order in which they ſtand in the work; and the ſum will be equal to the dividend when the work is right.

EXAMPLES.

1.	Quot.	2.	Quot.
3) 1234567	(411522	37) 12345678	(333666
12	mult. 3	111	37
3	1234566	124	2335662
3	add 1	111	1000998
4	1234567	135	rem. 36
3	Proof.	111	12345678
15		246	Proof.
15		222	
6		247	
6		222	
7		258	
6		222	
Rem. 1		Rem. 36	

3. Divide 73146085 by 4. Anſ. 18286521¼.
4. Divide 5317986027 by 7. Anſ. 759712289⅐.
5. Divide 570196382 by 12. Anſ. 47516365⅚.
6. Divide 74638105 by 37. Anſ. 2017246 3/37.
7. Divide 137896254 by 97 Anſ. 1421610 84/97.
8. Divide 35821649 by 764. Anſ. 46886 745/764.
9. Divide 72091365 by 5201. Anſ. 13861 3045/5201.
10. Divide 4637064283 by 57606 Anſ. 80496 17067/57606.

11. Suppoſe 471 men are formed into ranks of 3 deep, what is the number in each rank? Anſ. 157.

12. A party, at the diſtance of 378 miles from the head quarters, receive orders to join their corps in 18 days; what number of miles muſt they march each day to obey their orders? Anſ. 21.

13. The annual revenue of a gentleman being 38330l; how much per day is that equivalent to, there being 365 days in the year? Anſ. 104l.

CONTRACTIONS IN DIVISION.

THERE are certain contractions in Diviſion, by which the operation in particular caſes may be performed in a ſhorter manner; as follows:

1. *Divi-*

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work, except only the quotient immediately below the dividend.

EXAMPLES.

3) 56103961	4) 52019675	5) 1370192
Quot. 18701320$\frac{1}{3}$		
6) 38072940	7) 81390627	8) 23718620
9) 43081962	11) 57014230	12) 27980313

II. * *When Ciphers are annexed to the Divisor;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend; then divide the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder; otherwise, the figures cut off only will be the remainder.

EXAMPLES.

1. Divide 3704196 by 20.

2,0) 370419,6

Quot. 185209$\frac{16}{20}$

2. Divide 31086901 by 7100.

71,00) 310869,01 (4378$\frac{2101}{7100}$.

284

268

213

556

497

599

568

31

3. Divide

† This method is only to avoid a needless repetition of ciphers, which would happen in the common way. And the truth of the principle

3. Divide 7380964 by 23000. Anf. $320\frac{964}{23000}$.

4. Divide 2304109 by 5800. Anf. $397\frac{509}{5800}$.

III. *When the Diviſor is the exact Product of two or more of the ſmall Numbers not greater than* 12: * Divide by each of thoſe numbers ſeparately, inſtead of the whole diviſor at once.

N. B. There are commonly ſeveral remainders in working by this rule, one to each diviſion; and to find the true or whole remainder, the ſame as if the diviſion had been performed all at once, proceed as follows: Multiply the laſt remainder by the preceding diviſor, or laſt but one, and to the product add the preceding remainder; multiply this ſum by the next preceding diviſor, and to the product add the next preceding remainder; and ſo on, till you have gone backward through all the diviſors and remainders to the firſt, As in the example following:

EXAMPLES.

1. Divide 31046835 by 56 or 7 times 8.

7) 31046835 — 6 the laſt rem.

mult. 7 preced. diviſor,

8) 4435262—1 firſt rem. — 42

554407—6 ſecond rem. add 1 the 1ſt. rem.

Anf. $554407\frac{43}{56}$. — 43 whole rem.

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principle upon which it is founded, is evident; for, cutting off the ſame number of ciphers, or figures, from each, is the ſame as dividing each of them by 10, or 100, or 1000, &c. according to the number of ciphers cut off; and it is evident that as often as the whole diviſor is contained in the whole dividend, ſo often muſt any part of the former be contained in a like part of the latter.

* This follows from the 2d contraction in Multiplication, being only the converſe of it; for the half of the third part of any thing, is evidently the ſame as the ſixth part of the whole; and ſo of any other numbers.—The reaſon of the method of finding the whole remainder from the ſeveral particular ones, will beſt appear from the nature of Vulgar Fractions. Thus, in the firſt example above, the firſt remainder being 1, when the diviſor is 7, makes $\frac{1}{7}$; this muſt be added to the ſecond remainder 6, making $6\frac{1}{7}$ to the diviſor 8, or to be divided by 8. But $6\frac{1}{7} = \frac{6 \times 7 + 1}{7} = \frac{43}{7}$; and this divided by 8 gives $\frac{43}{7 \times 8} = \frac{43}{56}$

2. Divid

2. Divide 7014596 by 72. Anſ. $97424\frac{68}{72}$.
3. Divide 5130652 by 132. Anſ. $38868\frac{76}{132}$.
4. Divide 83016572 by 240. Anſ. $345902\frac{92}{240}$.

IV. *Common Diviſion may be performed more conciſely*, by omitting the ſeveral products, and ſetting down only the remainders; namely, multiply the diviſor by the quotient figures as before, and, without ſetting down the product, ſubtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 304679 by 833.

833) 3104679 ($3727\frac{88}{833}$.
6056
2257
5919
88

2. Divide 79165238 by 238. Anſ. $332627\frac{192}{238}$.
3. Divide 29137062 by 5317. Anſ. $5479\frac{5219}{5317}$.
4. Divide 62015735 by 7803. Anſ. $7947\frac{5294}{7803}$.

OF REDUCTION.

REDUCTION is the changing of numbers from one name or denomination to another, without altering their value.—This is chiefly concerned in reducing money, weights, and meaſures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction Deſcending*; but when, contrarywiſe, from a lower name to a higher, it is *Reduction Aſcending*.

Before proceeding to the rules and queſtions of Reduction, it will be proper to ſet down the uſual Tables of money, weights, and meaſures, which are as follow:

Of

Of MONEY, WEIGHTS, AND MEASURES.

TABLES OF MONEY.*

2 Farthings	=	1 Halfpenny	½	*qrs*		*d*			
4 Farthings	=	1 Penny	*d*	4	=	1		*s*	
12 Pence	=	1 Shilling	*s*	48	=	12	=	1	£
20 Shillings	=	1 Pound	£	960	=	240	=	20	= 1

PENCE TABLE.

d.		*s.*	*d.*
20	is	1	8
30	—	2	6
40	—	3	4
50	—	4	2
60	—	5	0
70	—	5	10
80	—	6	8
90	—	7	6
100	—	8	4
110	—	9	2
120	—	10	0

SHILLINGS TABLE.

s.		*d.*
1	is	12
2	—	24
3	—	36
4	—	48
5	—	60
6	—	72
7	—	84
8	—	96
9	—	108
10	—	120

TROY

£ denotes pounds, *s* ſhillings, and *d* denotes pence.

¼ denotes 1 farthing, or one quarter of any thing.

½ denotes a halfpenny, or the half of any thing.

¾ denotes 3 farthings, or 3 quarters of any thing.

The full weight and value of the Engliſh gold and ſilver coin, is as here below:

GOLD.	*Value.* £	*s*	*d*	*Weight.* *dwt.*	*gr.*	SILVER.	*Value.* *s*	*d*	*Weight.* *dwt.*	*gr.*
A Guinea	1	1	0	5	9½	A Crown	5	0	19	8½
Half-guinea	0	10	6	2	16¾	Half-crown	2	6	9	16¼
Seven Shillings	0	7	0	1	19⅙	Shilling	1	0	3	21
Quarter-guinea	0	5	3	1	8¼	Sixpence	0	6	1	22½

The uſual value of gold is nearly 4l an ounce, or 2d a grain; and that of ſilver is nearly 5s an ounce. Alſo, any quantity of gold is to the ſame weight of ſtandard ſilver, nearly as 15 to 1, or more nearly as 15 and 1-14th to 1.

Pure gold, free from mixture with other metals, uſually called fine gold, is of ſuch purity of nature, that it will endure the fire without

TROY WEIGHT.*

Grains - - - -	marked	*gr*		*gr*		*dwt*		
24 Grains make	1 Pennyweight	*dwt*		24	=	1	*oz*	
20 Pennyweights	1 Ounce	*oz*		480	=	20	= 1	*lb*
12 Ounces	1 Pound	*lb*		5760	=	240	= 12	= 1

By this weight are weighed Gold, Silver, and Jewels.

APOTHECARIES' WEIGHT.

Grains - - -	marked	*gr*				
20 Grains make	1 Scruple - - -	*sc*	or	℈		
3 Scruples - -	1 Dram - - -	*dr*	or	ʒ		
8 Drams - -	1 Ounce - - -	*oz*	or	℥		
12 Ounces - -	1 Pound - - -	*lb*	or	lb		

gr		*sc*		*dr*			
20	=	1					
60	=	3	=	1	*oz*		
480	=	24	=	8	= 1	*lb*	
5760	=	288	=	96	= 12	=	1

This is the same as Troy weight, only having some different divisions. Apothecaries make use of this weight in compounding their Medicines; but they buy and sell their Drugs by Avoirdupois weight.

AVOIR-

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without wasting, though it be kept continually melted. But silver, not having the purity of gold, will not endure the fire like it: yet fine silver will waste but a very little by being in the fire any moderate time; whereas copper, tin, lead, &c. will not only waste, but may be calcined or burnt to a powder.

Both gold and silver, in their purity, are so very soft and flexible (like new lead, &c.) that they are not so useful either in coin or otherwise (except to beat into leaf gold or silver) as when they are allayed, or mixed and hardened with copper or brass. And though most nations differ, more or less, in the quantity of such allay, as well as in the same place at different times, yet in England the standard for gold and silver coin has been for a long time as follows, viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, shall be esteemed the true standard for gold coin: And that 11 ounces and 2 pennyweights of fine silver, and 18 pennyweights of copper, being melted together, is esteemed the true standard for silver coin, called Sterling silver.

* The original of all weights used in England, was a grain or corn of wheat, gathered out of the middle of the ear, and, being well dried, 32 of them were to make one pennyweight, 20 pennyweights one

Avoirdupois Weight.

	Drams		marked	*dr*
16	Drams make	1	Ounce	*oz*
16	Ounces	1	Pound	*lb*
28	Pounds	1	Quarter	*qr*
4	Quarters	1	Hundred Weight	*cwt*
20	Hundred Weight	1	Ton	*ton*

Dr	*oz*	*lb*	*qr*	*cwt*	*ton*
16 =	1				
256 =	16 =	1			
7168 =	448 =	28 =	1		
28672 =	1792 =	112 =	4 =	1	
573440 =	35840 =	2240 =	80 =	20 =	1

By this Weight are weighed all things of a coarſe or droſſy nature, as Corn, Bread, Butter, Cheeſe, Fleſh, Grocery Wares, and ſome Liquids; alſo all Metals except Silver and Gold.

		oz	*dwt*	*gr*	
Note, that 1*lb* Avoirdupois	=	14	11	$15\frac{1}{2}$	Troy.
1*oz*	=	0	18	$5\frac{1}{2}$	
1*dr*	=	0	1	$3\frac{1}{2}$	

Long Measure.

3	Barley-corns make	1	Inch	*In*
12	Inches	1	Foot	*Ft*
3	Feet	1	Yard	*Yd*
6	Feet	1	Fathom	*Fth*
5	Yards and a half	1	Pole or Rod	*Pl*
40	Poles	1	Furlong	*Fur*
8	Furlongs	1	Mile	*Mile*
3	Miles	1	League	*Lea*
$69\frac{1}{6}$	Miles nearly	1	Degree	*Deg.* or °

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one ounce, and 12 ounces one pound. But in later times, it was thought ſufficient to divide the ſame pennyweight into 24 equal parts, ſtill called grains, being the leaſt weight now in common uſe; and from thence the reſt are computed, as in the Tables above.

In

In	*Fi*		*Yd*			
12 =	1					
36 =	3 =		1	*Pl*		
198 =	16½ =		5½ =	1	*Fur*	
7920 =	660 =		220 =	40 =	1	*Mile*
63360 =	5280 =		1760 =	320 =	8 =	1

CLOTH MEASURE.

2 Inches and a quarter make	1 Nail	-	*Nl*
4 Nails - -	1 Quarter of Yard		*Qr*
3 Quarters - -	1 Ell Flemiſh	-	*E F*
4 Quarters - -	1 Yard -	-	*Yd*
5 Quarters - -	1 Ell Engliſh	-	*E E*
4 Qrs 1⅕ Inc -	1 Ell Scotch	-	*E S*

SQUARE MEASURE.

144	Square Inches make	1 Sq Foot -	*Ft*
9	Square Feet - -	1 Sq Yard -	*Yd*
30¼	Square Yards -	1 Sq Pole -	*Pole*
40	Square Poles -	1 Rood -	*Rd*
4	Roods - -	1 Acre -	*Acr*

Sq Inc	*Sq Ft*		*Sq Yd*			
144 =	1					
1296 =	9 =		1	*Sq Pl*		
39204 =	272¼ =		30¼ =	1	*Rd*	
1568160 =	10890 =		1210 =	40 =	1	*Acr*
6272640 =	43560 =		4840 =	160 =	4 =	1

By this meaſure, Land, Huſbandmen and Gardeners work are meaſured; alſo Artificers works, ſuch as Board, Glaſs, Pavements, Plaſtering, Wainſcotting, Tiling, Flooring, and every dimenſion of length and breadth only.

When three dimenſions are concerned, namely, length, breadth, and depth or thickneſs, it is called cubic or ſolid meaſure, which is uſed to meaſure Timber, Stone, &c.

The cubic or ſolid Foot, which is 12 inches in length and breadth and thickneſs, contains 1728 cubic or ſolid inches, and 27 ſolid feet make one ſolid yard.

DRY

Dry, *or* Corn Measure.

2 Pints make	1 Quart	- -	*Qt*
2 Quarts -	1 Pottle	- -	*Pot*
2 Pottles -	1 Gallon	- -	*Gal*
2 Gallons -	1 Peck	- -	*Pec*
4 Pecks -	1 Buſhel	- -	*Bu*
8 Buſhels -	1 Quarter	- -	*Qr*
5 Quarters -	1 Weigh or Load		*Wey*
2 Weys -	1 Laſt		*Laſt*

Pts	*Gal*					
8 =	1 =	*Pec*				
16 =	2 =	1 =	*Bu*			
64 =	8 =	4 =	1	*Qr*		
512 =	64 =	32 =	8 =	1	*Wey*	
2560 =	320 =	160 =	40 =	5 =	1	*Laſt*
5120 =	640 =	320 =	80 =	10 =	2 =	1

By this are meaſured all dry wares, as Corn, Seeds, Roots, Fruits, Salt, Coals, Sand, Oyſters, &c.

The ſtandard Gallon dry-meaſure contains $268\frac{4}{5}$ cubic or ſolid inches, and the Corn or Wincheſter buſhel $2150\frac{2}{5}$ cubic inches; for the dimenſions of the Wincheſter buſhel, by the Statute, are 8 inches deep, and $18\frac{1}{2}$ inches wide or in diameter. But the Coal buſhel muſt be $19\frac{1}{2}$ inches in diameter; and 36 buſhels, heaped up, make a London chaldron of coals, the weight of which is 3156lb Avoirdupois.

Ale *and* Beer Measure.

2 Pints make -	1 Quart -		*Qt*
4 Quarts - -	1 Gallon -		*Gal*
36 Gallons -	1 Barrel -		*Bar*
1 Barrel and a half	1 Hogſhead		*Hhd*
2 Barrels - -	1 Puncheon		*Pun*
2 Hogſheads -	1 Butt -		*Butt*
2 Butts - -	1 Tun -		*Tun*

Pts	*Qt*			
2 =	1	*Gal*		
8 =	4 =	1	*Bar*	
288 =	144 =	36 = 1	*Hhd*	
432 =	216 =	54 = $1\frac{1}{2}$ = 1	*Butt*	
864 =	432 =	108 = 3	= 2 =	1

Note, The Ale Gallon contains 282 cubic or ſolid inches.

Wine

WINE MEASURE.

2 Pints make	- -	1 Quart	- - *Qt*
4 Quarts	- -	1 Gallon	- - *Gal*
42 Gallons	- -	1 Tierce	- - *Tier*
63 Gallons or 1½ Tier		1 Hogshead	- *Hhd*
2 Tierces	- -	1 Puncheon	- *Pun*
2 Hogsheads	- -	1 Pipe or Butt	*Pi*
2 Pipes	- -	1 Tun	- - *Tun*

Pts	*Qt*	*Gal*	*Tier*	*Hhd*	*Pun*	*Pi*	*Tun*
2 =	1						
8 =	4 =	1					
336 =	168 =	42 =	1				
504 =	252 =	63 =	1½ =	1			
672 =	336 =	84 =	2 =	1½ =	1		
1008 =	504 =	126 =	3 =	2 =	1½ =	1	
2016 =	1008 =	252 =	6 =	4 =	3 =	2 =	1

Note, By this are measured all Wines, Spirits, Strong-waters, Cyder, Mead, Perry, Vinegar, Oil, Honey, &c.

The Wine Gallon contains 231 cubic or solid inches. And it is remarkable, that the Wine and Ale Gallons have the same proportion to each other, as the Troy and Avoirdupois Pounds have; that is, as one Pound Troy is to one Pound Avoirdupois, so is one Wine Gallon to one Ale Gallon.

Of TIME.

60 Seconds make	-	1 Minute	- *M* or
60 Minutes	- -	1 Hour	- *Hr*
24 Hours	- -	1 Day	- *Day*
7 Days	- -	1 Week	- *Wk*
4 Weeks	- -	1 Month	*Mo*
13 Months, 1 Day, 6 Hours or 365 Days, 6 Hours		1 Julian Year,	*Yr*

Sec	*Min*	*Hr*	*Day*	*Wk*	*Mo*	*Year*
60 =	1					
3600 =	60 =	1				
86400 =	1440 =	24 =	1			
604800 =	10080 =	168 =	7 =	1		
2419200 =	40320 =	672 =	28 =	4 =	1	
31557600 =	525960 =	8766 =	365¼ =	1		*Year*

Wk

Wk Da Hr Mo Da Hr Julian Year
Or 52 1 6 = 13 1 6 = 1
Da Hr M Sec
But 365 5 48 48 = 1 *Solar Year.*

RULES FOR REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower:*

MULTIPLY the number in the higheſt denomination by as many of the next lower as make an integer, or 1, in that higher; to this product add the number, if any, which was in this lower denomination before, and ſet down the amount.

Reduce this amount in like manner, by multiplying it by as many of the next lower as make an integer of this, taking in the odd parts of this lower, as before. And ſo proceed through all the denominations, to the loweſt; ſo ſhall the number laſt found be the value of all the numbers which were in the higher denominations, taken together*.

EXAMPLE.

1. In 1234l 15s 7d, how many farthings?

l	s	d
1234	15	7
20		
24695	Shillings	
12		
296347	Pence	
4		
Anſwer 1185388	Farthings.	

* The reaſon of this rule is very evident; for pounds are brought into ſhillings by multiplying them by 20; ſhillings into pence, by multiplying them by 12; and pence into farthings, by multiplying by 4; and the reverſe of this rule by Diviſion.—And the ſame, it is evident, will be true in the reduction of numbers conſiſting of any denominations whatever.

II. *When*

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher,*

DIVIDE the given number by as many of that denomination as make 1 of the next higher, and ſet down what remains, as well as the quotient.

Divide the quotient by as many of this denomination as make 1 of the next higher; ſetting down the new quotient, and remainder, as before.

Proceed in the ſame manner through all the denominations, to the higheſt; and the quotient laſt found, together with the ſeveral remainders, if any, will be of the ſame value as the firſt number propoſed.

EXAMPLES.

2. Reduce 1185388 farthings, into pounds, ſhillings, and pence.

4) 1185388

12) 296347 d

2,0) 2469,5 s—7d

Anſwer 12341 15s 7d.

3. Reduce 23l to farthings. Anſ. 22080
4. Reduce 337587 farthings to pounds, &c.
Anſ. 351l 13s $0\frac{3}{4}$d
5. How many farthings are in 35 guineas? Anſ. 35280
6. In 35280 farthings how many guineas? Anſ. 35
7. In 59lb 13 dwts 5 gr how many grains? Anſ. 340157
8. In 8012131 grains how many pounds, &c?
Anſ. 1390lb 11oz 18dwt 19gr.
9. In 35 ton 17cwt 1qr 23lb 7oz 13dr how many drams?
Anſ. 20571005.
10. How many barley-corns will reach round the earth, ſuppoſing it, according to the beſt calculations, to be 25000 miles? Anſ. 4752000000.
11. How many ſeconds are in a ſolar year, or 365 days 5hrs 48 min 48 ſec? Anſ. 31556928.
12. In a lunar month, or 29ds 12hrs 44min 3ſec how many ſeconds? Anſ 2551443.

COM-

COMPOUND ADDITION.

COMPOUND ADDITION ſhews how to add or collect ſeveral numbers of different denominations into one ſum.

RULE.—Place the numbers ſo that thoſe of the ſame denomination may ſtand directly under each other, and draw a line below them. Add up the figures in the loweſt denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their ſum.—Set down the remainder below its proper column, and carry thoſe units or ones to the next denomination, which add up in the ſame manner as before.—Proceed thus through all the denominations, to the higheſt, whoſe ſum, together with the ſeveral remainders, will give the anſwer ſought.

The method of proof is the ſame as in Simple Addition.

EXAMPLES OF MONEY.

1.			2.			3.			4.		
l	s	d	l	s	d	l	s	d	l	s	d
7	13	3	14	7	5	15	17	10	53	14	8
3	5	10½	8	19	2¼	3	14	6	5	10	2¾
6	18	7	5	3	4½	23	6	2¾	93	11	6
9	2	5¾	21	2	9	8	3	5	7	5	0
4	0	3	7	16	8½	15	6	4	2	0	9
17	15	4½	0	4	3	6	12	9¾	0	18	7
39	15	9¾									
32	2	6¾									
39	15	9¾									

5.			6.			7.			8.		
l	s	d	l	s	d	l	s	d	l	s	d
14	0	7¼	37	15	8	61	3	2½	472	15	3
5	13	6	14	12	9¾	7	16	8	9	2	2½
62	4	7	5	6	11	29	13	10¾	27	12	6¼
4	17	8	23	10	9¼	8	14	0	370	16	2½
23	0	4¾	8	6	0	0	7	5¼	25	3	8
6	6	7	14	0	5½	24	13	0	6	10	5¾
91	0	10¼	54	2	7½	5	0	10¾	30	0	11¾

EXAM-

EXAM. 9. A nobleman, going out of town, is informed by his ſteward that his butcher's bill comes to 197l 13s 7½d; his baker's to 59l 5s 2¼d; his brewer's to 85l; his wine-merchant's to 103l 13s; to his corn-chandler is due 75l 3d; to his tallow-chandler and cheeſe-monger, 27l 15s 11¼d; and to his tailor, 55l 3s 5¾d; alſo for rent, ſervants wages, and other charges, 127l 3s: Now, ſuppoſing he would take 100l with him, to defray his charges on the road, for what ſum muſt he ſend to his banker? Anſ. 830l 14s 6¼d.

10. The ſtrength of a regiment of foot, of 10 companies, and the amount of their ſubſiſtence *, for a month of 30 days, according to the annexed Table, are required?

Numb.	Rank.	Subſiſtence for a Month. l	s	d
1	Colonel	27	0	0
1	Lieutenant Colonel	19	10	0
1	Major	17	5	0
7	Captains	78	15	0
11	Lieutenants	57	15	0
9	Enſigns	40	10	0
1	Chaplain	7	10	0
1	Adjutant	4	10	0
1	Quarter-maſter	5	5	0
1	Surgeon	4	10	0
1	Surgeon's Mate	4	10	0
30	Serjeants	45	0	0
30	Corporals	30	0	0
20	Drummers	20	0	0
2	Fifes	2	0	0
390	Private men	292	10	0
507	Total	656	10	0

* Subſiſtence Money, is the money paid to the ſoldiers weekly, ſhort of their full pay; becauſe their clothes, accoutrements, &c. are to be accounted for. It is likewiſe the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following Table ſhews the full pay and ſubſiſtence of each rank on the Engliſh eſtabliſhment:

DAILY PAY OF COMMISSIONED OFFICERS,

ACCORDING TO THE LATEST REGULATIONS.

RANK.	Horſe Artillery & corps of cap. com.	D. G. Dr. & F. Cav.	Foot Artillery.	Regular and Fenc. Inf. and Militia.	Life Guards.		Horſe Guards.		Foot Guards.	
					Subſiſt.	Full Pay.	Subſiſt.	Full Pay.	Subſiſt.	Full Pay.
Colonel (Comm.)	—	1 12 10	2 4 0	1 2 6	1 7 0	1 16 0	1 11 0	—	1 0 0	1 19 0
Colonel (en Second)	1 10 0	—	1 4 0							
1ſt Lieut. Col.	1 6 0	1 3 0	1 0 0	0 15 11	1 3 3	1 11 0	1 2 6	—	1 1 6	1 8 6
2d Ditto	—	—	0 17 0							
1ſt Major	1 1 0	0 19 3	0 15 0	0 14 1	0 19 6	1 6 0	1 1 6	—	0 18 6	1 4 6
2d Ditto	—	—	—	—	0 18 0	1 4 0				
Captain	0 15 0	0 14 7	0 10 0	0 9 5	0 12 1	0 16 0	0 16 6	—	0 12 6	0 16 6
Capt. Lieut.	0 10 0	0 9 0	0 7 0	0 5 8	0 8 2	0 11 0	0 11 6	0 15 0	0 6 0	0 7 10
1ſt Lieut.	0 9 0	0 9 0	0 6 0	0 5 8						
2d Ditto	0 8 0	—	0 5 0							
Cornet	—	0 8 0	—	—	—	—	0 11 0	0 14 0		
Enſign	—	—	—	0 4 8	—	—	—	—	0 4 6	0 5 10
Adjutant	—	0 5 0	—	0 5 0	0 8 6	0 11 0	0 4 6	0 5 0	0 3 0	0 4 0
Pay-maſter										
Quarter-maſter	—	0 5 0	—	0 5 8	—	—	0 6 6	0 8 6	—	0 5 8
Surgeon-major	0 12 0	0 11 4	0 10 0	0 9 5	—	—	—	—	0 12 6	0 15 0
Bat. Surg. or Surg.	—	—	—	—	0 6 0	0 8 0	0 9 0	0 12 0	0 7 6	0 10 0
Aſſiſt. Surg.	0 6 0	0 5 0	0 5 0	0 5 0	—	—	0 5 0	0 5 0	0 5 0	0 5 0
Veter. Surg.	0 8 0	0 8 0								
Solicitor										

N. B When a *Lieutenant*, *Enſign*, *Adjutant*, or *Quarter-maſter* of Foot, Militia, Fencible Infantry, or Invalids, holds two commiſſions, one ſhilling per day is to be deducted from the above rates for each commiſſion.

EXAM-

EXAMPLES *of* WEIGHTS, MEASURES, *&c.*

TROY WEIGHT.

1.			2.			3.				4.			
lb	oz	dwt	oz	dwt	gr	lb	oz	dr	sc	oz	dr	sc	gr
17	3	15	37	9	3	3	5	7	2	3	5	1	17
4	6	3	9	5	3	13	7	3	0	7	3	2	5
0	10	7	3	16	21	9	11	0	1	16	7	0	12
9	5	0	17	7	8	0	9	1	2	9	5	1	5
176	2	17	5	9	0	36	3	5	0	4	1	2	18
23	11	12	3	0	19	5	8	6	1	36	4	1	14

AVOIRDUPOIS WEIGHT.

5.			6.			7.			8.		
lb	oz	dr	cwt	qr	lb	mls	fur	pls	yds	feet	inc
17	10	13	15	2	15	29	3	14	127	1	5
5	14	8	6	3	24	19	6	29	12	2	9
8	6	15	7	0	10	5	4	20	0	2	6
27	1	6	9	1	17	9	1	37	54	1	11
0	4	0	10	2	6	7	0	3	5	2	7
6	14	10	3	0	3	4	5	9	23	0	5

CLOTH MEASURE.

9.			10.			
yds	qr	nls	el	en	qrs	nls
26	3	1	270	1	0	
13	1	2	57	4	3	
6	2	0	8	2	1	
217	0	3	0	3	2	
9	1	0	10	1	0	
55	3	1	4	4	1	

LAND MEASURE.

11.			12.		
ac	ro	p	ac	ro	p
225	3	37	19	0	16
16	1	25	270	3	29
9	0	13	9	1	3
4	2	9	23	0	34
42	1	19	7	2	16
7	0	6	75	0	23

WINE MEASURE.

13.			14.		
t	hds	gal	hds	gal	pts
13	3	15	15	61	5
8	1	37	7	16	3
4	2	26	29	23	7
25	0	12	3	15	1
3	1	9	16	8	0
72	3	21	4	36	6

ALE and BEER MEASURE.

15.			16.		
hds	gal	pts	hds	gal	pts
17	37	3	29	43	5
4	13	5	7	9	2
3	6	2	14	16	6
5	14	0	6	8	1
12	9	6	57	13	4
8	42	4	5	6	0

COMPOUND SUBTRACTION.

COMPOUND SUBTRACTION ſhews how to find the difference between any two numbers of different denominations.—To perform which, obſerve the following Rule:

* RULE. *

PLACE the leſs number below the greater, ſo that thoſe parts which are of the ſame denomination may ſtand directly under each other; and draw a line below them.—Begin at the right-hand, and ſubtract each number or part in the lower line, from the one juſt above it, and ſet the remainder ſtraight below it.—But if any number in the lower line be greater than that above it, add ſo many to the upper number as make 1 of the next higher denomination; then take the lower number from the upper one thus increaſed, and ſet down the remainder. Carry the unit borrowed to the next number in the lower line; after which ſubtract this number from the one above it, as before; and ſo proceed till the whole is finiſhed. Then the ſeveral remainders, taken together, will be the whole difference ſought.

The method of Proof is the ſame as in Simple Subtraction.

EXAMPLES OF MONEY.

	1.			2.			3.			4.		
	l	s	d	l	s	d	l	s	d	l	s	d
From	79	17	$8\frac{3}{4}$	103	3	$2\frac{1}{2}$	57	0	10	251	13	0
Take	35	12	$4\frac{1}{4}$	71	12	$5\frac{3}{4}$	29	13	$3\frac{1}{4}$	35	4	$7\frac{3}{4}$
Rem.	44	5	$4\frac{1}{2}$	31	10	$8\frac{3}{4}$						
Proof	79	17	$8\frac{3}{4}$	103	3	$2\frac{1}{2}$						

5. What is the difference between 73l $5\frac{1}{4}$d and 19l 13s 10d?
Anſ. 53l 6s $7\frac{1}{4}$d.

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* The reaſon of this Rule will eaſily appear from what has been ſaid in Simple Subtraction; for the borrowing depends upon the ſame principle, and is only different as the numbers to be ſubtracted are of different denominations.

Ex. 6.

Ex. 6. A lends to B 100l, how much is B in his debt after A has taken goods of him to the amount of 73l 12s $4\frac{3}{4}$d?

Anſ. 26l 7s $7\frac{1}{4}$d.

7. Suppoſe that my rent for half a year is 10l 12s, and that I have laid out for the land-tax 14s 6d, and for ſeveral repairs 1l 3s $3\frac{1}{4}$d, what have I to pay of my half year's rent?

Anſ. 8l 14s $2\frac{3}{4}$.

8. A trader, failing, owes to A 35l 7s 6d, to B 91l 13s $\frac{1}{2}$d, to C 53l $7\frac{1}{4}$d, to D 87l 5s, and to E 111l 3s $5\frac{3}{4}$d. When this happened, he had by him in caſh 23l 7s 5d, in wares 53l 11s $10\frac{1}{4}$d, in houſehold furniture 63l 17s $7\frac{3}{4}$d, and in recoverable book-debts 25l 7s 5d. What will his creditors loſe by him, ſuppoſe theſe things delivered to them?

Anſ. 212l 5s $6\frac{1}{2}$d.

EXAMPLES *of* WEIGHTS, MEASURES, *&c.*

TROY WEIGHT.

	1. lb	oz	dwt	gr	2. lb	oz	dwt	gr
From	7	3	14	11	4	9	1	13
Take	3	7	5	19	3	7	16	12
Rem.								
Proof								

APOTHECAR. WEIGHT.

	3. lb	oz	dr	ſcr	gr
From	73	4	7	0	14
Take	26	7	2	1	16
Rem.					
Proof					

AVOIRDUPOIS WEIGHT.

	4. c	qrs	lb	5. lb	oz	dr
From	5	0	17	71	5	9
Take	3	2	11	14	6	14
Rem.						
Proof						

LONG MEASURE.

	6. m	fu	pl	7. yd	ft	in
From	14	3	17	96	1	4
Take	3	7	9	41	2	7
Rem.						
Proof						

CLOTH MEASURE.

	8. yd	qr	nl	9. yd	qr	nl
From	17	2	1	9	0	2
Take	5	2	1	6	1	2
Rem.						
Proof						

LAND MEASURE.

	10. ac	ro	p	11. ac	ro	p
From	17	1	14	57	1	16
Take	6	3	6	24	2	25
Rem.						
Proof						

WINE MEASURE.

	12. t	hd	gal	13. hd	gal	pt
From	17	2	23	5	0	4
Take	4	3	39	3	2	7
Rem.						
Proof						

ALE & BEER MEASURE.

	14. hd	gal	pt	15. hd	gal	pt
From	14	29	3	71	16	5
Take	7	34	5	17	3	2
Rem.						
Proof						

DRY MEASURE

	16. la	qr	bu	17. bu	gal	pt
From	9	4	7	13	7	1
Take	3	7	2	7	3	4
Rem.						
Proof						

TIME.

	18. mo	we	da	19. ds	hrs	min
From	71	2	5	114	17	26
Take	14	3	0	75	12	35
Rem.						
Proof						

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and ſhoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the baſtion? Anſ. 89 yds 1 f 8 in.

COMPOUND MULTIPLICATION

COMPOUND MULTIPLICATION ſhews how to find the amount of any given number of different denominations repeated a certain propoſed number of times.

RULE.

SET the multiplier under the loweſt denomination of the multiplicand, and draw a line below it.—Multiply the number in the loweſt denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, ſetting down what remains.—In like manner multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount,

amount, which carry in like manner to the next product, ſetting down the overplus.—Proceed thus to the higheſt denomination propoſed; ſo ſhall the laſt product, with the ſeveral remainders, taken as one compound number, be the whole *amount* required.

The method of Proof, and the reaſon of the Rule, are the ſame as in Simple Multiplication.

EXAMPLES OF MONEY.

1. To find the amount of 8lb of Tea, at 5s $8\frac{1}{2}$d per lb.

s	d
5	$8\frac{1}{2}$
	8
£2	5 8

Anſwer.

			l	s	d
2.	4lb of Tea, at 7s 8d per lb.	Anſ.	1	10	8
3.	6lb of Butter, at $9\frac{1}{2}$d per lb.	Anſ.	0	4	9
4.	7lb of Tobacco, at 1s $8\frac{1}{2}$d per lb.	Anſ.	0	11	$11\frac{1}{2}$
5.	9 cwt of Cheeſe, at 1l 11s 5d per cwt.	Anſ.	14	2	9
6.	10 cwt of Cheeſe, at 2l 17s 10d per cwt.	Anſ.	28	18	4
7.	12 cwt of Sugar, at 3l 7s 4d per cwt.	Anſ.	40	8	0

CONTRACTIONS.

I. If the multiplier exceed 12, multiply ſucceſſively by its component parts, inſtead of the whole number at once.

EXAMPLES.

1. 15 cwt of Cheeſe, at 17s 6d per cwt.

l	s	d
0	17	6
		3
2	12	6
		5
13	2	6

Anſwer.

			l	s	d
2.	20 cwt. of Hops, at 4l 7s 2d per cwt.	Anſ.	87	3	4
3.	24 tons of Hay, at 3l 7s 6d per ton.	Anſ.	81	0	0
4.	45 ells of Cloth, at 1s 6d per ell.	Anſ.	3	7	6

Ex. 5.

		l	s	d
Ex. 5.	63 gallons of Oil, at 2s 3d per gall. Anſ.	7	1	9
6.	70 barrels of Ale, at 1l 4s per barrel. Anſ.	84	0	0
7.	84 quarters of Oats, at 1l 12s 8d per qr. Anſ.	137	4	0
8.	96 quarters of Barley, at 1l 3s 4d per qr. Anſ.	112	0	0
9.	120 days Wages, at 5s 9d per day. Anſ.	34	10	0
10.	144 reams of Paper, at 1s 4d per ream. Anſ.	96	0	0

II. If the multiplier cannot be exactly produced by the multiplication of ſimple numbers, take the neareſt number to it, either greater or leſs, which can be ſo produced, and multiply by its parts, as before.—Then multiply the given multiplicand by the difference between this aſſumed number and the multiplier, and add the product to that before found when the aſſumed number is leſs than the multiplier, but ſubtract the ſame when it is greater.

EXAMPLES.

1. 26 yards of Cloth, at 3s $0\frac{3}{4}$d per yard.

l	s	d
0	3	$0\frac{3}{4}$
		5
0	15	$3\frac{3}{4}$
		5
3	16	$6\frac{3}{4}$
	3	$0\frac{3}{4}$
£ 3	19	$7\frac{1}{2}$ Anſwer.

		l	s	d
2.	29 quarters of Corn, at 2l 5s $3\frac{1}{4}$d per qr. Anſ.	65	12	$10\frac{1}{4}$
3.	53 loads of Hay, at 3l 15s 2d per load. Anſ.	199	3	10
4.	79 buſhels of Wheat, at 11s $5\frac{3}{4}$d per buſh. Anſ.	45	6	$10\frac{1}{4}$
5.	94 caſks of Beer, at 12s 2d per caſk. Anſ.	57	3	8
6.	114 ſtone of Meat, at 15s $3\frac{3}{4}$d per ſtone. Anſ.	87	5	$7\frac{1}{2}$

EXAMPLES OF WEIGHTS AND MEASURES.

1.				2.					3.			
lb	oz	dwt	gr	lb.	oz	dr	ſc	gr	cwt	qr	lb	oz
21	1	7	13	2	4	2	1	0	27	1	13	12
			4					7				12

4.				5.			6.		
mls	fu	pls	yds	yds	qrs	na	ac	ro	po.
24	3	20	2	127	2	2	27	2	1
			6			8			9

7.				8.				9.				
tuns	hhd	gal	pts	we	qr	bu	pe	mo	we	da	ho	min
29	1	20	3	27	1	7	2	175	3	6	20	59
			5				7					pt.

COMPOUND DIVISION.

COMPOUND DIVISION teaches how to divide a number of feveral denominations by any given number, or into any number of equal parts.

RULE.

PLACE the diviſor on the left of the dividend, as in Simple Diviſion.—Begin at the left-hand, and divide the number of the higheſt denomination by the diviſor, ſetting down the quotient in its proper place.—If there be any remainder after this diviſion, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the ſum by the diviſor.—Set down again this quotient, reduce its remainder to the next lower denomination again, and ſo on through all the denominations to the laſt.

EXAMPLES OF MONEY.

1. Divide 225l 2s 4d by 2.

l	s	d
2) 225	2	4
£112	11	2 the Quotient.

2. Di-

		l	s	d			l	s	d
2.	Divide	751	14	7¾	by 3.	Anſ.	250	11	6½
3.	Divide	821	17	9¾	by 4.	Anſ.	205	9	5¼
4.	Divide	2382	13	5½	by 5.	Anſ.	476	10	8¼
5.	Divide	28	2	1½	by 6.	Anſ.	4	13	8¼
6.	Divide	55	14	0¾	by 7.	Anſ.	7	19	1¾
7.	Divide	6	5	4	by 8.	Anſ.	0	15	8
8.	Divide	135	10	7	by 9.	Anſ.	15	1	2
9.	Divide	21	18	4	by 10.	Anſ.	2	3	10
10.	Divide	227	10	5	by 11.	Anſ.	20	13	8
11.	Divide	1332	11	8½	by 12.	Anſ.	111	0	11½

CONTRACTIONS.

I. If the diviſor exceed 12, find what ſimple numbers, multiplied together, will produce it, and divide by them ſeparately, as in Simple Diviſion.

EXAMPLES.

1. What is Cheeſe per cwt if 16 cwt coſt 30l 18s 8d?

l	s	d
4) 30	18	8
4) 7	14	8
£ 1	18	8 the Anſwer.

		l	s	d
2.	If 20 cwt of Tobacco come to 120l 10s, what is that per cwt?	Anſ.	6	0	6
3.	Divide 57l 3s 7d by 35.	Anſ.	1	12	8
4.	Divide 85l 6s by 72.	Anſ.	1	3	8¼
5.	Divide 31l 2s 10½d by 99.	Anſ.	0	6	3½
6.	At 18l 18s per cwt, how much per lb?	Anſ.	0	3	4½

II. If the diviſor cannot be produced by the multiplication of ſmall numbers, divide by the whole diviſor at once, after the manner of Long Diviſion.

EXAM-

EXAMPLES.

1. Divide 74l 13s 6d by 17.

l	s	d		l	s	d	
17) 74	13	6	(	4	7	10	Anſwer.
68							
6							
20							
133							
119							
14							
12							
174							
170							
4							

	l	s	d		l	s	d
2. Divide	23	15	7¼	by 37.	Anſ. 0	12	10¼
3. Divide	199	3	10	by 53.	Anſ. 3	15	2
4. Divide	675	12	6	by 138.	Anſ. 4	17	11
5. Divide	315	3	10¼	by 365.	Anſ. 0	17	3¼

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 23lb 7oz 6dwts 12gr by 7.
Anſ. 3lb 4oz 9dwts 12gr.
2. Divide 13lb 10z 2dr oſcr 10gr by 12.
Anſ. 1lb 10z odr 2ſcr 10gr.
3. Divide 106 1cwt 2qrs by 28. Anſ. 37cwt 3qrs 18lb.
4. Divide 375mi 2fur 7po 2yds 1ft 2in by 39.
Anſ. 9mi 4fur 39po oyds 2ft 8in.
5. Divide 571yds 2qrs 1na by 47. Anſ. 12yds oqrs 2na.
6. Divide 51ac 2ro 3po by 51. Anſ. 1ac oro 1po.
7. Divide 10tu 2hhds 17 gall 2pi by 67. Anſ. 39gal 3p.
8. Divide 120la 1qr 1bu 2pe by 74. Anſ. 1la 6qrs 1bu 3pe.
9. Divide 120mo 2we 3da 5ho 20min by 111.
Anſ. 1mo owe 2da 10ho 12mi.

GOLDEN

GOLDEN RULE, OR RULE OF THREE.

THE RULE OF THREE teaches how to find a fourth proportional to three numbers given. Whence it is alſo ſometimes called the Rule of Proportion. It is called the Rule of Three, becauſe three terms or numbers are given, to find a fourth. And becauſe of its great and extenſive uſefulneſs, it is often called the Golden Rule.

This Rule is uſually conſidered as of two kinds, namely, Direct, and Inverſe.

The Rule of Three Direct, is that in which more requires more, or leſs requires leſs. As in this, if 3 men dig 21 yards of trench in a certain time, how much will 6 men dig in the ſame time? Here more requires more, that is, 6 men, which are more than 3 men, will alſo perform more work, in the ſame time. Or when it is thus: if 6 men dig 42 yards, how much will 3 men dig in the ſame time? Here, then, leſs requires leſs, or 3 men will perform proportionally leſs work than 6 men, in the ſame time. In both theſe caſes, then, the Rule, or the Proportion, is Direct; and the ſtating muſt be

thus, As 3 : 21 :: 6 : 42,
or thus, As 6 : 42 :: 3 : 21.

But the Rule of Three Inverſe, is when more requires leſs, or leſs requires more. As in this; if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform an equal quantity of work in leſs time, or fewer hours. Or thus; if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the ſame? Here leſs requires more, for 3 men will take more hours than 6 to perform the ſame work. In both theſe caſes, then, the Rule, or the Proportion, is Inverſe; and the ſtating muſt be

thus, As 6 : 14 :: 3 : 7,
or thus, As 3 : 7 :: 6 : 14.

And in all theſe ſtatings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1ſt term.

Of the three given numbers, two of them contain the ſuppoſition, and the third a demand. And for ſtating and working queſtions of theſe kinds, obſerve the following general Rule:

RULE.

RULE.

STATE the queſtion, by ſetting down in a ſtraight line the three given numbers, in the following manner, viz. ſo that the 2d term be that number of ſuppoſition which is of the ſame kind that the anſwer or 4th term is to be; making the other number of ſuppoſition the 1ſt term, and the demanding number the 3d term, when the queſtion is in direct proportion; but contrariwiſe, the other number of ſuppoſition the 3d term, and the demanding number the 1ſt term, when the queſtion has inverſe proportion.

Then, in both caſes, multiply the 2d and 3d terms together, and divide the product by the 1ſt, which will give the anſwer, or 4th term ſought, of the ſame denomination as the ſecond term.

Note, If the firſt and third terms conſiſt of different denominations, reduce them both to the ſame: and if the ſecond term be a compound number, is is moſtly convenient to reduce it to the loweſt denomination mentioned.—If, after diviſion, there be any remainder, reduce it to the next lower denomination, and divide by the ſame diviſor as before, and the quotient will be of this laſt denomination. Proceed in the ſame manner with all the remainders, till they be reduced to the loweſt denomination which the ſecond term admits of, and the ſeveral quotients taken together will be the anſwer required.

Note alſo, The reaſon for the foregoing Rules will appear when we come to treat of the nature of Proportions.—Sometimes alſo two or more ſtatings are neceſſary, which may always be known from the nature of the queſtion.

EXAMPLES.

1. If 8 yards of Cloth coſt 11 4s, what will 96 yards coſt?

yds	l	s	yds	l	s	
As 8	: 1	4	:: 96	: 14	8	the Anſwer.

20
24
96
144
216
8) 2304
2,0) 28,8s
£14 8 Anſwer

Ex. 2.

Ex. 2. An engineer having raiſed 100 yards of a certain work in 24 days with 5 men; how many men muſt he employ to finiſh a like quantity of work in 15 days?

	ds		men		ds		men	
As	15	:	5	::	24	:	8	Anſ.

5

15) 120 (8 Anſwer.
120

3. What will 72 yards of Cloth coſt, at the rate of 9 yards for 5l 12s? Anſ. 44l 16s.

4. A perſon's annual income being 146l; how much is that per day? Anſ. 8s.

5. If 3 paces or common ſteps of a certain perſon be equal to 2 yards; how many yards will 160 of his paces make? Anſ. 106yds 2ft.

6. What length muſt be cut off a board, that is 9 inches broad, to make a ſquare foot, or as much as 12 inches in length and 12 in breadth contains? Anſ. 16 inches.

7. If 750 men require 22500 rations of bread for a month; how many rations will a garriſon of 1200 men require? Anſ. 36000.

8. If 7cwt 1qr of ſugar coſt 26l 10s 4d, what will be the price of 43cwt 2qrs? Anſ. 159l 2s.

9. The clothing of a regiment of foot of 750 men amounting to 2831l 5s; what will the clothing of a body of 3500 men amount to? Anſ. 13212l 10s.

10. How many yards of matting, that is 2ft 6in broad, will cover a floor that is 27 feet long and 20 feet broad? Anſ. 72yds.

11. What is the value of 6 buſhels of coals, at the rate of 1l 14s 6d the chaldron? Anſ. 5s 9d.

12. If 6352 ſtones of 3 feet long complete a certain quantity of walling; how many ſtones of 2 feet long will raiſe a like quantity? Anſ. 9528.

13. What muſt be given for a piece of ſilver, weighing 73lb 5oz 15dwts, at the rate of 5s 9d per ounce? Anſ. 253l 10s $0\frac{3}{4}$d.

14. A garriſon of 536 men have proviſion for 12 months; how long will thoſe proviſions laſt, if the garriſon be increaſed to 1124 men? Anſ. 174 days and $\frac{64}{1124}$.

15. What will the tax upon 763l 15s be, at the rate of 3s 6d per pound ſterling? Anſ. 133l 13s $1\frac{1}{2}$d.

16. A

16. A certain work being raiſed in 12 days, by working 4 hours each day; how long would it have been in raiſing by working 6 hours per day? Anſ. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6s the buſhel? Anſ. 39qrs 3bu.

18. A perſon, failing in trade, owes in all 977l; at which time he has in money, goods, and recoverable debts, 420l 6s 3¼d; now ſuppoſing theſe things delivered to his creditors, how much will they get per pound? Anſ. 8s 7¼d.

19. A plain of a certain extent having ſupplied a body of 3000 horſe with forage for 18 days; then how many days would the ſame plain have ſupplied a body of 2000 horſe? Anſ. 27 days.

20. Suppoſe a gentleman's income is 500 guineas a year, and that he ſpends 19s 7d per day, one day with another; how much will he have ſaved at the year's end? Anſ. 167l 12s 1d.

21. What coſt 30 pieces of lead, each weighing 1cwt 12lb, at the rate of 16s 4d the cwt? Anſ. 27l 2s 6d.

22. The governor of a beſieged place having proviſion for 54 days, at the rate of 1½lb of bread; but being deſirous to prolong the ſiege to 80 days, in expectation of ſuccour, in that caſe what muſt the ration of bread be? Anſ. 1$\frac{1}{80}$lb.

23. At half a guinea per week, how long can I be boarded for 20 pounds? Anſ. 38$\frac{1}{126}$wks.

24. How much will 75 chaldrons 7 buſhels of coals come to, at the rate of 1l 13s 6d per chaldron? Anſ. 125l 19s 1½d.

25. If the penny loaf weigh 9 ounces when the buſhel of wheat coſt 6s 3d; what ought the penny loaf to weigh when the wheat is at 8s 2½d? Anſ. 6oz 13$\frac{17}{19}$dr.

26. How much a year will 173 acres 2 roods 14 poles of land give, at the rate of 1l 7s 8d per acre? Anſ. 240l 2s 7$\frac{1}{20}$d.

27. To how much amounts 172 pieces of lead, each weighing 3cwt 2qrs 17½lb, at 8l 17s 6d per fother of 19½cwt? Anſ. 286l 4s 4½d.

28. How many yards of ſtuff, of 3qrs wide, will line a cloak that is 5½ yards in length and 1¼ yard wide? Anſ. 9yds 2qrs 2$\frac{3}{5}$nl.

29. If 5 yards of cloth coſt 14s 2d, what muſt be given for 9 pieces, containing each 21 yards 1 quarter? Anſ. 27l 1s 10½d.

30. If a gentleman's eſtate be worth 2107l 12s a year; what may he ſpend per day, to ſave 500l in the year? Anſ. 4l 8s 1$\frac{19}{365}$d.

31. Want-

31. Wanting juſt an acre of land cut off from a piece which is $13\frac{1}{2}$ poles in breadth, what length muſt the piece be?
Anſ. 11po 4yds 2ft $0\frac{8}{17}$in.

32. At 13s $2\frac{1}{2}$d per yard, what is the value of a piece of cloth containing $52\frac{3}{4}$ ells Engliſh? Anſ. 43l 10s $11\frac{6}{16}$d.

33. If the carriage of 5cwt 14lb for 96 miles be 1l 12s 6d; how far may I have 3cwt 1qr carried for the ſame money?
Anſ. 151m 3fur $3\frac{1}{13}$pol.

34. Bought a ſilver tankard, weighing 1lb 7oz 14dwts; what did it coſt me at 6s 4d the ounce? Anſ. 6l 4s $9\frac{1}{5}$d.

35. What is the half year's rent of 547 acres of land, at 15s 6d the acre? Anſ. 211l 19s 3d.

36. A wall that is to be built to the height of 27 feet, was raiſed 9 feet high by 12 men in 6 days; then how many men muſt be employed to finiſh the wall in 4 days, at the ſame rate of working? Anſ. 36 men.

37. What will be the charge of keeping 11 horſes for a year, at the rate of $11\frac{1}{2}$d per day for each horſe?
Anſ. 192l 7s $8\frac{1}{2}$d.

38. If 15 ells of ſtuff that is $\frac{1}{4}$ yard wide coſt 37s 6d; what will 40 ells, of the ſame goodneſs, coſt, being yard wide?
Anſ. 61 13s 4d

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit, and 9 feet high?
Anſ. 72 yds.

40. If a gentleman's eſtate be worth 384l 16s a year, and the land-tax be aſſeſſed at 2s $9\frac{1}{2}$d per pound, what is his net annual income? Anſ. 331l 17s $9\frac{1}{4}$d.

41. The circumference of the earth is about 25000 miles; at what rate per hour is a perſon at the middle of its ſurface carried round, one whole rotation being made in 23 hours 56 minutes? Anſ. $1044\frac{8}{14}\frac{16}{36}$ miles.

42. If a perſon drink 20 bottles of wine per month, when it coſts 8s a gallon; how many bottles per month may he drink, without increaſing the expence, when wine coſts 10s the gallon? Anſ. 16 bottles.

43. What coſt 43qrs 5 buſhels of corn, at 1l 8s 6d the quarter? Anſ. 62l 3s. $3\frac{3}{4}$d.

44. How many yards of canvas that is ell wide, will line 20 yards of ſay that is 3 quarters wide? Anſ. 12yds.

45. If an ounce of gold coſt 4 guineas, what is the value of a grain? Anſ. $2\frac{1}{10}$d.

46. If 3cwt of tea coſt 40l 12s; at how much a pound muſt it be retailed, to gain 10l by the whole? Anſ. $3\frac{4}{336}$s.

COMPOUND PROPORTION.

COMPOUND PROPORTION teaches how to reſolve ſuch queſtions as require two or more ſtatings by Simple Proportion; and that, whether they be Direct or Inverſe.

In theſe queſtions, there is always given an odd number of terms, either five, or ſeven, or nine, &c. Theſe are diſtinguiſhed into terms of ſuppoſition, and terms of demand, there being always one term more of the former than of the latter, which is of the ſame kind with the anſwer ſought.

RULE.

SET down in the middle place that term of ſuppoſition which is of the ſame kind with the anſwer ſought.—Take one of the other terms of ſuppoſition, and one of the demanding terms which is of the ſame kind with it; then place one of them for a firſt term, and the other for a third, according to the directions given in the Rule of Three.—Do the ſame with another term of ſuppoſition, and its correſponding demanding term; and ſo on if there be more terms of each kind; ſetting the numbers under each other which fall all on the left-hand ſide of the middle term, and the ſame for the others on the right-hand ſide.—Then, to work

By ſeveral Operations.—Take the two upper terms and the middle term, in the ſame order as they ſtand, for the firſt Rule of Three queſtion to be worked, whence will be found a fourth term. Then take this fourth number, ſo found, for the middle term of a ſecond Rule of Three queſtion, and the next two under terms in the general ſtating, in the ſame order as they ſtand, finding a fourth term from them. And ſo on, as far as there are any numbers in the general ſtating, making always the fourth number reſulting from each ſimple ſtating to be the ſecond term of the next following one. So ſhall the laſt reſulting number be the anſwer to the queſtion.

By One Operation.—Multiply together all the terms ſtanding under each other, on the left-hand ſide of the middle term; and in like manner, multiply together all thoſe on the right-hand ſide of it. Then multiply the middle term by the latter product, and divide the reſult by the former product, ſo ſhall the quotient be the anſwer ſought.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, when 16 men can dig 54 yards in 6 days?

General Stating.

yds 54 : 16 men :: 135 yds
days 8 6 days
432 810
16
4860
81 men
432) 12960 (30 Anf. by one operat.
1296
0

The ſame by two operations.

1ſt.	2d.
As 54 : 16 :: 135 : 40	As 8 : 40 :: 6 : 30
16	6
810	8) 240 (30 Anf.
135	24
54) 2160 (40	0
216	
0	

2. If 100l in one year gain 5l intereſt, what will be the intereſt of 750l for 7 years? Anf. 262l 10s

3. If a family of 9 perſons expend 120l in 8 months; how much will ſerve a family of 24 people 16 months? Anf. 640l.

4. If 27s be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days? Anf. 6l 15s.

5. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Anf. $9\frac{6}{13}$ days.

Ex. 6.

Ex. 6. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses?

Ans. $102\frac{6}{7}$ days.

7. If 3000lb of beef serve 340 men 15 days; how many lbs will serve 120 men for 25 days? Ans. 1764lb $11\frac{4}{7}$oz.

8. If a barrel of beer be sufficient to last a family of 7 persons 12 days; how many barrels will be drank by 14 persons in the space of a year? Ans. $60\frac{6}{7}$ barrels.

9. If 240 men, in 5 days, of 11 hours each, can dig a trench 230 yards long, 3 wide, and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide, and 3 deep? Ans. $288\frac{5}{207}$ days.

OF VULGAR FRACTIONS.

A FRACTION, or broken number, is an expression of a part, or some parts, of something considered as a whole.

It is denoted by two numbers, placed one below the other, with a line between them:

Thus, $\left.\begin{matrix}3 \text{ numerator} \\ 4 \text{ denominator}\end{matrix}\right\}$ which is named 3-fourths.

The Denominator, or number placed below the line, shews how many equal parts the whole quantity is divided into; and represents the Divisor in Division.—And the Numerator, or number set above the line, shews how many of these parts are expressed by the Fraction; being the remainder after division.—Also, both these numbers are, in general, named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, or Mixed.

A Proper Fraction, is when the numerator is less than the denominator; as $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{1}{4}$, &c.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator; as, $\frac{3}{3}$, or $\frac{5}{4}$, or $\frac{7}{5}$, &c.

A Simple Fraction, is a single expression, denoting any number of parts of the integer; as $\frac{2}{3}$, or $\frac{3}{2}$.

A Compound Fraction, is the fraction of a fraction, or several fractions connected with the word *of* between them; as $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{5}{6}$ of 3, &c.

A Mixed Number, is composed of a whole number and a fraction together; as $3\frac{1}{4}$, or $12\frac{4}{5}$, &c.

A whole or integer number may be expreſſed like a fraction, by writing 1 below it, as a denominator; ſo 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$, &c.

A fraction denotes diviſion; and its value is equal to the quotient obtained by dividing the numerator by the denominator; ſo $\frac{12}{4}$ is equal to 3, and $\frac{20}{5}$ is equal to 4.

Hence then, if the numerator be leſs than the denominator, the value of the fraction is leſs than 1. If the numerator be the ſame as the denominator, the fraction is juſt equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

REDUCTION OF VULGAR FRACTIONS.

REDUCTION of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c. of which there are ſeveral caſes.

PROBLEM.

To find the Greateſt Common Meaſure of Two or more Numbers.

THE Common Meaſure of two or more numbers, is that number which will divide them both without remainder: ſo, 3 is a common meaſure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greateſt number that will do this, is the greateſt common meaſure: ſo 6 is the greateſt common meaſure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide farther.

RULE.

IF there be two numbers only; divide the greater by the leſs; then divide the diviſor by the remainder; and ſo on, dividing always the laſt diviſor by the laſt remainder, till nothing remains; ſo ſhall the laſt diviſor of all be the greateſt common meaſure ſought.

When there are more than two numbers; find the greateſt common meaſure of two of them, as before; then do the ſame for that common meaſure and another of the numbers; and

and so on, through all the numbers; so will the greatest common measure last found be the answer.

If it happen that the common measure thus found is 1; then the numbers are said to be incommensurable, or not having any common measure.

EXAMPLES.

1. To find the greatest common measure of 1998, 918, and 522.

```
918 ) 1998 ( 2                    So, 54 is the greatest common
      1836                          measure of 1998 and 918.
      ----
       162 ) 918 ( 5   Hence 54 ) 522 ( 9
             810                  486
             ---                  ---
             108 ) 162 ( 1         36 ) 54 ( 1
                   108                   36
                   ---                   ---
                    54 ) 108 ( 2         18 ) 36 ( 2
                         108                   36
                         ---                   ---
```

So that 18 is the answer required.

2. What is the greatest common measure of 246 and 372? Ans. 6.

3. What is the greatest common measure of 336, 720, and 1736? Ans. 8.

CASE I.

To abbreviate or reduce Fractions to their Lowest Terms.

RULE*.

DIVIDE the terms of the given fraction by any number that will divide them without a remainder; then divide these quotients

---

* That dividing both the terms of the fraction by the same number, whatever it be, will give another fraction equal to the former, is evident. And when these divisions are performed as often as can be done, or when the common divisor is the greatest possible, the terms of the resulting fraction must be the least possible.

Note, 1. Any number ending with an even number, or a cipher, is divisible, or can be divided, by 2.

2. Any number ending with 5, or 0, is divisible by 5.

3. If

quotients again in the ſame manner; and ſo on, till it appears that there is no number greater than 1 which will divide them; then the fraction will be in its loweſt terms.

Or, Divide both the terms of the fraction by their greateſt common meaſure at once, and the quotients will be the terms of the fraction required, of the ſame value as at firſt.

EXAMPLES.

1. Reduce $\frac{144}{240}$ to its leaſt terms.

$\frac{144}{240} = \frac{72}{120} = \frac{36}{60} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$, the Anſwer.

Or thus;

144) 240 (1
144
96) 144 (1
96
48) 96 (2
96

Therefore 48 is the greateſt common meaſure; and 48) $\frac{144}{240} = \frac{3}{5}$ the Anſwer, the ſame as before.

2. Reduce

---

3. If the right-hand place of any number be 0, the whole is diviſible by 10; if there be two ciphers, it is diviſible by 100; if three ciphers, by 1000; and ſo on; which is only cutting off thoſe ciphers.

4. If the two right-hand figures of any number be diviſible by 4, the whole is diviſible by 4. And if the three right-hand figures be diviſible by 8, the whole is diviſible by 8. And ſo on.

5. If the ſum of the digits in any number be diviſible by 3, or by 9, the whole is diviſible by 3, or by 9.

6. If the right-hand digit be even, and the ſum of all the digits be diviſible by 3, then the whole will be diviſible by 6.

7. A number is diviſible by 11, when the ſum of the 1ſt, 3d, 5th, &c, or all the odd places, is equal to the ſum of the 2d, 4th, 6th, &c, or of all the even places of digits.

8. If a number cannot be divided by ſome quantity leſs than the ſquare of the ſame, that number is a prime, or cannot be divided by any number whatever.

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9 in the place of units; and all other numbers are compoſite, or can be divided.

10. When

2. Reduce $\frac{198}{576}$ to its loweſt terms. Anſ. $\frac{1}{3}$.
3. Reduce $\frac{855}{1064}$ to its loweſt terms. Anſ. $\frac{9}{112}$.
4. Reduce $\frac{1344}{1568}$ to its loweſt terms. Anſ. $\frac{7}{8}$.

CASE II.

To reduce a Mixed Number to its equivalent Improper Fraction.

RULE*

MULTIPLY the whole number by the denominator of the fraction, and add the numerator to the product; then ſet that ſum above the denominator for the fraction required.

EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

$$\begin{array}{r} 23 \\ 5 \\ \hline 115 \\ 2 \\ \hline 117 \\ 5 \end{array}$$

Or,

$\frac{(23 \times 5) + 2}{5} = \frac{117}{5}$, the Anſwer.

2. Reduce $12\frac{7}{9}$ to a fraction. Anſ. $\frac{115}{9}$.
3. Reduce $14\frac{7}{10}$ to a fraction. Anſ. $\frac{147}{10}$.
4. Reduce $183\frac{5}{21}$ to a fraction. Anſ. $\frac{3848}{21}$.

---

10. When numbers, with the ſign of addition or ſubtraction between them, are to be divided by any number, then each of thoſe numbers muſt be divided by it. Thus $\frac{10+8-4}{2} = 5+4-2=7$.

11. But if the numbers have the ſign of multiplication between them, only one of them muſt be divided. Thus,

$$\frac{10\times8\times3}{6\times2} = \frac{10\times4\times3}{6\times1} = \frac{10\times4\times1}{2\times1} = \frac{10\times2\times1}{1\times1} = \frac{20}{1} = 20.$$

* This is no more than firſt multiplying a quantity by ſome number, and then dividing the reſult back again by the ſame, which it is evident does not alter the value; for any fraction repreſents a diviſion of the numerator by the denominator.

CASE

CASE III.

To reduce an Improper Fraction to its equivalent Whole or Mixed Number.

RULE*.

DIVIDE the numerator by the denominator, and the quotient will be the whole or mixed number ſought.

EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.

   Here $\frac{12}{3}$ or $12 \div 3 = 4$, the Anſwer.

2. Reduce $\frac{15}{7}$ to its equivalent number.

   Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the Anſwer.

3. Reduce $\frac{749}{17}$ to its equivalent number.

   Thus, 17) 749 ($44\frac{1}{17}$
   68
   69
   68
   1

   So that $\frac{749}{17} = 44\frac{1}{17}$, the Anſwer.

4. Reduce $\frac{56}{7}$ to its equivalent number. Anſ. 8.
5. Reduce $\frac{3362}{25}$ to its equivalent number. Anſ. $54\frac{12}{25}$.
6. Reduce $\frac{2018}{17}$ to its equivalent number. Anſ. $171\frac{11}{17}$.

CASE IV.

To reduce a Whole Number to an equivalent Fraction, having a Given Denominator.

RULE†.

MULTIPLY the whole number by the given denominator; then ſet the product over the ſaid denominator, and it will form the fraction required.

---

* This Rule is evidently the reverſe of the former; and the reaſon of it is manifeſt from the nature of Common Diviſion.

† Multiplication and Diviſion being here equally uſed, the reſult muſt be the ſame as the quantity firſt propoſed.

EXAM-

EXAMPLES.

1. Reduce 9 to a fraction whoſe denominator ſhall be 7.
Here $9 \times 7 = 63$, then $\frac{63}{7}$ is the Anſwer.
For $\frac{63}{7} = 63 \div 7 = 9$, the Proof.
2. Reduce 13 to a fraction whoſe denominator ſhall be 12.
Anſ. $\frac{156}{12}$.
3. Reduce 27 to a fraction whoſe denominator ſhall be 11.
Anſ. $\frac{297}{11}$.

CASE V.

To reduce a Compound Fraction to an equivalent Simple One.

RULE*.

MULTIPLY all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the ſimple fraction ſought.

When part of the compound fraction is a whole or mixed number, it muſt firſt be reduced to a fraction by one of the former caſes.

And, when it can be done, any two terms of the fraction may be divided by the ſame number, and the quotients uſed inſtead of them. Or when there are terms that are common, they may be omitted.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a ſimple fraction.

Here $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4}$, the Anſwer.

Or, $\frac{1 \times \cancel{2} \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 4} = \frac{1}{4}$, by omitting the 2's and 3's.

* The truth of this Rule may be ſhewn as follows: Let the compound fraction be $\frac{2}{3}$ of $\frac{5}{7}$. Now $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{7} \div 3$, which is $\frac{5}{21}$; conſequently $\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21} \times 2$ or $\frac{10}{21}$; that is, the numerators are multiplied together, and alſo the denominators, as in the Rule.—When the compound fraction conſiſts of more than two ſingle ones; having firſt reduced two of them as above, then the reſulting fraction and a third will be the ſame as a compound fraction of two parts; and ſo on to the laſt of all.

2. Reduce

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a ſimple fraction.

Here $\frac{2\times3\times10}{3\times5\times11} = \frac{60}{165} = \frac{12}{33} = \frac{4}{11}$, the Anſwer.

Or $\frac{2\times\cancel{3}\times\cancel{10}}{\cancel{3}\times\cancel{5}\times11} = \frac{4}{11}$, the ſame as before.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a ſimple fraction. Anſ. $\frac{12}{35}$.

4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a ſimple fraction. Anſ. $\frac{1}{4}$.

5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a ſimple fraction. Anſ. $\frac{7}{8}$.

6. Reduce $\frac{2}{7}$ of $\frac{5}{8}$ of $\frac{7}{2}$ of 4 to a ſimple fraction. Anſ. $\frac{5}{2}$.

7. Reduce $2\frac{2}{5}$ of $\frac{5}{6}$ to a fraction. Anſ. $\frac{7}{3}$.

CASE VI.

To reduce Fractions of Different Denominators to equivalent Fractions, having a Common Denominator.

RULE*.

MULTIPLY each numerator by all the denominators except its own, for the new numerators; and multiply all the denominators together for a common denominator.

Note, It is evident, that in this and ſeveral other operations, when any of the propoſed quantities are integers, or mixed numbers, or compound fractions, they muſt be reduced, by their proper Rules, to the form of ſimple fractions.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to a common denominator.

$1 \times 3 \times 4 = 12$ the new numerator for $\frac{1}{2}$.
$2 \times 2 \times 4 = 16$ - - ditto for $\frac{2}{3}$.
$3 \times 2 \times 3 = 18$ - - ditto for $\frac{3}{4}$.
$2 \times 3 \times 4 = 24$ the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may be very well performed mentally, and only ſetting down the reſults and

* This is evidently no more than multiplying each numerator and its denominator by the ſame quantity, and conſequently the value of the fraction is not altered.

given

given fractions thus: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4} = \frac{12}{24}, \frac{16}{24}, \frac{18}{24} = \frac{6}{12}, \frac{8}{12}, \frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator.

Anſ. $\frac{18}{63}, \frac{35}{63}$.

3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{3}{4}$ to a common denominator.

Anſ. $\frac{40}{60}, \frac{36}{60}, \frac{45}{60}$.

4. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4 to a common denominator.

Anſ. $\frac{25}{30}, \frac{78}{30}, \frac{120}{30}$.

Note 1. When the denominators of two given fractions have a common meaſure, let them be divided by it; then multiply the terms of each given fraction by the quotient ariſing from the other's denominator.

2. When the leſs denominator of two fractions exactly divides the greater, multiply the terms of that which hath the leſs denominator by the quotient.

3. When more than two fractions are propoſed; it is ſometimes convenient, firſt to reduce two of them to a common denominator; then theſe and a third; and ſo on till they be all reduced to their leaſt common denominator.

CASE VII.

To find the Value of a Fraction in Parts of the Integer.

RULE.

MULTIPLY the integer by the numerator, and divide the product by the denominator, by Compound Multiplication and Diviſion, if the integer be a compound quantity.

Or, if it be a ſingle integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before; and ſo on as far as neceſſary; ſo ſhall the quotients, placed in order, be the value of the fraction required*.

---

* The numerator of a fraction being conſidered as a remainder, in Diviſion, and the denominator as the diviſor, this rule is of the ſame nature as Compound Diviſion, or the valuation of remainders in the Rule of Three, before explained.

EXAM-

EXAMPLES.

1. What is the $\frac{4}{5}$ of 21 6s?

By the former part of the Rule,

21 6s
4

5) 9 4

Anſ. 11 16s 9d $2\frac{2}{5}$q.

2. What is the value of $\frac{2}{3}$ of 1l?

By the 2d part of the Rule,

2
20

3) 40 (13s 4d Anſ.
1
12

3) 12 (4d

3. Find the value of $\frac{3}{8}$ of a pound ſterling? Anſ. 7s 6d.
4. What is the value of $\frac{2}{9}$ of a guinea? Anſ. 4s 8d.
5. What is the value of $\frac{1}{4}$ of a half crown? Anſ. 1s $10\frac{1}{2}$d.
6. What is the value of $\frac{2}{5}$ of 4s 10d? Anſ. 1s $11\frac{1}{5}$d.
7. What is the value of $\frac{3}{5}$lb. troy? Anſ. 7oz 4dwts.
8. What is the value of $\frac{5}{16}$ of a cwt? Anſ. 1qr 7lb.
9. What is the value of $\frac{5}{8}$ of an acre? Anſ. 2ro 20po.
10. What is the value of $\frac{3}{10}$ of a day? Anſ. 7hrs 12min.

CASE VIII.

To reduce a Fraction from one Denomination to another.

RULE*.

CONSIDER how many of the leſs denomination make one of the greater; then multiply the numerator by that number, if the reduction be to a leſs name, or multiply the denominator, if to a greater.

EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny.

$\frac{2}{9} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{9} = 160$, the Anſwer.

* This is the ſame as the Rule of Reduction in whole numbers, from one denomination to another.

2. Re-

2. Reduce $\frac{5}{6}$ of a penny to the fraction of a pound.

$\frac{5}{6} \times \frac{1}{12} \times \frac{1}{20} = \frac{5}{1440} = \frac{1}{288}$, the Anſwer.

3. Reduce $\frac{2}{15}$l to the fraction of a penny. Anſ. $\frac{8}{1}$d.
4. Reduce $\frac{2}{3}$qr. to the fraction of a pound. Anſ. $\frac{1}{112}$.
5. Reduce $\frac{2}{7}$cwt to the fraction of a lb. Anſ. $\frac{8}{7}$.
6. Reduce $\frac{4}{5}$dwt to the fraction of a lb troy. Anſ. $\frac{1}{300}$.
7. Reduce $\frac{5}{8}$crown to the fract. of a guinea. Anſ. $\frac{25}{168}$.
8. Reduce $\frac{5}{6}$half-crown to the fract. of a ſhilling. Anſ. $\frac{25}{12}$.
9. Reduce 2s 6d to the fraction of a £. Anſ. $\frac{1}{8}$.

ADDITION OF VULGAR FRACTIONS.

RULE.

IF the fractions have a common denominator; add all the numerators together, then place the ſum over the common denominator, and that will be the ſum of the fractions required.

If the propoſed fractions have not a common denominator; they muſt be reduced to one. Alſo compound fractions muſt be reduced to ſimple ones; and mixed numbers to improper fractions: alſo fractions of different denominations to thoſe of the ſame denomination*.

* Before fractions are reduced to a common denominator, they are quite diſſimilar, as much as ſhillings and pence are, and therefore cannot be incorporated with one another, any more than theſe can. But when they are reduced to a common denominator, and made parts of the ſame thing, their ſum, or difference, may then be as properly expreſſed by the ſum or difference of the numerators, as the ſum or difference of any two quantities whatever, by the ſum or difference of their individuals. Whence the reaſon of the Rule is manifeſt, both for Addition and Subtraction.

When ſeveral fractions are to be collected, it is commonly beſt firſt to add two of them together that moſt eaſily reduce to a common denominator; then add their ſum and a third, and ſo on.

EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.
Here $\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$, the Anſwer.

2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.
$\frac{3}{5} + \frac{5}{6} = \frac{18}{30} + \frac{25}{30} = \frac{43}{30} = 1\frac{13}{30}$, the Anſwer.

3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.
$\frac{5}{8} + 7\frac{1}{2} + \frac{1}{3}$ of $\frac{3}{4} = \frac{5}{8} + \frac{15}{2} + \frac{1}{4} = \frac{5}{8} + \frac{60}{8} + \frac{2}{8} = \frac{67}{8} = 8\frac{3}{8}$.

4. To add $\frac{3}{7}$ and $\frac{6}{7}$ together. Anſ. $1\frac{2}{7}$.

5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together. Anſ. $1\frac{11}{36}$.

6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together. Anſ. $\frac{9}{14}$.

7. What is the ſum of $\frac{2}{3}$ and $\frac{3}{5}$ and $\frac{5}{7}$? Anſ. $1\frac{103}{105}$.

8. What is the ſum of $\frac{5}{9}$ and $\frac{3}{5}$ and $2\frac{1}{6}$? Anſ. $3\frac{29}{90}$.

9. What is the ſum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$; and $9\frac{3}{20}$? Anſ. $10\frac{1}{60}$.

10. What is the ſum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a ſhilling?
Anſ. $\frac{125}{9}$s or 13s 10d $2\frac{2}{3}$q.

11. What is the ſum of $\frac{3}{5}$ of a ſhilling and $\frac{4}{15}$ of a penny.
Anſ. $\frac{112}{15}$d or 7d $1\frac{13}{15}$q.

12. What is the ſum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a ſhilling, and $\frac{5}{12}$ of a penny? Anſ. $\frac{3139}{1008}$s or 3s 1d $1\frac{10}{21}$q.

13. To ſum $7\frac{3}{5}$ of $\frac{4}{7} + \frac{3}{5}$ of $\frac{4}{7}$ of $7 + 5\frac{2}{3} + \frac{9}{11}$.
Anſ. $16\frac{263}{1155}$.

SUBTRACTION OF VULGAR FRACTIONS.

RULE.

PREPARE the fractions the ſame as for Addition; then ſubtract the one numerator from the other, and ſet the remainder over the common denominator, for the difference of the fractions ſought.

EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$.
Here $\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$, the Anſwer.

2. To find the difference between $\frac{3}{4}$ and $\frac{5}{7}$.
$\frac{3}{4} - \frac{5}{7} = \frac{21}{28} - \frac{20}{28} = \frac{1}{28}$, the Anſwer.

3. What

3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$? Anſ. $\frac{1}{6}$.

4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$? Anſ. $\frac{5}{39}$.

5. What is the difference between $\frac{5}{12}$ and $\frac{7}{13}$? Anſ. $\frac{19}{156}$.

6. What is the diff. between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$? Anſ. $4\frac{341}{672}$.

7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a ſhilling? Anſ. $\frac{19}{36}$s or 10s 7d $1\frac{1}{3}$q.

8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a ſhilling? Anſ. $\frac{3037}{2100}$l or 1l 8s $11\frac{3}{35}$d.

MULTIPLICATION OF VULGAR FRACTIONS.

RULE*.

REDUCE mixed numbers, if there be any, to equivalent fractions; then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

EXAMPLES.

1. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.

Here $\frac{3}{4} \times \frac{2}{9} = \frac{6}{36} = \frac{1}{6}$, the Anſwer.

Or $\frac{3}{4} \times \frac{2}{9} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

2. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{5}$ of $\frac{3}{5}$.

Here $\frac{\cancel{2}}{\cancel{3}} \times \frac{\overset{13}{\cancel{13}}}{\cancel{4}} \times \frac{\cancel{5}}{1} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{3}{\cancel{5}} = \frac{13 \times 3}{4 \times 2} = \frac{39}{8}$

$= 4\frac{7}{8}$, Anſwer.

3. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$. Anſ. $\frac{5}{28}$.

4. Required the product of $\frac{4}{15}$ and $\frac{5}{24}$. Anſ. $\frac{1}{18}$.

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* Multiplication of any thing by a fraction implies the taking ſome part or parts of the thing; it may therefore be truly expreſſed by a compound fraction; which is reſolved by multiplying together the numerators and the denominators.

Note, A fraction is beſt multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

5. Re-

5. Required the product of $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{14}{15}$. Anſ. $\frac{8}{45}$.
6. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3. Anſ. 1.
7. Required the product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$. Anſ. $2\frac{1}{30}$.
8. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$. Anſ. $\frac{10}{21}$.
9. Required the product of 6, and $\frac{2}{3}$ of 5. Anſ. 20.
10. Required the product of $\frac{2}{9}$ of $\frac{3}{5}$, and $\frac{5}{8}$ of $3\frac{2}{7}$. Anſ $\frac{23}{84}$.
11. Required the product of $3\frac{2}{7}$, and $4\frac{14}{33}$. Anſ. $14\frac{124}{231}$.
12. Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$, and $4\frac{1}{6}$. Anſ. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

RULE*

PREPARE the fractions as before in Multiplication; then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide; but if not, then invert the terms of the diviſor, and multiply the dividend by it, as in Multiplication.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$.

Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the firſt method.

2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.

Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} = 4\frac{1}{6}$.

3. It is required to divide $\frac{16}{25}$ by $\frac{4}{5}$. Anſ. $\frac{4}{5}$.
4. It is required to divide $\frac{7}{16}$ by $\frac{3}{4}$. Anſ. $\frac{7}{12}$.
5. It is required to divide $\frac{14}{9}$ by $\frac{7}{6}$. Anſ. $1\frac{1}{3}$.
6. It is required to divide $\frac{5}{6}$ by $\frac{15}{7}$. Anſ. $\frac{7}{18}$.

* Diviſion being the reverſe of Multiplication, the reaſon of the Rule is evident.

Note, A fraction is beſt divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

7. Di-

7. It is required to divide $\frac{12}{35}$ by $\frac{3}{5}$. Anſ. $\frac{4}{7}$.
8. It is required to divide $\frac{2}{7}$ by $\frac{3}{5}$. Anſ. $\frac{10}{21}$.
9. It is required to divide $\frac{9}{16}$ by 3. Anſ. $\frac{3}{16}$.
10. It is required to divide $\frac{3}{5}$ by 2. Anſ. $\frac{3}{10}$.
11. It is required to divide $7\frac{1}{3}$ by $9\frac{5}{8}$. Anſ. $\frac{88}{115}$.
12. It is required to divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{3}{5}$. Anſ. $\frac{7}{171}$.

RULE-OF-THREE IN VULGAR FRACTIONS.

RULE*.

MAKE the neceſſary preparations as before directed; then multiply continually together, the ſecond and third terms, and the firſt with its parts inverted as in Diviſion, for the anſwer.

EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet coſt $\frac{2}{5}$ of a pound ſterling; what will $\frac{5}{16}$ of a yard coſt?

$$\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{\cancel{5}}{3} \times \frac{\cancel{2}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{16}} = \frac{1}{3}\text{l} = 6\text{s } 8\text{d}.$$ Anſwer.

2. What will $3\frac{3}{8}$oz of ſilver coſt, at 6s 4d an ounce?
Anſ. 1l 1s $4\frac{1}{2}$d.

3. If $\frac{3}{16}$ of a ſhip be worth 273l 2s 6d, what is $\frac{5}{32}$ of her worth?
Anſ. 227l 12s 1d.

4. What is the purchaſe of 1230l bank-ſtock, at $108\frac{5}{8}$ per cent?
Anſ. 1336l 1s 9d.

5. What is the intereſt of 273l 15s for a year, at $3\frac{1}{4}$ per cent?
Anſ. 8l 17s $11\frac{1}{4}$d.

6. If $\frac{1}{8}$ of a ſhip be worth 73l 1s 3d, what part of her is worth 25l 10?
Anſ. $\frac{2}{7}$.

* This is only multiplying the 2d and 3d terms together, and dividing the product by the firſt, as in the Rule-of-Three in whole numbers.

7. What length muſt be cut off a board, that is $7\frac{1}{4}$ inches broad, to contain a ſquare foot, or as much as another piece of 12 inches long and 12 broad? Anſ. $18\frac{8}{31}$ inches.

8. What quantity of ſhalloon, that is $\frac{3}{4}$ of a yard wide, will line $9\frac{1}{2}$ yards of cloth that is $2\frac{1}{4}$ yards wide? Anſ. $31\frac{2}{3}$ yds.

9. If the penny-loaf weigh $6\frac{9}{10}$ oz when the price of wheat is 5s the buſhel; what ought it to weigh when the wheat is at 8s 6d the buſhel? Anſ. $4\frac{4}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{1}{2}\frac{1}{2}$ poles broad, will make an acre of land, or as much as 40 poles in length and 4 in breadth? Anſ. $13\frac{61}{143}$ poles.

11. If a courier perform a certain journey in $35\frac{1}{2}$ days, travelling $13\frac{5}{8}$ hours a-day; how long would he be in performing the ſame, travelling only $11\frac{9}{10}$ hours a-day?
Anſ. $40\frac{6}{9}\frac{1}{5}\frac{5}{2}$ days.

12. A regiment of ſoldiers, conſiſting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth that is $1\frac{5}{8}$ yard wide, and lined with ſhalloon $\frac{7}{8}$ yard wide; how many yards of ſhalloon will line them?
Anſ. 4531yds 1qr $2\frac{6}{7}$ nails.

DECIMAL FRACTIONS.

A DECIMAL FRACTION is that which has for its denominator, an unit (1) with as many ciphers annexed as the numerator has places; and it is uſually expreſſed by ſetting down the numerator only, with a point before it, on the left-hand. Thus, $\frac{5}{10}$ is ·5, and $\frac{25}{100}$ is ·25, and $\frac{75}{1000}$ is ·075, and $\frac{124}{100000}$ is ·00124; where ciphers are prefixed to make up as many places as are in the numerator, when there is a deficiency of figures.

A mixed number is made up of a whole number with ſome decimal fraction, the one being ſeparated from the other by a point. Thus 3·25 is the ſame as $3\frac{25}{100}$, or $3\frac{2}{4}\frac{5}{0}$.

Ciphers on the right-hand of decimals make no alteration in their value; for ·5 or ·50 or ·500, are decimals having all the ſame value, being each $= \frac{5}{10}$ or $\frac{1}{2}$. But if they are placed on the left-hand, they decreaſe the value in a tenfold proportion: Thus ·5 is $\frac{5}{10}$ or 5 tenths, but ·05 is only $\frac{5}{100}$ or 5 hundreds, and ·005 is but $\frac{5}{1000}$ or 5 thouſandths.

The

The 1ſt place of decimals, counted from the left-hand towards the right, is called the place of primes, or 1oths; the 2d is the place of ſeconds, or 1ooths; the third is the place of thirds, or 1oooths; and ſo on. For, in decimals, as well as in whole numbers, the values of the places increaſe towards the left-hand, and decreaſe towards the right, both in the ſame tenfold proportion; as in the following Scale or Table of Notation.

millions	hundred thouſands	ten thouſands	thouſands	hundreds	tens	units	tenth parts	hundredth parts	thouſandth parts	ten thouſandth parts	hundred thouſandth parts	millionth parts
3	3	3	3	3	3	3	3	3	3	3	3	3

ADDITION of DECIMALS.

RULE.

SET the numbers under each other according to the value of their places, like as in whole numbers; in which ſtate the decimal ſeparating points will ſtand all exactly under each other. Then, beginning at the right hand, add up all the columns of numbers as in integers; and point off as many places, for decimals, as are in the greateſt number of decimal places in any of the lines that are added; or place the point directly below all the other points.

EXAMPLES.

1. To add together 29·0146, and 3146·5, and 2109, and ·62417, and 14·16.

29·0146
3146·5
2109·
·62417
14·16

5299·29877 the Sum.

Ex. 2. To find the ſum of 376·25 + 86·125 + 637·4725 + 6·5 + 41·02 + 358·865. Anſ. 1506·2325.

3. Required the ſum of 3·5 + 47·25 + 2·0073 + 927·01 + 1·5. Anſ. 981·2673.

4. Required the ſum of 276 + 54·321 + 112 + 0·65 + 12·5 + ·0463. Anſ. 455·5173.

SUBTRACTION OF DECIMALS.

RULE.

PLACE the numbers under each other according to the value of their places, as in the laſt Rule. Then, beginning at the right-hand, ſubtract as in whole numbers, and point off the decimals as in Addition.

EXAMPLES.

1. To find the difference between 91·73 and 2·138.

91·73
2·138

Anſ. 89·592 the Difference.

2. Find the diff. between 1·9185 and 2·73. Anſ. 0·8115.
3. To ſubtract 4·90142 from 214·81. Anſ. 209·90858.
4. Find the diff. between 2714 and ·916. Anſ. 2713·084.

MULTIPLICATION OF DECIMALS

RULE*.

PLACE the factors, and multiply them together the ſame as if they were whole numbers.—Then point off in the pro-

* The Rule will be evident from this example: Let it be required to multiply ·12 by ·361; theſe numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$; the product of which is $\frac{4332}{100000} = ·04332$, by the nature of Notation, which conſiſts of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner for any other numbers.

duct

duct juſt as many places of decimals as there are decimals in both the factors. But if there be not ſo many figures in the product, then ſupply the defect by prefixing ciphers.

EXAMPLES.

1. Multiply ·321096
by ·2465

1605480
1926576
1284384
642192

Anſ. ·079150164o the Product.

2. Multiply ·79·347 by 23·15. Anſ. ·1836·88305.
3. Multiply ·63478 by ·8204. Anſ. ·520773512.
4. Multiply ·385746 by ·00464. Anſ. ·0017898614.

CONTRACTION I.

To multiply Decimals by 1 *with any number of Ciphers, as* 10, *or* 100, *or* 1000, *&c.*

THIS is done by only removing the decimal point ſo many places farther to the right-hand as there are ciphers in the multiplier; and ſubjoining ciphers if need be.

EXAMPLES.

1. The product of 51·3 and 1000 is 51300.
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

CONTRACTION II.

To Contract the Operation, ſo as to retain only as many Decimals in the Product as may be thought Neceſſary, when the Product would naturally contain ſeveral more Places.

SET the units place of the multiplier under that figure of the multiplicand whoſe place is the ſame as is to be retained for the laſt in the product; and diſpoſe of the reſt of the figures

figures in the inverted or contrary order to what they are uſually placed in.—Then, in multiplying, reject all the figures that are more to the right than each multiplying figure and ſet down the products, ſo that their right-hand figures may fall in a column ſtraight below each other; but obſerving to increaſe the firſt figure of every line with what would ariſe from the figures omitted, in this manner, namely 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c; and the ſum of all the lines will be the product as required, commonly to the neareſt unit in the laſt figure.

EXAMPLES.

1. To multiply 27·14986 by 92·41035, ſo as to retain only four places of decimals in the product,

Contracted Way.	*Common Way.*
27·14986	27·14986
53014·29	92·41035
24434874	13 574930
542997	81 44958
108599	2714986
2715	108599 44
81	542997 2·
14	24434874
2508·9280	2508·9280 650510

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.

3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.

4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

DIVI-

DIVISION OF DECIMALS.

RULE.

DIVIDE as in whole numbers; and point off in the quotient as many places for decimals, as the decimal places in the dividend exceed thoſe in the diviſor *.

Another way to know the place for the decimal point is this: The firſt figure of the quotient muſt be made to occupy the ſame place, of integers or decimals, as doth that figure of the dividend which ſtands over the unit's figure of the firſt product.

When the places of the quotient are not ſo many as the Rule requires, let the defect be ſupplied by prefixing ciphers.

When there happens to be a remainder after the diviſion; or when the decimal places in the diviſor are more than thoſe in the dividend; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

EXAMPLES.

1. 179) ·48624697 (·00271643
1282
294
1150
769
537
900

2. 2685) 27·00000 (100·55865
15000
15750
23250
17700
15900
24750

3. Divide 234·70525 by 64·25. Anſ. 3·653.
4. Divide 14 by ·7854. Anſ. 17·825.
5. Divide 2175·68 by 100. Anſ. 21·7568.
6. Divide ·8727587 by ·162. Anſ. 5·38739.

CONTRACTION I.

WHEN the diviſor is an integer, with any number of ciphers annexed; cut off thoſe ciphers, and remove the deci-

* The reaſon of this Rule is evident; for ſince the diviſor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend is equal to thoſe in the diviſor and quotient, taken together, by the nature of Multiplication; and conſequently the quotient itſelf muſt contain as many as the dividend exceeds the diviſor.

mal

mal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers if need be; then proceed as before *.

EXAMPLES.

1. Divide 45·5 by 2100.

21·00) ·455 (·0216 &c.
35
140
14

2. Divide 41020 by 32000.
3. Divide 953 by 21600.
4. Divide 61 by 79000.

CONTRACTION II.

HENCE, if the divisor be 1 with ciphers, as 10, or 100, or 1000, &c; then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left as the divisor has ciphers; prefixing ciphers if need be.

EXAMPLES.

So, 217·3 ÷ 100 = 2·173. And 419 ÷ 10 =
And 5·16 ÷ 100 = And ·21 ÷ 1000 =

CONTRACTION III.

WHEN there are many figures in the divisor; or when only a certain number of decimals are necessary to be retained in the quotient; then take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

* This is no more than dividing both divisor and dividend by the same number, either 10, or 100, or 1000, &c, according to the number of ciphers cut off; which, leaving them in the same proportion, does not affect the quotient. And, in the same way, the decimal point may be moved the same number of places in both the divisor and dividend, either to the right or left, whether they have ciphers or not.

Let

Let each remainder be a new dividend; and for every ſuch dividend, leave out one figure more on the right-hand ſide of the diviſor; remembering to carry for the increaſe of the figures cut off, as in the 2d contraction in Multiplication.

Note. When there are not ſo many figures in the diviſor as are required to be in the quotient, begin the operation with all the figures, and continue it as uſual till the number of figures in the diviſor be equal to thoſe remaining to be found in the quotient, after which begin the contraction.

EXAMPLES.

1. Divide 2508·92806 by 92·41035, ſo as to have only four decimals in the quotient, in which caſe the quotient will contain ſix figures.

Contracted.	*Common Way.*
92·4103,5)2508·928,06(27·1498	92·4103,5)2508·928,06(27·1498
660721	66072106
13849	13848610
4608	46075750
912	91116100
80	79467850
6	5539570

2. Divide 4109·2351 by 230·409, ſo that the quotient may contain only four decimals. Anſ. 17·8345.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals. Anſ. ·00649.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

REDUCTION OF DECIMALS.

CASE I.

To reduce a Vulgar Fraction to its equivalent Decimal.

RULE.

DIVIDE the numerator by the denominator as in Diviſion of Decimals, annexing ciphers to the numerator as far as neceſſary; ſo ſhall the quotient be the decimal required.

EXAM-

EXAMPLES.

1. Reduce $\frac{7}{24}$ to a decimal.

$24 = 4 \times 6$. Then $4)\ 7\cdot$
$6)\ 1\cdot750000$
$\cdot291666$ &c

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$ to decimals.
Anſ. ·25, and ·5, and ·75.

3. Reduce $\frac{3}{8}$ to a decimal. Anſ. ·375.

4. Reduce $\frac{1}{25}$ to a decimal. Anſ. ·04.

5. Reduce $\frac{3}{192}$ to a decimal. Anſ. ·015625.

6. Reduce $\frac{275}{3842}$ to a decimal. Anſ. ·071577 &c.

CASE II.

To find the Value of a Decimal in terms of the Inferior Denominations.

RULE.

MULTIPLY the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder, to the right-hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the ſame manner through all the parts of the integer; then the ſeveral denominations ſeparated on the left-hand, will make up the Anſwer.

Note, This operation is the ſame as Reduction Deſcending in whole numbers.

EXAMPLES.

1. Required to find the value of ·775 pounds ſterling.

·775
20

s 15·500
12

d 6·000

Anſ. 15s 6d.

Ex. 2.

Ex. 2. What is the value of ·625 ſhil? Anſ. 7½d.
3. What is the value of ·86351? Anſ. 17s 3·24d.
4. What is the value of ·0125lb troy? Anſ. 3 dwts.
5. What is the value of ·4694lb troy?
Anſ. 5oz 12dwt 15·744gr.
6. What is the value of ·625 cwt? Anſ. 2qr 14lb.
7. What is the value of ·009943 miles?
Anſ. 17yd 1ft 5·98848inc.
8. What is the value of ·6875 yd? Anſ. 2qr 3nls.
9. What is the value of ·3375 acr? Anſ. 1rd 14poles.
10. What is the value of ·2083hhd of wine?
Anſ. 13·1229gal.

CASE III.

To reduce Integers or Decimals to Equivalent Decimals of Higher Denominations.

RULE.

DIVIDE by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be neceſſary, the ſame as in Reduction Aſcending of whole numbers.

EXAMPLES.

1. Reduce 1dwt. to the decimal of a pound troy

20	1 dwt
12	0·05 oz
	0·004166 &c lb Anſ.

2. Reduce 9d to the decimal of a pound. Anſ. ·0375l.
3. Reduce 7 drams to the decimal of a pound avoird.
Anſ. ·02734375lb.
4. Reduce ·26d to the decimal of a l. Anſ. ·0010833 &c l.
5. Reduce 2·15lb to the decimal of a cwt.
Anſ. ·019196 + cwt.
6. Reduce 24 yards to the decimal of a mile.
Anſ. ·013636 &c mile.
7. Reduce ·056 pole to the decimal of an acre.
Anſ. ·00035 ac.
8. Reduce 1·2 pint of wine to the decimal of a hhd.
Anſ. ·00238 + hhd.
9. Reduce 14 minutes to the decimal of a day.
Anſ. ·009722 &c da.
10. Reduce ·21 pint to the decimal of a peck.
Anſ. ·013125 pec.

NOTE,

NOTE, *When there are ſeveral numbers, to be reduced all to the decimal of the higheſt,*

Set the given numbers directly under each other, for dividends, proceeding orderly from the loweſt denomination to the higheſt.

Oppoſite to each dividend, on the left-hand, ſet ſuch a number for a diviſor as will bring it to the next higher name; drawing a perpendicular line between all the diviſors and dividends.

Begin at the uppermoſt, and perform all the diviſions; only obſerving to ſet the quotient of each diviſion, as decimal parts, on the right-hand of the dividend next below it; ſo ſhall the laſt quotient be the decimal required.

EXAMPLES.

1. Reduce 15s $9\frac{1}{4}$d to the decimal of a pound.

4	3.
12	9·75
20	15·8125
	£ 0·790625 Anſ.

2. Reduce 19l 17s $3\frac{1}{4}$d to l. Anſ. 19·86354166 &c l.
3. Reduce 15s 6d to the decimal of a l. Anſ. ·775l.
4. Reduce $7\frac{1}{2}$d to the decimal of ſhil. Anſ. ·625s.
5. Reduce 5oz 12dwts 16gr to lbs. Anſ. ·46944 &c lb.

RULE OF THREE IN DECIMALS.

RULE.

PREPARE the terms by reducing, the vulgar fractions to decimals, any compound numbers either to decimals of the higher denominations, or to integers of the lower, alſo the firſt and third terms to the ſame name: Then multiply and divide as in whole numbers.

Note, Any of the convenient Examples in the Rule-of-Three or Rule-of-Five in Integers, or Vulgar Fractions, may be taken as proper examples to the ſame rules in Decimals. —The following Example, which is the firſt in Vulgar Fractions, is wrought out here to ſhew the method.

If

If $\frac{3}{8}$ of a yard of velvet coſt $\frac{4}{5}$l, what will $\frac{5}{16}$yd coſt?

	yd	l	yd	l	s d
$\frac{3}{8} = .375$	.375 : .4 :: .3125 : .333 &c or				6 8
$\frac{4}{5} = .4$			.4		
	.375) .12500 (.333333 &c.				
			1250	20	
$\frac{5}{16} = .3125$			125	s6.66666 &c.	
				12	
	Anſ. 6s 8d.			d7.99999 &c = 8d	

DUODECIMALS.

DUODECIMALS, or CROSS MULTIPLICATION, is a rule made uſe of by workmen and artificers, in computing the contents of their works.

Dimenſions are uſually taken in feet, inches, and quarters; any parts ſmaller than theſe being neglected as of no conſequence. And the ſame in multiplying them together, or caſting up the contents.

RULE.

SET down the two dimenſions, to be multiplied together, one under the other, ſo that feet ſtand under feet, inches under inches, &c.

Multiply each term in the multiplicand, beginning at the loweſt, by the feet in the multiplier, and ſet the reſult of each ſtraight under its correſponding term, obſerving to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and ſet the reſult of each term one place removed to the right-hand of thoſe in the multiplicand; omitting however what is below parts of inches, only carrying to theſe the proper number of units from the loweſt denomination.

Or, inſtead of multiplying by the inches, take ſuch parts of the multiplicand as theſe are of a foot.

Then add the two lines together, after the manner of Compound Addition, carrying 1 to the feet for 12 inches, when theſe come to ſo many.

EXAMPLES.

1. Multiply 4f 7inc by 6 4

	27	6
	1	$6\frac{1}{3}$
Anf.	29	$0\frac{1}{3}$

2. Multiply 14f 9inc. by 4 6

	59	0
	7	$4\frac{1}{2}$
Anf.	66	$4\frac{1}{2}$

3. Multiply 4 feet 7 inches by 9f 6inc. Anf. 43f $6\frac{1}{2}$inc.
4. Multiply 12f 5inc by 6f 8inc. Anf. 82 $9\frac{1}{3}$
5. Multiply 35f $4\frac{1}{2}$inc by 12f 3inc. Anf. 433 $4\frac{1}{8}$
6. Multiply 64f 6inc by 8f $9\frac{1}{4}$inc. Anf. 565 $8\frac{5}{8}$

INVOLUTION.

INVOLUTION is the raiſing of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itſelf. Thus,

$2 =$ 2 is the root, or 1ſt power of 2.
$2 \times 2 =$ 4 is the 2d power, or ſquare of 2.
$2 \times 2 \times 2 =$ 8 is the 3d power, or cube of 2.
$2 \times 2 \times 2 \times 2 =$ 16 is the 4th power of 2, &c.

And in this manner may be calculated the following Table of the firſt nine powers of the firſt 9 numbers.

TABLE

TABLE of the firſt NINE POWERS of NUMBERS.

1ſt	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

The Index or Exponent of a Power, is the number denoting the height or degree of that power; and it is 1 more than the number of multiplications uſed in producing the ſame. So 1 is the index or exponent of the 1ſt power or root, 2 of the 2d power or ſquare, 3 of the 3d power or cube, 4 of the 4th power, and ſo on.

Powers, that are to be raiſed, are uſually denoted by placing the index above the root or firſt power.

So $2^2 = 4$ is the 2d power of 2.
$2^3 = 8$ is the 3d power of 2.
$2^4 = 16$ is the 4th power of 2.
540^4 is the 4th power of 540, &c.

When two or more powers are multiplied together, their product will be that power whoſe index is the ſum of the exponents of the factors or powers multiplied. Or the multiplication of the powers, anſwers to the addition of the indices. Thus, in the following powers of 2,

1ſt	2d	3d	4th	5th	6th	7th	8th	9th	10th
2	4	8	16	32	64	128	256	512	1024
or 2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Here,

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
alſo $16 \times 64 = 1024$, and $4 + 6 = 10$ its index.

OTHER EXAMPLES.

1. What is the 2d power of 45? Anſ. 2025.
2. What is the ſquare of 4·16? Anſ. 17·3056.
3. What is the 3d power of 3·5? Anſ. 42·875.
4. What is the 5th power of ·029? Anſ. ·00000020511149.
5. What is the ſquare of $\frac{2}{3}$? Anſ. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{9}$? Anſ. $\frac{125}{729}$.
7. What is the 4th power of $\frac{3}{4}$? Anſ. $\frac{81}{256}$.

EVOLUTION.

EVOLUTION, or the reverſe of Involution, is the extracting or finding the roots of any given powers.

The root of any number, or power, is ſuch a number, as being multiplied into itſelf a certain number of times, will produce that power. Thus, 2 is the ſquare-root or 2d root of 4, becauſe $2^2 = 2 \times 2 = 4$; and 3 is the cube-root or 3d root of 27, becauſe $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found exactly, namely, by multiplying the number continually into itſelf. But there are many numbers of which a propoſed root can never be exactly found. Yet, by means of decimals we may approximate or approach towards the root, to any degree of exactneſs.

Thoſe roots which only approximate, are called Surd roots; but thoſe which can be found quite exact, are called Rational roots. Thus, the ſquare root of 3 is a ſurd root; but the ſquare root of 4 is a rational root, being equal to 2: alſo the cube root of 8 is rational, being equal to 2; but the cube root of 9 is ſurd or irrational.

Roots are ſometimes denoted by writing the character $\sqrt{}$ before the power, with the index of the root againſt it. Thus,

Thus, the 3d root of 20 is expreſſed by $\sqrt[3]{20}$; and the ſquare root or 2d root of it is $\sqrt{20}$, the index 2 being always omitted, when the ſquare root is deſigned.

When the power is expreſſed by ſeveral numbers, with the ſign + or — between them, a line is drawn from the top of the ſign over all the parts of it: thus the third root of $45 - 12$ is $\sqrt[3]{\overline{45 - 12}}$, or thus $\sqrt[3]{(45 - 12)}$, incloſing the numbers in parentheſes.

But all roots are now often deſigned like powers, with fractional indices: thus, the ſquare-root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of $45 - 18$ is $\overline{45 - 18}^{\frac{1}{4}}$, or $(45 - 18)^{\frac{1}{4}}$.

TO EXTRACT THE SQUARE ROOT.

RULE*.

DIVIDE the given number into periods of two figures each, by ſetting a point over the place of units, another over the place of hundreds, and ſo on, over every ſecond figure, both to the left-hand in integers, and to the right in decimals.

Find

* The reaſon for ſeparating the figures of the dividend into periods or portions of two places each, is, that the ſquare of any ſingle figure never conſiſts of more than two places; the ſquare of a number of two figures, of not more than four places, and ſo on. So that there will be as many figures in the root as the given number contains periods ſo divided or parted off.

And the reaſon of the ſeveral ſteps in the operation, appears from the algebraic form of the ſquare of any number of terms, whether two or three or more. Thus,

$\overline{a + b}^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a + b} \cdot b$, the ſquare of two terms; where it appears, that a is the firſt term of the root, and b the ſecond term; alſo a the firſt diviſor, and the new diviſor is $2a + b$, or double the firſt term increaſed by the ſecond. And hence the manner of extraction is thus:

1ſt diviſor a) $a^2 + 2ab + b^2$ ($a + b$ the root.

a^2

2d diviſor $2a + b$ | $2ab + b^2$

b | $2ab + b^2$

Find the greateſt ſquare in the firſt period on the left-hand, and ſet its root on the right-hand of the given number, after the manner of a quotient figure in Diviſion.

Subtract the ſquare thus found from the ſaid period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a diviſor; and find how often it is contained in the ſaid dividend, excluſive of its right-hand figure; and ſet that quotient figure both in the quotient and diviſor.

Multiply the whole augmented diviſor by this laſt quotient figure, and ſubtract the product from the ſaid dividend, bringing down to it the next period of the given number, for a new dividend.

Repeat the ſame proceſs over again, viz. find another new diviſor, by doubling all the figures now found in the root; from which, and the laſt dividend, find the next figure of the root as before; and ſo on through all the periods, to the laſt.

Note, The beſt way of doubling the root, to form the new diviſors, is by adding the laſt figure always to the laſt diviſor, as appears in the following examples.—Alſo, after the figures belonging to the given number are all exhauſted, the operation may be continued into decimals at pleaſure, by adding any number of periods of ciphers, two in each period.

Again, for a root of three parts a, b, c, thus:

$\overline{a+b+c}^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 =$

$a^2 + \overline{2a+b}.b + \overline{2a+2b+c}.c$, the ſquare of three terms; where a is the firſt term of the root, b the ſecond, and c the third term; alſo a the firſt diviſor, $2a+b$ the ſecond, and $2a+2b+c$ the third, each conſiſting of the double of the root increaſed by the next term of the ſame. And the mode of extraction is thus:

1ſt diviſor a) $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$ ($a+b+c$ the root.
a^2

2d diviſor $2a+b$ | $2ab + b^2$
b | $2ab + b^2$

3d diviſor $2a+2b+c$ | $2ac + 2bc + c^2$
c | $2ac + 2bc + c^2$.

EXAM-

EXAMPLES.

1. To find the square root of 29506624.

```
      29506624 ( 5432 the root.
      25
  104 | 450
    4 | 416
 1083 | 3466
    3 | 3249
10862 | 21724
    2 | 21724
```

NOTE, *When the root is to be extracted to many places of figures, the work may be considerably shortened, thus:*

Having proceeded in the extraction after the common method till there be found half the required number of figures in the root, or one figure more; then, for the rest, divide the last remainder by its corresponding divisor after the manner of the third contraction in Division of Decimals; thus,

2. To find the root of 2 to nine places of figures.

```
        2 ( 1·4142
        1
   24 | 100
    4 |  96
  281 |  400
    1 |  281
 2824 | 11900
    4 | 11296
28282 |  60400
    2 |  56564
28284 )   3836 ( 1356
          1008
           160
            19
             2
```

Ans. 1·41421356 the root required.

 3. What

3. What is the ſquare root of 2025? Anſ. 45.
4. What is the ſquare root of 17'3056? Anſ. 4'16.
5. What is the ſquare root of '000729? Anſ. '027.
6. What is the ſquare root of 3? Anſ. 1'732050.
7. What is the ſquare root of 5? Anſ. 2'236068.
8. What is the ſquare root of 6? Anſ. 2'449489.
9. What is the ſquare root of 7? Anſ. 2'645751.
10. What is the ſquare root of 10? Anſ. 3'162277.
11. What is the ſquare root of 11? Anſ. 3'316624.
12. What is the ſquare root of 12? Anſ. 3'464101.

RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

FIRST prepare all vulgar fractions, by reducing them to their leaſt terms, both for this and all other roots. Then

1. Take the root of the numerator and of the denominator for the reſpective terms of the root required. And this is the beſt way if the denominator be a complete power: but if it be not, then

2. Multiply the numerator and denominator together; take the root of the product: this root being made the numerator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

That is, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}$.

And this rule will ſerve whether the root be finite or infinite.

3. Or reduce the vulgar fraction to a decimal, and extract its root.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the firſt or ſecond rule; or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

EXAMPLES.

1. What is the root of $\frac{25}{36}$? Anſ. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$? Anſ. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$? Anſ. 0·866025.
4. What is the root of $\frac{5}{12}$? Anſ. 0·645497.
5. What is the root of $17\frac{3}{8}$? Anſ. 4·168333.

By means of the ſquare root alſo may readily be found the 4th root, or the 8th root, or the 16th root, &c; that is, the root of any power whoſe index is ſome power of the number 2;

number 2; namely, by extracting so often the square root as is denoted by that power of 2; that is, two extractions for the fourth root, three for the 8th root, and so on.

So, to find the 4th root of the number 21035·8, extract the square root two times as follows:

```
      21035·8000      ( 145·037237 (12·0431407 the 4th root.
      1                 1
24 | 110            22 | 45
 4 |  96             2 |  44
285 | 1435         2404 | 10372
  5 | 1425            4 |  9616
29003 | 108000    24083 |  75637
    6 |  87009        6 |  72249
        20991 ( 7237        3388 ( 1407
          687                980
          107                 17
           20
```

TO EXTRACT THE CUBE ROOT.

I. *By the Common Rule**.

1. HAVING divided the given number into periods of three figures each, find the nearest less cube to the first period; set its root in the quotient, and subtract the said cube from the first period; to the remainder bring down the second period, and call this the resolvend.

---

* The reason for pointing the given number into periods of three figures each, is because the cube of one figure never amounts to more than three places. And, for a similar reason, a given number is pointed into periods of four figures for the 4th root, of five figures for the 5th root, and so on.

And the reason for the other parts of the rule depends on the algebraic formation of a cube: for, if the root consist of the two parts $a+b$, then its cube is as follows: $\overline{a+b}|^3 = a^3 + 3a^2b + 3ab^2 + b^3$; where a is the root of the first part a^3; the resolvend is $3a^2b + 3ab^2 + b^3$, which is also the same as the three parts of the subtrahend; also the divisor is $3a^2 + 3a$, by which dividing the first two terms of the resolvend $3a^2b + 3ab^2$, gives b for the second part of the root; and so on.

2. To

2. To three times the ſquare of the root, juſt found, add three times the root itſelf, ſetting this one place more to the right than the former, and call this ſum the diviſor. Then divide the reſolvend, wanting the laſt figure, by the diviſor, for the next figure of the root, which annex to the former; calling this laſt figure e, and the part of the root before found let be called a.

3. Add all together theſe three products, namely, thrice a ſquare multiplied by e, thrice a multiplied by e ſquare, and e cube, ſetting each of them one place more to the right than the former, and call the ſum the ſubtrahend; which muſt not exceed the reſolvend; but if it does, then make the laſt figure e leſs, and repeat the operation for finding the ſubtrahend, till it be leſs than the reſolvend.

4. From the reſolvend take the ſubtrahend, and to the remainder join the next period of the given number for a new reſolvend; to which form a new diviſor from the whole root now found; and from thence another figure of the root, as directed in Article 2, and ſo on.

EXAMPLE.

To extract the cube root of 48228·544.

$3 \times 3^2 = 27$ | 48228·544 (36·4 root.
$3 \times 3 = 09$ | 27

Diviſor 279 | 21228 reſolvend.

$3 \times 3^2 \times 6 = 162$
$3 \times 3 \times 6^2 = 324$ } add
$6^3 = 216$

$3 \times 36^2 = 3888$ | 19656 ſubtrahend.
$3 \times 36 = 108$ |

38988 | 1572544 reſolvend.

$3 \times 36^2 \times 4 = 15552$
$3 \times 36 \times 4^2 = 1728$ } add
$4^3 = 64$

1572544 ſubtrahend.

0000000 remainder.

Ex. 2. Extract the cube root of 571482·19.
Ex. 3. Extract the cube root of 1628·1582.

II. To

II. *To extract the Cube Root by a ſhort Way*.*

1. By trials take the neareſt rational cube to the given number, whether it be greater or leſs; and call it the aſſumed cube.

2. Then ſay, by the Rule-of-Three, As the ſum of the given number and double the aſſumed cube, is to the ſum of the aſſumed cube and double the given number, ſo is the root of the aſſumed cube, to the root required, nearly. Or, As the firſt ſum is to the difference of the given and aſſumed cube, ſo is the aſſumed root, to the difference of the roots nearly.

3. Again, by uſing, in like manner, the cube of the root laſt found as a new aſſumed cube, another root will be obtained ſtill nearer. And ſo on as far as we pleaſe; uſing always the cube of the laſt found root, for the aſſumed cube.

EXAMPLE.

To find the Cube root of 21035·8.

Here we ſoon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the aſſumed cube. Then

19683		21035·8			
2		2			
39366		42071·6			
21035·8		19683			
As 60401·8	:	61754·6	::	27	: 27·6047
		27			
		4322822			
		1235092			
60401·8)		1667374·2	(	27·6047	the root nearly,
		459338			
		36525			
		284			
		42			

Again,

* The method uſually given for extracting the cube root, is ſo exceedingly tedious, and difficult to be remembered, that various other approximating rules have been invented, by Newton, Raphſon, Halley, De Lagny, Simpſon, Emerſon, and ſeveral other

Again, for a ſecond operation, the cube of this root is 21035·318645155823, and the proceſs by the latter method will be thus:

21035·318645 &c
2

-42070·637290 21035·8
21035·8 21035·318645 &c

As 63106·43729 : diff. ·481355 the dif. :: 27·6047 : ·000210834

conſeq. the root req. is 27·604910834

TO EXTRACT ANY ROOT WHATEVER*.

LET N be the given power or number, n the index of the power, A the aſſumed power, r its root, R the required root of N.

Then, as the ſum of $n + 1$ times A and $n - 1$ times N, is to the ſum of $n + 1$ times N and $n - 1$ times A, ſo is the aſſumed root r, to the required root R.

Or, as half the ſaid ſum of $n + 1$ times A and $n - 1$ times N, is to the difference between the given and aſſumed powers, ſo is the aſſumed root r, to the difference between the true and aſſumed roots: which difference, added or ſubtracted, as the caſe requires, gives the true root nearly.

That is, $\overline{n+1}.\ A + \overline{n-1}.N : n+1.N + \overline{n-1}.A :: N : R.$

Or, $\overline{n+1}.\frac{1}{2}A + \overline{n-1}.\frac{1}{2}N : N \sim A :: r : R \sim r.$

mathematicians; but no one that I have yet ſeen, is ſo ſimple in its form, or ſeems ſo well adapted for general uſe, as that above given. This rule, as far as I have learned, firſt came from Mr. James Dodſon, and is the ſame in effect as Dr. Halley's rational formula, but ſomewhat more commodiouſly expreſſed; and the firſt inveſtigation of it was given in my Tracts, pa. 49. The algebraic form of it is this:

As N + 2A : A + 2N :: r : R. Or
As N + 2A : N ~ A :: r : R ~ r.

where N is the given number, A the aſſumed neareſt cube, r the cube root of A, and R the root of N ſought.

* This is a very general approximating rule, of which that for the cube root is a particular caſe, and is the beſt adapted for practice and for memory, of any that I have yet ſeen. It was firſt diſcovered in this form by myſelf, and the inveſtigation and uſe of it were given at large in my Tracts, vol. 1, pa. 45, &c.

And

And the operation may be repeated as often as we please, by using always the last found root for the assumed root, and its nth power for the assumed power A.

EXAMPLE.

To extract the 5th root of 21035·8.

Here it appears that the 5th root is between 7·3 and 7·4. Taking 7·3, its 5th power is 20730·71593. Hence then we have

$N = 21035{\cdot}8$; $r = 7{\cdot}3$; $n = 5$; $\frac{1}{2}.\overline{n+1} = 3$; $\frac{1}{2}.\overline{n-1} = 2$.

$A = 20730{\cdot}716$

$305{\cdot}084 = N - A$

$A = 20730{\cdot}716$ N = 21035·8

3

$3A = 62192{\cdot}148$

$2N = 42071{\cdot}6$

As 104263·7 : 305·084

73

915252

2135588

104263·7) 2227·1132

14184

3758

630

5

OTHER EXAMPLES.

1. What is the 3d root
2. What is the 4th root
3. What is the 4th root
4. What is the 5th root
5. What is the 6th root
6. What is the 6th root
7. What is the 7th root
8. What is the 7th root
9. What is the 8th root
10. What is the 8th root
11. What is the 9th root
12. What is the 9th root

A TABLE

A TABLE *of* SQUARES *and* CUBES, *also* SQUARE ROOTS *and* CUBE ROOTS.

Number.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1·0000000	1·000000
2	4	8	1·4142136	1·259921
3	9	27	1·7320508	1·442250
4	16	64	2·0000000	1·587401
5	25	125	2·2360680	1·709976
6	36	216	2·4494897	1·817121
7	49	343	2·6457513	1·912933
8	64	512	2·8284271	2·000000
9	81	729	3·0000000	2·080084
10	100	1000	3·1622777	2·154435
11	121	1331	3·3166248	2·223980
12	144	1728	3·4641016	2·289428
13	169	2197	3·6055513	2·351335
14	196	2744	3·7416574	2·410142
15	225	3375	3·8729833	2·466212
16	256	4096	4·0000000	2·519842
17	289	4913	4·1231056	2·571282
18	324	5832	4·2426407	2·620741
19	361	6859	4·3588989	2·668402
20	400	8000	4·4721360	2·714418
21	441	9261	4·5825757	2·758923
22	484	10648	4·6904158	2·802039
23	529	12167	4·7958315	2·843867
24	576	13824	4·8989795	2·884499
25	625	15625	5·0000000	2·924018
26	676	17576	5·0990195	2·962496
27	729	19683	5·1961524	3·000000
28	784	21952	5·2915026	3·036589
29	841	24389	5·3851648	3·072317
30	900	27000	5·4772256	3·107232
31	961	29791	5·5677644	3·141381
32	1024	32768	5·6568542	3·174802
33	1089	35937	5·7445626	3·207534
34	1156	39304	5·8309519	3·239612
35	1225	42875	5·9160798	3·271066
36	1296	46656	6·0000000	3·301927
37	1369	50653	6·0827625	3·332222
38	1444	54872	6·1644140	3·361975
39	1521	59319	6·2449980	3·391211
40	1600	64000	6·3245553	3·419952

Number.	Square.	Cube.	Square Root.	Cube Root.
41	1681	68921	6·4031242	3·448217
42	1764	74088	6·4807407	3·476027
43	1849	79507	6·5574385	3·503398
44	1936	85184	6.6332496	3·530348
45	2025	91125	6·7082039	3·556893
46	2116	97336	6·7823300	3·583048
47	2209	103823	6·8556546	3·608826
48	2304	110592	6·9282032	3·634241
49	2401	117649	7·0000000	3·659306
50	2500	125000	7·0710678	3·684031
51	2601	132651	7·1414284	3·708430
52	2704	140608	7·2111026	3·732511
53	2809	148877	7·2801099	3·756286
54	2916	157464	7·3484692	3·779763
55	3025	166375	7·4161985	3·802953
56	3136	175616	7·4893148	3·825862
57	3249	185193	7·5498344	3·848501
58	3364	195112	7·6157731	3·870877
59	3481	205379	7·6811457	3·892996
60	3600	216000	7·7459667	3·914867
61	3721	226981	7·8102497	3·936497
62	3844	238328	7·8740079	3·957892
63	3969	250047	7·9372539	3·979057
64	4096	262144	8·0000000	4·000000
65	4225	274625	8·0622577	4·020726
66	4356	287496	8·1240384	4·041240
67	4489	300763	8·1853528	4·061548
68	4624	314432	8·2462113	4·081656
69	4761	328509	8·3066239	4·101566
70	4900	343000	8·3666003	4·121285
71	5041	357911	8·4261498	4·140818
72	5184	373248	8·4852814	4·160168
73	5329	389017	8·5440037	4·179339
74	5476	405224	8·6023253	4·198336
75	5625	421875	8·6602540	4·217163
76	5776	438976	8·7177979	4·235824
77	5929	456533	8·7749644	4·254321
78	6084	474552	8·8317609	4·272659
79	6241	493039	8·8881944	4·290841
80	6400	512000	8·9442719	4·308870
81	6561	531441	9·0000000	4·326749
82	6724	551368	9·0553851	4·344481
83	6889	571787	9·1104336	4·362071
84	7056	592704	9·1651514	4·379519
85	7225	614125	9·2195445	4·396830

Number.	Square.	Cube.	Square Root.	Cube Root.
86	7396	636056	9·2736185	4·414005
87	7569	658503	9·3273791	4·431047
88	7744	681472	9·3808315	4·447960
89	7921	704969	9·4339811	4·464745
90	8100	729000	9·4868330	4·481405
91	8281	753571	9·5393920	4·497942
92	8464	778688	9·5916630	4·514357
93	8649	804357	9·6436508	4·530655
94	8836	830584	9·6953597	4·546836
95	9025	857375	9·7467943	4·562903
96	9216	884736	9·7979590	4·578857
97	9409	912673	9·8488578	4·594701
98	9604	941192	9·8994949	4·610436
99	9801	970299	9·9498744	4·626065
100	10000	1000000	10·0000000	4·641589
101	10201	1030301	10·0498756	4·657010
102	10404	1061208	10·0995049	4·672330
103	10609	1092727	10·1488916	4·687548
104	10816	1124864	10·1980390	4·702669
105	11025	1157625	10·2469508	4·717694
106	11236	1191016	10·2956301	4·732624
107	11449	1225043	10·3440804	4·747459
108	11664	1259712	10·3923048	4·762203
109	11881	1295029	10·4403065	4·776856
110	12100	1331000	10·4880885	4·791420
111	12321	1367631	10·5356538	4·805896
112	12544	1404928	10·5830052	4·820284
113	12769	1442897	10·6301458	4·834588
114	12996	1481544	10·6770783	4·848808
115	13225	1520875	10·7238053	4·862944
116	13456	1560896	10·7703296	4·876999
117	13689	1601613	10·8166538	4·890973
118	13924	1643032	10·8627805	4·904868
119	14161	1685159	10·9087121	4·918685
120	14400	1728000	10·9544512	4·932424
121	14641	1771561	11·0000000	4·946088
122	14884	1815848	11·0453610	4·959675
123	15129	1860867	11·0905365	4·973190
124	15376	1906624	11·1355287	4·986631
125	15625	1953125	11·1803399	5·000000
126	15876	2000376	11·2249722	5·013298
127	16129	2048383	11·2694277	5·026526
128	16384	2097152	11·3137085	5·039684
129	16641	2146689	11·3578167	5·052774
130	16900	2197000	11·4017543	5·065797

Number.	Square.	Cube.	Square Root.	Cube Root.
131	17161	2248091	11·4455231	5·078753
132	17424	2299968	11·4891253	5·091643
133	17689	2352637	11·5325626	5·104469
134	17956	2406104	11·5758369	5·117230
135	18225	2460375	11·6189500	5·129928
136	18496	2515456	11·6619038	5·142563
137	18769	2571353	11·7046999	5·155137
138	19044	2628072	11·7473444	5·167649
139	19321	2685619	11·7898261	5·180101
140	19600	2744000	11·8321596	5·192494
141	19881	2803221	11·8743421	5·204828
142	20164	2863288	11·9163753	5·217103
143	20449	2924207	11·9582607	5·229321
144	20736	2985984	12·0000000	5·241482
145	21025	3048625	12·0415946	5·253588
146	21316	3112136	12·0830460	5·265637
147	21609	3176523	12·1243557	5·277632
148	21904	3241792	12·1655251	5·289572
149	22201	3307949	12·2065556	5·301459
150	22500	3375000	12·2474487	5·313293
151	22801	3442951	12·2882057	5·325074
152	23104	3511808	12·3288280	5·336803
153	23409	3581577	12·3693169	5·348481
154	23716	3652264	12·4096736	5·360108
155	24025	3723875	12·4498996	5·371685
156	24336	3796416	12·4899960	5·383213
157	24649	3869893	12·5299641	5·394690
158	24964	3944312	12·5698051	5·406120
159	25281	4019679	12·6095202	5·417501
160	25600	4096000	12·6491106	5·428835
161	25921	4173281	12·6885775	5·440122
162	26244	4251528	12·7279221	5·451362
163	26569	4330747	12·7671453	5·462556
164	26896	4410944	12·8062485	5·473703
165	27225	4492125	12·8452326	5·484806
166	27556	4574296	12·8840987	5·495865
167	27889	4657463	12·9228480	5·506879
168	28224	4741632	12·9614814	5·517848
169	28561	4826809	13·0000000	5·528775
170	28900	4913000	13·0384048	5·539658
171	29241	5000211	13·0766968	5·550499
172	29584	5088448	13·1148770	5·561298
173	29929	5177717	13·1529464	5·572054
174	30276	5268024	13·1909060	5·582770
175	30625	5359375	13·2287566	5·593445

Number.	Square.	Cube.	Square Root.	Cube Root.
176	30976	5451776	13·2664992	5·604079
177	31329	5545233	13·3041347	5·614673
178	31684	5639752	13·3416641	5·625226
179	32041	5735339	13·3790882	5·635741
180	32400	5832000	13·4164079	5·646216
181	32761	5929741	13·4536240	5·656652
182	33124	6028568	13·4907376	5·667051
183	33489	6128487	13·5277493	5·677411
184	33856	6229504	13·5646600	5·687734
185	34225	6331625	13·6014705	5·698019
186	34596	6434856	13·6381817	5·708267
187	34969	6539203	13·6747943	5·718479
188	35344	6644672	13·7113092	5·728654
189	35721	6751269	13·7477271	5·738794
190	36100	6859000	13·7840488	5·748897
191	36481	6967871	13·8202750	5·758965
192	36864	7077888	13·8564065	5·768998
193	37429	7189057	13·8924440	5·778996
194	37636	7301384	13·9283883	5·788960
195	38025	7414875	13·9642400	5·798890
196	38416	7529536	14·0000000	5·808786
197	38809	7645373	14·0356688	5·818648
198	39204	7762392	14·0712473	5·828476
199	39601	7880599	14·1067360	5·838272
200	40000	8000000	14·1421356	5·848035
201	40401	8120601	14·1774469	5·857765
202	40804	8242408	14·2126704	5·867464
203	41209	8365427	14·2478068	5·877130
204	41616	8489664	14·2828569	5·886765
205	42025	8615125	14·3178211	5·896368
206	42436	8741816	14·3527001	5·905941
207	42849	8869743	14·3874946	5·915481
208	43264	8998912	14·4222051	5·924991
209	43681	9123329	14·4568323	5·934473
210	44100	9261000	14·4913767	5·943911
211	44521	9393931	14·5258390	5·953341
212	44944	9528128	14·5602198	5·962731
213	45369	9663597	14·5945195	5·972091
214	45796	9800344	14·6287388	5·981426
215	46225	9938375	14·6628783	5·990727
216	46656	10077696	14·6969385	6·000000
217	47089	10218313	14·7309199	6·009244
218	47524	10360282	14·7648231	6·018463
219	47961	10503459	14·7986486	6·027650
220	48400	10648000	14·8323970	6·036811

Of Ratios, Proportions, and Progressions.

Numbers are compared to each other in two different ways: the one compariſon conſiders the difference of the two numbers, and is named Arithmetical Relation; and the difference ſometimes the Arithmetical Ratio: the other conſiders their quotient, which is called Geometrical Relation, and the quotient the Geometrical Ratio. So, of theſe two numbers 6 and 3, the difference, or arithmetical ratio, is $6 - 3$ or 3; but the geometrical ratio is $\frac{6}{3}$ or 2.

There muſt be two numbers to form a compariſon: the number which is compared, being placed firſt, is called the Antecedent; and that to which it is compared, the Conſequent. So, in the two numbers above, 6 is the antecedent, and 3 the conſequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4, 2 and 8, 6, are arithmetical proportionals, becauſe $4 - 2 = 8 - 6 = 2$; and the two couplets 4, 2 and 6, 3, are geometrical proportionals, becauſe $\frac{4}{2} = \frac{6}{3} = 2$, the ſame ratio.

To denote numbers as being geometrically proportional, a colon is ſet between the terms of each couplet, to denote their ratio; and a double colon, or elſe a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3 are ſet thus, $4 : 2 :: 6 : 3$, which means, that 4 is to 2 as 6 is to 3; or thus, $4 : 2 = 6 : 3$; or thus, $\frac{4}{2} = \frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3.

Proportion is diſtinguiſhed into Continued and Diſcontinued. When the difference or ratio of the conſequent of one couplet and the antecedent of the next couplet, is not the ſame as the common difference or ratio of the couplets, the proportion is diſcontinued. So, 4, 2, 8, 6 are in diſcontinued arithmetical proportion, becauſe $4 - 2 = 8 - 6 = 2$, whereas $8 - 2 = 6$: and 4, 2, 6, 3 are in diſcontinued geometrical proportion, becauſe $\frac{4}{2} = \frac{6}{3} = 2$, but $\frac{6}{2} = 3$, which is not the ſame.

But when the difference or ratio of every two ſucceeding terms is the ſame quantity, the proportion is ſaid to be Continued, and the numbers themſelves a ſeries of Continued Propor-

Proportionals, or a Progreſſion. So, 2, 4, 6, 8 form an arithmetical progreſſion, becauſe $4 - 2 = 6 - 4 = 8 - 6 = 2$, all the ſame common difference; and 2, 4, 8, 16 a geometrical progreſſion, becauſe $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$, all the ſame ratio.

When the following terms of a Progreſſion exceed each other, it is called an Aſcending Progreſſion or Series; but when the terms decreaſe, it is a Deſcending one.

So, 0, 1, 2, 3, 4, &c, is an aſcending arithmetical progreſſion, but 9, 7, 5, 3, 1, &c, is a deſcending arithmetical progreſſion: Alſo 1, 2, 4, 8, 16, &c, is an aſcending geometrical progreſſion, and 16, 8, 4, 2, 1, &c, is a deſcending geometrical progreſſion.

ARITHMETICAL PROPORTION AND PROGRESSION.

THE firſt and laſt terms of a Progreſſion, are called the Extremes; and the other terms, lying between them, the Means.

The moſt uſeful part of arithmetical proportions, is contained in the following theorems:

THEOREM 1. If four quantities be in arithmetical proportion, the ſum of the two extremes will be equal to the ſum of the two means.

Thus, of the four 2, 4, 6, 8, here $2 + 8 = 4 + 6 = 10$.

THEOREM 2. In any continued arithmetical progreſſion, the ſum of the two extremes, is equal to the ſum of any two means that are equally diſtant from them, or equal to double the middle term when there is an uneven number of terms.

Thus, in the terms 1, 3, 5, it is $1 + 5 = 3 + 3 = 6$.

And in the ſeries 2, 4, 6, 8, 10, 12, 14, it is $2 + 14 = 4 + 12 = 6 + 10 = 8 + 8 = 16$.

THEOREM 3. The difference between the extreme terms of an arithmetical progreſſion, is equal to the common difference of the ſeries multiplied by one leſs than the number of the terms.

So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one leſs than the number of terms 9; then the difference of the extremes is $20 - 2 = 18$, and $2 \times 9 = 18$ alſo.

Conſe-

Conſequently, the greateſt term is equal to the leaſt term added to the product of the common difference multiplied by 1 leſs than the number of terms.

THEOREM 4. The ſum of all the terms, of any arithmetical progreſſion, is equal to the ſum of the two extremes multiplied by the number of terms, and divided by 2; or the ſum of the two extremes multiplied by the number of the terms, gives double the ſum of all the terms in the ſeries.

This is made evident by ſetting the terms of the ſeries in an inverted order under the ſame ſeries in a direct order, and adding the correſponding terms together in that order. Thus,

in the ſeries 1, 3, 5, 7, 9, 11, 13, 15;
ditto inverted 15, 13, 11, 9, 7, 5, 3, 1;
the ſums are 16 + 16 + 16 + 16 + 16 + 16 + 16 + 16,

which muſt be double the ſum of the ſingle ſeries, and is equal to he ſum of the extremes repeated ſo often as are the number of the terms.

From theſe theorems may readily be found any one of theſe five parts; the two extremes, the number of terms, the common difference, and the ſum of all the terms, when any three of them are given; as in the following Problems:

PROBLEM I.

Given the Extremes, and the Number of Terms; to find the Sum of all the Terms.

RULE.

ADD the extremes together, multiply the ſum by the number of terms, and divide by 2.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the ſum of the terms?

19
3
22
9
2) 198
Anſ. 99

Or $\frac{19+3}{2} \times 9 = \frac{22}{2} \times 9 = 11 \times 9 = 99$.

2. It is required to find the number of all the ſtrokes a clock ſtrikes in one whole revolution of the index, or in 12 hours? Anſ. 78.

Ex. 3. How many ſtrokes do the clocks of Venice ſtrike in the compaſs of the day, which go right on from 1 to 24 o'clock? Anſ. 300.

4. What debt can be diſcharged in a year, by weekly payments in arithmetical progreſſion, the firſt payment being 1s, and the laſt or 52d payment 5l 3s? Anſ. 135l 4s.

PROBLEM II.

Given the Extremes, and the Number of Terms; to find the Common Difference.

RULE.

SUBTRACT the leſs extreme from the greater, and divide the remainder by 1 leſs than the number of terms, for the common difference.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the common difference?

$$\begin{array}{r} 19 \\ 3 \\ \hline 8 \,)\, 16 \\ \hline \text{Anſ.} \quad 2 \end{array} \qquad \text{Or } \frac{19-3}{9-1} = \frac{16}{8} = 2.$$

2. If the extremes be 10 and 70, and the number of terms 21; what is the common difference, and the ſum of the ſeries?
Anſ. the com. diff. is 3, and the ſum is 840.

3. A certain debt can be diſcharged in one year, by weekly payments in arithmetical progreſſion, the firſt payment being 1s, and the laſt 5l 3s; what is the common difference of the terms? Anſ. 2.

PROBLEM III.

Given one of the Extremes, the Common Difference, and the Number of Terms; to find the other Extreme, and the Sum of the Series.

RULE.

MULTIPLY the common difference by 1 leſs than the number of terms, and the product will be the difference of the extremes: Therefore add the product to the leſs extreme, to give the greater; or ſubtract it from the greater, to give the leſs.

EXAM-

EXAMPLES.

1. Given the leaſt term 3, the common difference 2, of an arithmetical ſeries of 9 terms; to find the greateſt term, and the ſum of the ſeries?

2
8
16
3
19 the greateſt term
3 the leaſt
22 ſum
9 number of terms.
2) 198
99 the ſum of the ſeries.

2. If the greateſt term be 70, the common difference 3, and the number of terms 21; what is the leaſt term and the ſum of the ſeries?

Anſ. The leaſt term is 10, and the ſum is 840.

3. A debt can be diſcharged in a year, by paying 1 ſhilling the firſt week, 3 ſhillings the ſecond, and ſo on, always 2 ſhillings more every week; what is the debt, and what will the laſt payment be?

Anſ. The laſt payment will be 5l 3s, and the debt is 1351 4s.

PROBLEM IV.

To find an Arithmetical Mean Proportional between Two Given Terms.

RULE.

ADD the two given extremes or terms together, and take half their ſum for the arithmetical mean required. Or, ſubtract the leſs extreme from the greater, and half the remainder will be the common difference; which being added to the leſs extreme, or ſubtracted from the greater, will give the mean required.

EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

Here	Or		Or
14	14		14
4	4		5
2) 18	2) 10		9
Anſ. 9	5	the com. dif.	
	4	the leſs extreme.	
	9		

So that 9 is the mean required, by both methods.

PROBLEM V.

To find Two Arithmetical Means between Two Given Extremes.

RULE.

SUBTRACT the leſs extreme from the greater, and divide the difference by 3, ſo will the quotient be the common difference; which being continually added to the leſs extreme, or taken from the greater, gives the means.

EXAMPLE.

To find two arithmetical means between 2 and 8.

Here 8
2
3) 6
com. dif. 2

Then 2 + 2 = 4 the one mean,
and 4 + 2 = 6 the other mean.

PROBLEM VI.

To find any Number of Arithmetical Means between Two Given Terms or Extremes.

RULE.

SUBTRACT the leſs extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference; then this being added continually to the leaſt term, or ſubtracted from the greateſt, will give the mean terms required.

EXAM-

EXAMPLE.

To find five arithmetical means between 2 and 14.

Here 14
2
6) 12
com. dif. 2

Then by adding this com. dif. continually, the means are found 4, 6, 8, 10, 12.

See more of Arithmetical Progreſſion in the Algebra.

GEOMETRICAL PROPORTION AND PROGRESSION.

THE moſt uſeful part of Geometrical Proportion, is contained in the following theorems.

THEOREM 1. If four quantities be in geometrical proportion, the product of the two extremes will be equal to the product of the two means.

Thus, in the four 2, 4, 3, 6, it is $2 \times 6 = 3 \times 4 = 12$.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. So, of the above numbers, the product of the means $12 \div 2 = 6$ the one extreme, and $12 \div 6 = 2$ the other extreme; and this is the foundation and reaſon of the practice in the Rule-of-Three.

THEOREM 2. In any continued geometrical progreſſion, the product of the two extremes is equal to the product of any two means that are equally diſtant from them, or equal to the ſquare of the middle term when there is an uneven number of terms.

Thus, in the terms 2, 4, 8, it is $2 \times 8 = 4 \times 4 = 16$.

And in the ſeries 2, 4, 8, 16, 32, 64, 128,
it is $2 \times 128 = 4 \times 64 = 8 \times 32 = 16 \times 16 = 256$.

THEOREM 3. The quotient of the extreme terms of a geometrical progreſſion, is equal to the common ratio of the ſeries raiſed to the power denoted by 1 leſs than the number of the terms.

So,

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, one leſs than the number of terms 9; then the quotient of the extremes is $\frac{1024}{2} = 512$, and $2^9 = 512$ alſo.

Conſequently the greateſt term is equal to the leaſt term multiplied by the ſaid power of the ratio whoſe index is 1 leſs than the number of terms.

THEOREM 4. The ſum of all the terms, of any geometrical progreſſion, is found by adding the greateſt term to the difference of the extremes divided by 1 leſs than the ratio.

So, the ſum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whoſe ratio is 2), is $1024 + \frac{1024 - 2}{2 - 1} = 1024 + 1022 = 2046$.

The foregoing, and ſeveral other properties of geometrical proportion, are demonſtrated more at large in the Algebraic part of this work. A few examples may here be added of the theorems, juſt delivered, with ſome problems concerning mean proportionals.

EXAMPLES.

1. The leaſt of ten terms, in geometrical progreſſion, being 1, and the ratio 2; what is the greateſt term, and the ſum of all the terms?

Anſ. The greateſt term is 512, and the ſum 1023.

2. What debt may be diſcharged in a year, or 12 months, by paying 1l the firſt month, 2l the ſecond, 4l the third, and ſo on, each ſucceeding payment being double the laſt; and what will the laſt payment be?

Anſ. The debt 4095l, and the laſt payment 2048l.

PROBLEM 1.

To find One Mean Geometrical Proportional between any Two Numbers.

RULE.

MULTIPLY the two numbers together, and extract the ſquare root of the product, which will give the mean proportional ſought.

Or, Divide the greater term by the leſs, and extract the ſquare-root of the quotient, which will give the common ratio of the three terms: then multiply the leſs term by the ratio, or divide the greater term by it, either of theſe will give the middle term required.

EXAM-

EXAMPLES.

To find a geometrical mean between the two numbers 3 and 12.

First Way.

12
3
36 (6 the mean.
36

Second Way.

3) 12 (4, its root is 2 the ratio.
Then 3 × 2 = 6 the mean,
Or 12 ÷ 2 = 6 ditto.

PROBLEM II.

To find Two Geometrical Mean Proportionals between any Two Numbers.

RULE.

DIVIDE the greater number by the leſs, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the leaſt given term by the ratio for the firſt mean, and this mean again by the ratio for the ſecond mean: or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the leſs mean.

EXAMPLE.

To find two geometrical mean proportionals between 3 and 24.

Here 3) 24 (8; its cube root 2 is the ratio.
Then 3 × 2 = 6, and 6 × 2 = 12, the two means.
Or 24 ÷ 2 = 12, and 12 ÷ 2 = 6, the ſame.
That is, the two means between 3 and 24, are 6 and 12.

PROBLEM III.

To find any Number of Geometrical Mean Proportionals between Two Numbers.

RULE.

DIVIDE the greater number by the leſs, and extract ſuch root of the quotient whoſe index is 1 more than the number of means required, that is, the 2d root for one mean, the 3d root for two means, the 4th root for three means, and ſo on; and that root will be the common ratio of all the terms. Then, with the ratio multiply continually from the firſt term, or divide continually from the laſt or greateſt term.

EXAM-

EXAMPLE.

To find four geometrical mean proportionals between 3 and 96.

Here 3) 96 (32 ; the 5th root of which is 2, the ratio.

Then $3 \times 2 = 6$, and $6 \times 2 = 12$, and $12 \times 2 = 24$, and $24 \times 2 = 48$.

Or $96 \div 2 = 48$, and $48 \div 2 = 24$, and $24 \div 2 = 12$, and $12 \div 2 = 6$.

That is 6, 12, 24, 48 are the four means between 3 and 96.

OF MUSICAL PROPORTION.

THERE is alſo a third kind of proportion, called Muſical, which being but of little or no common uſe, a very ſhort account of it may here ſuffice.

Muſical Proportion is when, of three numbers, the firſt has the ſame proportion to the third, as the difference between the firſt and ſecond, hath to the difference between the ſecond and third.

As in theſe three, 6, 8, 12;

Where $6 : 12 :: 8 - 6 : 12 - 8$,

that is $6 : 12 :: 2 : 4$.

When four numbers are in muſical proportion; then the firſt has the ſame proportion to the fourth, as the difference between the firſt and ſecond hath to the difference between the third and fourth.

As in theſe, 6, 8, 12, 18;

where $6 : 18 :: 8 - 6 : 18 - 12$,

that is $6 : 18 :: 2 : 6$.

When numbers are in muſical progreſſion, their reciprocals are in arithmetical progreſſion; and the converſe, that is, when numbers are in arithmetical progreſſion, their reciprocals are in muſical progreſſion.

So in theſe muſicals 6, 8, 12, their reciprocals, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$ are in arithmetical progreſſion; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the ſum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding out numbers in muſical proportion, is beſt expreſſed by letters in Algebra.

FELLOWSHIP, OR PARTNERSHIP.

FELLOWSHIP is a rule, by which any ſum or quantity may be divided into any number of parts, which ſhall be in any given proportion to one another.

By this rule are adjuſted the gains or loſs or charges of partners in company; or the effects of bankrupts, or legacies in caſe of a deficiency of aſſets or effects; or the ſhares of prizes, or the numbers of men to form certain detachments; or the diviſion of waſte lands among a number of proprietors.

Fellowſhip is either Single or Double. It is Single, when the ſhares or portions are to be proportional each to one ſingle given number only; as when the ſtocks of partners are all employed for the ſame time; And Double, when each portion is to be proportional to two or more numbers; as when the ſtocks of partners are employed for different times.

SINGLE FELLOWSHIP

GENERAL RULE.

ADD together the numbers that denote the proportion of the ſhares. Then ſay,

As the ſum of the ſaid proportional numbers,
Is to the whole ſum to be parted or divided,
So is each ſeveral proportional number,
To the correſponding ſhare or part.

OR, As the whole ſtock, is to the whole gain or loſs,
So is each man's particular ſtock,
To his particular ſhare of the gain or loſs.

TO PROVE THE WORK. Add all the ſhares or parts together, and the ſum will be equal to the whole number to be ſhared, when the work is right.

EXAMPLES.

1. To divide the number 240 into three ſuch parts, as ſhall be in proportion to each other as the three numbers 1, 2, and 3.

Here

Here 1 + 2 + 3 = 6 the ſum of the numbers.

Then, as 6 : 240 :: 1 : 40 the 1ſt part,
and as 6 : 240 :: 2 : 80 the 2d part,
alſo as 6 : 240 :: 3 : 120 the 3d part.
Sum of all 240, the proof.

2. Three perſons, A, B, C, freighted a ſhip with 340 tuns of wine; of which, A loaded 110 tuns, B 97, and C the reſt: in a ſtorm the ſeamen were obliged to throw overboard 85 tuns; how much muſt each perſon ſuſtain of the loſs?

Here 110 + 97 = 207 tuns, loaded by A and B
theref. 340 — 207 = 133 tuns, loaded by C.

Hence, as 340 : 85 :: 110
or as 4 : 1 :: 110 : $27\frac{1}{2}$ tuns = A's loſs;
and as 4 : 1 :: 97 : $24\frac{1}{4}$ tuns = B's loſs;
alſo as 4 : 1 :: 133 : $33\frac{1}{4}$ tuns = C's loſs.

Sum 85 tuns, the proof.

3. Two merchants, C and D, made a ſtock of 120l; of which C contributed 75l, and D the reſt; by trading they gained 30l; what muſt each have of it?

Anſ. C 18l 15s, and D 11l 5s.

4. Three merchants, E, F, G, make a ſtock of 700l, of which E contributed 123l, F 358l, and G the reſt; by trading they gain 125l 10s; what muſt each have of it?

Anſ. E muſt have 22l 1s od $2\frac{2}{3}\frac{2}{5}$q,
F — — 64 3 8 $0\frac{3}{3}\frac{2}{5}$.
G — — 39 5 3 $1\frac{1}{3}\frac{1}{5}$.

5. A General impoſing a contribution * of 700l. on four villages, to be paid in proportion to the number of inhabitants contained in each; the 1ſt containing 250, the 2d 350, the 3d 400, and the 4th 500 perſons: what part muſt each village pay?

Anſ. the 1ſt to pay 116l 13s 4d.
the 2d — — 163 6 8
the 3d — — 186 13 4
the 4th — — 233 6 8

* Contribution is a tax paid by provinces, towns, villages, &c. to excuſe them from being plundered. It is paid in proviſions or in money, and ſometimes in both.

Ex. 6,

Ex. 6. A piece of ground, conſiſting of 37ac 2ro 14ps, is to be divided among three perſons, L, M, and N, in proportion to their eſtates: now if L's eſtate be worth 500l a year, M's 320l, and N's 75l; what quantity of land muſt each one have? Anſ. L muſt have 20ac 3ro $39\frac{3}{179}$ps.

M — 13 1 $30\frac{46}{179}$

N — 3 0 $23\frac{173}{179}$.

7. A perſon is indebted to Q 571 15s, to P 1081 3s 8d, to Q 221 10d, and to R 73l; but at his deceaſe, his effects are found to be worth no more than 170l 14s: how muſt it be divided among his creditors?

Anſ. Q muſt have 37l 15s 5d $2\frac{530}{1043}$q.

P — 70 15 2 $2\frac{749}{1043}$

Q — 14 8 4 $0\frac{470}{1043}$

R — 47 14 11 $2\frac{3368}{1043}$.

8. A ſhip worth 900l, being entirely loſt, of which $\frac{1}{6}$ belonged to S, $\frac{1}{4}$ to T, and the reſt to V; what loſs will each ſuſtain, ſuppoſing 540l of her were inſured?

Anſ. S will looſe 45l, T 90l, and V 225l.

9. Four perſons, W, X, Y, and Z, ſpent among them 25s. and agree that W ſhall pay $\frac{1}{2}$ of it. X $\frac{1}{3}$, Y $\frac{1}{4}$, and Z $\frac{1}{5}$; that is, their ſhares are to be in proportion as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$; what are their ſhares? Anſ. W muſt pay 9s 8d $3\frac{41}{77}$q.

X — 6 5 $3\frac{53}{77}$

Y — 4 10 $1\frac{59}{77}$

Z — 3 10 $3\frac{1}{77}$.

10. A detachment, conſiſting of 5 companies, being ſent into a garriſon, in which the duty required 76 men a day; what number of men muſt be furniſhed by each company, in proportion to their ſtrength; the 1ſt conſiſting of 54 men, the 2d of 51 men, the 3d of 48 men, the 4th of 39, and the 5th of 36 men?

Anſ. The 1ſt muſt furniſh 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men*.

* Queſtions of this nature frequently occurring in military ſervice, General Haviland, an officer of great merit, contrived an ingenious inſtrument, for more expeditiouſly reſolving them; which is diſtinguiſhed by the name of the inventor, being called a Haviland.

DOUBLE

DOUBLE FELLOWSHIP.

DOUBLE FELLOWSHIP, as has been ſaid, is concerned in caſes in which the ſtocks of partners are employed or continued for different times.

RULE*.

MULTIPLY each perſon's ſtock by the time of its continuance; then divide the quantity, as in Single Fellowſhip, into ſhares in proportion to theſe products, by ſaying,

As the total ſum of all the ſaid products,
Is to the whole gain or loſs, or quantity to be parted,
So is each particular product,
To the correſpondent ſhare of the gain or loſs.

EXAMPLES.

1. A had in company 50l for 4 months, and B had 60l for 5 months; at the end of which they find 24l gained; how muſt it be divided between them?

Here $\begin{matrix} 50 \\ 4 \\ \hline 200 \end{matrix} + \begin{matrix} 60 \\ 5 \\ \hline 300 \end{matrix} = 500$.

Then, as 500 : 24 :: 200 : $9\frac{3}{5}$ = 9l 12s = A's ſhare,
and as 500 : 24 :: 300 : $14\frac{2}{5}$ = 14 8 = B's ſhare.

2. C and D hold a piece of ground in common, for which they are to pay 36l. C put in 23 horſes for 27 days, and D 21 horſes for 39 days; how much ought each man to pay of the rent? Anſ. C muſt pay 15l 10s 6d.
D muſt pay 20 9 6.

3. Three perſons, E, F, G, hold a paſture in common, for which they are to pay 30l per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put

* The proof of this rule is as follows: When the times are equal, the ſhares of the gain or loſs are evidently as the ſtocks, as in Single Fellowſhip; and when the ſtocks are equal, the ſhares are as the times; therefore, when neither are equal, the ſhares muſt be as their products.

in

in 4 oxen for 12 months; how much muſt each perſon pay of the rent? Anſ. E muſt pay 5l 10s 6d $1\frac{5}{19}$q.

F — 11 16 10 $0\frac{8}{19}$

G — 12 12 7 $2\frac{16}{19}$.

4. A ſhip's company take a prize of 1000l, which they agree to divide among them according to their pay and the time they have been on board: now the officers and midſhipmen have been on board 6 months, and the ſailors 3 months; the officers have 40s a month, the midſhipmen 30s, and the ſailors 22s a month; moreover there are 4 officers, 12 midſhipmen, and 110 ſailors: what will each man's ſhare be?

Anſ. each officer muſt have 23l 2s 5d $0\frac{92}{173}$q.

each midſhip. — 17 6 9 $3\frac{69}{173}$

each ſeaman — 6 7 2 $0\frac{8}{173}$.

5. H, with a capital of 1000l, began trade the firſt of January, and, meeting with ſucceſs in buſineſs, took in I as a partner, with a capital of 1500l, on the firſt of March following. Three months after that, they admit K as a third partner, who brought into ſtock 2800l. After trading together till the end of the year, they find there has been gained 1776l 10s: how muſt this be divided among the partners?

Anſ. H muſt have 457l 9s $4\frac{1}{4}$d.

I — 571 16 $8\frac{1}{4}$

K — 747 3 $11\frac{1}{4}$.

6. X, Y, and Z made a joint-ſtock for 12 months; X at firſt put in 20l, and 4 months after 20l more; Y put in at firſt 30l, at the end of 3 months he put in 20l more, and 2 months after he put in 40l more; Z put in at firſt 60l, and 5 months after he put in 10l more, 1 month after which he took out 30l; during the 12 months they gained 50l; how much of it muſt each have?

Anſ. X muſt have 10l 18s 6d $3\frac{49}{61}$q.

Y — 22 8 1 $0\frac{12}{61}$

Z — 16 13 4 0

SIMPLE

SIMPLE INTEREST.

INTEREST is the premium or ſum allowed for the loan, or forbearance, of money.

The money lent, or forborn, is called the Principal.

The ſum of the principal and its intereſt, added together, is called the Amount.

Intereſt is allowed at ſo much per cent. per annum; which premium per cent. per annum, or intereſt of 100l for a year, is called the rate of Intereſt:—So,

When intereſt is at 3 per cent. the rate is 3;
— — — 4 per cent. — 4;
— — — 5 per cent. — 5;
— — — 6 per cent. — 6.

But, by law, intereſt ought not to be taken higher than at the rate of 5 per cent.

Intereſt is of two ſorts; Simple and Compound.

Simple Intereſt is that which is allowed for the principal lent or forborn only, for the whole time of forbearance.

As the intereſt of any ſum, for any time, is directly proportional to the principal ſum, and alſo to the time of continuance; hence ariſes the following general rule of calculation.

GENERAL RULE.

As 100l is to the rate of intereſt, ſo is any given principal, to its intereſt for one year. And again,

As 1 year is to any given time, ſo is the intereſt for a year, juſt found, to the intereſt of the given ſum for that time.

OTHERWISE. Take the intereſt of 1 pound for a year, which multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the intereſt of the propoſed ſum for that time.

Note. When there are certain parts of years in the time, as quarters, or months, or days; they may be worked for either by taking the aliquot or like parts of the intereſt of a year, or by the Rule-of-Three, in the uſual way. Alſo, to divide, by 100, is done by only pointing off two figures for decimals.

EXAMPLES.

1. To find the intereſt of 230l 10s, for 1 year, at the rate of 4 per cent. per annum.

Here,

Here, As 100 : 4 :: 2301 10s : 91 4s $4\frac{1}{4}$d.

4

100) 9,22 0

20

4·40

12

4·80

4

3·20 Anſ. 91 4s $4\frac{1}{4}$d.

2. To find the intereſt of 547l 15s, for 3 years, at 5 per cent. per annum.

As 100 : 5 :: 547·75 :

Or 20 : 1 :: 547·75 : 27·3875 intereſt for 1 year.

3

1 82·1625 ditto for 3 years.

20

s 3·2500

12

d 3·00 Anſ. 82l 3s 3d.

3. To find the intereſt of 200 guineas, for 4 years 7 months and 25 days, at $4\frac{1}{2}$ per cent. per annum.

210

$4\frac{1}{2}$

840

105

9·45 intereſt for 1 yr.

4

37·80 ditto 4 years.

6mo = $\frac{1}{2}$ 4·725 ditto 6 months.

1mo = $\frac{1}{6}$ ·7875 ditto 1 month.

·6472 ditto 25 days.

143·9597

20

s 19·1940

12

d 2·3280

4

q 1·3120

ds	l	ds	
As 365	: 9·45	:: 25	: 1
or 73	: 9·45	:: 5	: ·6472

5

73) 47·25 (·6472

345

530

19

Anſ. 43l 19s $2\frac{1}{4}$d.

Ex. 4.

Ex. 4. To find the intereſt of 450l, for a year, at 5 per cent. per annum. Anſ. 22l 10s.

5. To find the intereſt of 230l 10s, for a year, at 4 per cent. per annum. Anſ. 9l 4s $4\frac{3}{4}$d.

6. To find the intereſt of 715l 12s 6d, for a year, at $4\frac{1}{2}$ per cent. per annum. Anſ. 32l 4s $0\frac{3}{4}$d.

7. To find the intereſt of 720l, for 3 years, at 5 per cent. per annum. Anſ. 108l.

8. To find the intereſt of 355l 15s, for 4 years, at 4 per cent. per annum. Anſ. 56l 18s $4\frac{3}{4}$d.

9. To find the intereſt of 32l 5s 8d, for 7 years, at $4\frac{1}{2}$ per cent. per annum. Anſ. 9l 12s 1d.

10. To find the intereſt of 170l, for $1\frac{1}{2}$ year, at 5 per cent. per annum. Anſ. 12l 15s.

11. To find the inſurance of 205l 15s, for $\frac{1}{4}$ of a year, at 4 per cent. per annum. Anſ. 2l 1s $1\frac{1}{4}$d.

12. To find the intereſt of 319l 6d, for $5\frac{3}{4}$ years, at $3\frac{3}{4}$ per cent. per annum. Anſ. 68l 15s $9\frac{1}{2}$d.

13. To find the inſurance on 107l, for 117 days, at $4\frac{3}{4}$ per cent. per annum. Anſ. 1l 12s 7d.

14. To find the intereſt of 17l 5s, for 117 days, at $4\frac{3}{4}$ per cent. per annum. Anſ. 5s 3d.

15. To find the inſurance on 712l 6s, for 8 months, at $7\frac{1}{2}$ per cent. per annum. Anſ. 35l 12s $3\frac{1}{2}$d.

Note. The Rules for Simple Intereſt, ſerve alſo to calculate Inſurances, or the Purchaſe of Stocks, or any thing elſe that is rated at ſo much per cent.

See alſo more on the ſubject of Intereſt, with the algebraical expreſſion and inveſtigation of the rules, at the end of the Algebra, next following.

COMPOUND INTEREST.

Compound Interest, called alſo Intereſt upon Intereſt, is that which ariſes from the principal and intereſt, taken together, as it becomes due, at the end of each ſtated time of payment.

Although

Although it be not lawful to lend money at Compound Intereſt, yet in purchaſing annuities, penſions, or leaſes in reverſion, it is uſual to allow Compound Intereſt to the purchaſer for his ready money.

RULES.

1. FIND the amount of the given principal, for the time of the firſt payment, by Simple Intereſt. Then conſider this amount as a new principal for the ſecond payment, whoſe amount calculate as before. And ſo on through all the payments to the laſt, always accounting the laſt amount as a new principal for the next payment. The reaſon of which is evident from the definition of Compound Intereſt. *Or elſe*,

2. Find the amount of 1 pound for the time of the firſt payment, and raiſe or involve it to the power whoſe index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount. From which the ſaid principal being ſubtracted, leaves the Compound Intereſt of the ſame. As is evident from the firſt Rule.

EXAMPLES.

1. To find the amount of 720l, for 4 years, at 5 per cent. per annum.

Here 5 is the 20th part of 100, and the intereſt of 1l for a year is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. *By the 1ſt Rule.*

l	s	d	
20) 720	0	0	1ſt yr's princip.
36	0	0	1ſt yr's intereſt.
20) 756	0	0	2d yr's princip.
37	16	0	2d yr's intereſt.
20) 793	16	0	3d yr's princip.
39	13	$9\frac{1}{2}$	3d yr's intereſt.
20) 833	9	$9\frac{1}{2}$	4th yr's princip.
41	13	$5\frac{1}{4}$	4th yr's intereſt.
£875	3	$3\frac{1}{4}$	the whole amo^{t}. or anſ. required.

2. *By the 2d Rule.*

1·05 amount of 1l.
1·05
1·1025 2d power of it.
1·1025 ditto.
1·21550625 4th power of it
720
875·1645
20
s 3·2900
12
d 3·4800

2. To find the amount of 50l, in 5 years, at 5 per cent. per annum, compound intereſt. Anſ. 62l 16s $3\frac{1}{4}$d.

Ex. 3. To find the amount of 50l, in 5 years, or 10 half-years, at 5 per cent. per annum, compound intereſt, the intereſt payable half-yearly. Anſ. 64l 0s 1d.

4. To find the amount of 50l, in 5 years, or 20 quarters, at 5 per cent. per annum, compound intereſt, the intereſt payable quarterly. Anſ. 64l 2s $0\frac{1}{4}$d.

5. To find the compound intereſt of 370l, forborn for 6 years, at 4 per cent. per annum. Anſ. 98l 3s $4\frac{1}{4}$d.

6. To find the compound intereſt of 410l, forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, the intereſt payable half-yearly. Anſ. 48l 4s $11\frac{1}{4}$d.

7. To find the amount, at compound intereſt, of 217l, forborn for $2\frac{1}{4}$ years, at 5 per cent. per annum, the intereſt payable quarterly. Anſ. 242l 13s $4\frac{1}{2}$d.

Note. See the Rules for Compound Intereſt algebraically inveſtigated, at the end of the Algebra.

ALLIGATION.

ALLIGATION teaches how to compound or mix together ſeveral ſimples of different qualities, ſo that the compoſition may be of ſome intermediate quality, or rate. It is commonly diſtinguiſhed into two caſes, Alligation Medial, and Alligation Alternate.

ALLIGATION MEDIAL.

ALLIGATION MEDIAL is the method of finding the rate or quality of the compoſition, from having the quantities and rates or qualities of the ſeveral ſimples given.

RULE*.

MULTIPLY the quantity of each ingredient by its rate or quality; then add all the products together, and add alſo all the

Demonſtration. The Rule is thus proved by Algebra.

Let a, b, c be the quantities of the ingredients,

and

the quantities together into another ſum: then divide the former ſum by the latter, that is, the ſum of the products by the ſum of the quantities, and the quotient will be the rate or quality of the compoſition required.

EXAMPLES.

1. If three ſorts of gunpowder be mixed together, viz. 50lb at 12d a pound, 44lb at 9d, and 26lb at 8d a pound; how much a pound is the compoſition worth?

Here 50, 44, 26 are the quantities,
and 12, 9, 8 the rates or qualities;
then $50 \times 12 = 600$
$44 \times 9 = 396$
$26 \times 8 = 208$

120) 1204 ($10, \frac{4}{120} = 10\frac{1}{30}$.

Anſ. The rate or price is $10\frac{1}{30}$d the pound.

2. A compoſition being made of 5lb of tea at 7s per lb, 9lb at 8s 6d per lb, and $14\frac{1}{2}$lb at 5s 10d per lb; what is a lb of it worth? Anſ. 6s $10\frac{1}{4}$d.

3. Mixed 4 gallons of wine at 4s 10d per gall. with 7 gallons at 5s 3d per gall, and $9\frac{3}{4}$ gallons at 5s 8d per gall; what is a gallon of this compoſition worth? Anſ. 5s $4\frac{1}{4}$d.

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and m, n, p their rates, or qualities, or prices;
then am, bn, cp are their ſeveral values,
and $am + bn + cp$ the ſum of their values,
alſo $a + b + c$ is the ſum of the quantities,
and if r denote the rate of the whole compoſition,
then $\overline{a + b + c} \times r$ will be the value of the whole,
conſeq. $\overline{a + b + c} \times r = am + bn + cp$,
and $r = \overline{am + bn + cp} \div \overline{a + b + c}$, which is the Rule.

Note, If an ounce or any other quantity of pure gold be reduced into 24 equal parts, theſe parts are called Carats; but gold is often mixed with ſome baſer metal, which is called the Alloy, and the mixture is ſaid to be of ſo many carats fine, according to the proportion of pure gold contained in it; thus, if 22 carats of pure gold, and 2 of alloy be mixed together, it is ſaid to be 22 carats fine.

If any one of the ſimples be of little or no value with reſpect to the reſt, its rate is ſuppoſed to be nothing; as water mixed with wine, and alloy with gold and ſilver.

Ex. 4. A mealman would mix 3 buſhels of flour at 3s 5d per buſhel, 4 buſhels at 5s 6d per buſhel, and 5 buſhels at 4s 8d per buſhel: what is the worth of a buſhel of this mixture? Anſ. 4s $7\frac{1}{2}$d.

5. A farmer mixes 20 buſhels of wheat at 5s the buſhel, with 36 buſhels of rye at 3s the buſhel, and 40 buſhels of barley at 2s per buſhel: how much is a buſhel of the mixture worth? Anſ. 3s.

6. Having melted together 7 oz of gold of 22 carats fine, $12\frac{1}{2}$ oz of 21 carats fine, and 17 oz of 19 carats fine: I would know the fineneſs of the compoſition?

Anſ. $20\frac{9}{13}$ carats fine.

7. Of what fineneſs is that compoſition, which is made by mixing 3lb of ſilver of 9 oz fine, with 5lb 8oz of 10 oz fine, and 11b 10 oz of alloy? Anſ. $7\frac{6}{13}$oz fine.

ALLIGATION ALTERNATE.

ALLIGATION ALTERNATE is the method of finding what quantity of any number of ſimples, whoſe rates are given, will compoſe a mixture of a given rate. So that it is the reverſe of Alligation Medial, and may be proved by it.

RULE I.*

1. SET the rates of the ſimples in a column under each other.

2. Connect, or link with a continued line, the rate of each ſimple, which is leſs than that of the compound, with one, or any number, of thoſe that are greater than the compound; and each greater rate with one or any number of the leſs.

3. Write

* *Demonſt.* By connecting the leſs rate to the greater, and placing the differences between them and the rate alternately, the quantities reſulting are ſuch, that there is preciſely as much gained by one quantity as is loſt by the other, and therefore the gain and loſs upon the whole is equal, and is exactly the propoſed rate: and the ſame will be true of any other two ſimples managed according to the Rule.

In

3. Write the difference between the mixture rate, and that of each of the ſimples, oppoſite the rates with which they are linked.

4. Then if only one difference ſtand againſt any rate, it will be the quantity belonging to that rate; but if there be ſeveral, their ſum will be the quantity.

EXAMPLES.

1. A merchant would mix wines at 17s, 18s, and 22s per gallon, ſo as that the mixture may be worth 20s the gallon: what quantity of each muſt be taken?

Here 20 { 17, 18, 22 }

2 at 17s
2 at 18s
3 + 2 = 5 at 22s

Anſ. 2 gallons at 17s, 2 gallons at 18s, and 5 at 22s.

2. How much wine at 6s per gallon, and at 4s per gallon, muſt be mixed together, that the compoſition may be worth 5s per gallon? Anſ. 1qt, or 1gall, &c.

3. How much corn at 2s 6d, 3s 8d, 4s, and 4s 8d per buſhel, muſt be mixed together, that the compound may be worth 3s 10d per buſhel?

Anſ. 12 at 2s 6d, 12 at 3s 8d, 18 at 4s, and 18 at 4s 8d.

4. A goldſmith has gold of 17, 18, 22, and 24 carats fine: how much muſt he take of each, to make it 21 carats fine?

Anſ. 3 of 17, 1 of 18, 3 of 22, and 4 of 24.

In like manner, let the number of ſimples be what they will, and with how many ſoever every one is linked, ſince it is always a leſs with a greater than the mean price, there will be an equal balance of loſs and gain between every two, and conſequently an equal balance on the whole. Q. E. D.

It is obvious, from the Rule, that queſtions of this ſort admit of a great variety of anſwers; for, having found one anſwer, we may find as many more as we pleaſe, by only multiplying or dividing each of the quantities found, by 2, or 3, or 4, &c. the reaſon of which is evident; for, if two quantities, of two ſimples, make a balance of loſs and gain, with reſpect to the mean price, ſo muſt alſo the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of theſe quantities, and ſo on *ad infinitum*.

Theſe kinds of queſtions are called by algebraiſts *indeterminate* or *unlimited* problems; and by an analytical proceſs, theorems may be raiſed that will give all the *poſſible* anſwers.

Ex. 5.

5. It is required to mix brandy at 8s, wine at 7s, cyder at 1s, and water at 0 per gallon together, ſo that the mixture may be worth 5s per gallon?

Anſ. 9 gals of brandy, 9 of wine, 5 of cyder, and 5 of water.

6. How much ſugar at 4d, at 6d, and at 11d per lb, muſt be mixed together, ſo that the compoſition formed by them may be worth 7d per lb?

Anſ. 1 lb, or 1 ſtone, or 1 cwt, or any other equal quantity of each ſort.

RULE 2.

WHEN the whole compoſition is limited to a certain quantity:

Find an anſwer as before by linking; then ſay, as the ſum of the quantities, or differences thus determined, is to the given quantity; ſo is each ingredient, found by linking, to the required quantity of each.

EXAMPLES.

1. How much gold of 15, 17, 18 and 22 caracts fine, muſt be mixed together, to form a compoſition of 40 oz of 20 caracts fine?

Here 20 { 15 — — — — — 2
17 — — — — — 2
18 — — — — — 2
22 5 + 3 + 2 = 10

Then, 16

as 16 : 40 :: 2 : 5
and 16 : 40 :: 10 : 25

Anſ. 5 oz of 15, 17, and 18 caracts fine, and 25 oz of 22 caracts fine*.

Ex. 2.

* A great number of queſtions might be here given relating to the ſpecific gravities of metals, &c. but one of the moſt curious may here ſuffice.

Hiero, king of Syracuſe, gave orders for a crown to be made entirely of pure gold; but ſuſpecting the workman had debaſed it by mixing it with ſilver or copper, he recommended the diſcovery of the fraud to the famous Archimedes, and deſired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the impoſition, procured two other maſſes, the one of pure gold, the other of ſilver or copper, and each of the ſame weight with the former; and by putting each ſeparately

Ex. 2. A grocer has currants at 4d, 6d, 9d, and 11d per lb, and he would make a mixture of 240 lb, ſo that it might be afforded at 8d per lb; how much of each ſort muſt he take?

Anſ. 72lb at 4d, 24 at 6d, 48 at 9d, and 96 at 11d.

RULE 3*.

When one of the ingredients is limited to a certain quantity; Take the difference between each price, and the mean rate as before; then ſay,

As the difference of that ſimple, whoſe quantity is given, is to the reſt of the differences ſeverally, ſo is the quantity given, to the ſeveral quantities required.

EXAMPLES.

1. How much wine at 5s, at 5s 6d, and 6s the gallon, muſt be mixed with 3 gallons at 4s per gallon, ſo that the mixture may be worth 5s 4d per gallon?

Here 64 { 48 — 8 + 2 = 10
60 — 8 + 2 = 10
66 — 16 + 4 = 20
72 — 16 + 4 = 20

---

parately into a veſſel full of water, the quantity of water expelled by them determined their ſpecific gravities: from which, and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppoſe the weight of each crown to be 10lb, and that the water expelled by the copper or ſilver was ·92lb, by the gold ·52lb, and by the compound crown ·64lb; what will be the quantities of gold and alloy in the crown?

The rates of the ſimples are 92 and 52, and of the compound 64; therefore

64 | 92 — 12 of copper
52 — 28 of gold

And the ſum of theſe is 12 + 28 = 40, which ſhould have been but 10; therefore by the Rule,

40 : 10 :: 12 : 3lb of copper
40 : 10 :: 28 : 7lb of gold } the anſwer.

* In the very ſame manner queſtions may be wrought when ſeveral of the ingredients are limited to certain quantities, by finding firſt for one limit, and then for another.

The two laſt Rules can want no demonſtration, as they evidently reſult from the firſt, the reaſon of which has been already explained.

Then

Then 10 : 10 :: 3 : 3
10 : 20 :: 3 : 6
10 : 20 :: 3 : 6

Anſ. 3 gallons at 5s, 6 at 5s 6d, and 6 at 6s.

2. A grocer would mix teas at 12s, 10s, and 6s per lb, with 20lb at 4s per lb: how much of each ſort muſt he take to make the compoſition worth 8s per lb?

Anſ. 20lb at 4s, 10lb at 6s, 10lb at 10s, and 20lb at 12s.

3. How much gold of 15, of 17, and of 22 carats fine, muſt be mixed with 5 oz of 18 carats fine, ſo that the compoſition may be 20 carats fine?

Anſ. 5 oz of 15 carats fine, 5 oz of 17, and 25 of 22.

POSITION.

POSITION is a method of performing certain queſtions, which cannot be reſolved by the common direct rules. It is ſometimes called Falſe Poſition, or Falſe Suppoſition, becauſe it makes a ſuppoſition of falſe numbers, to work with the ſame as if they were the true ones, and by their means diſcovers the true numbers ſought. It is ſometimes alſo called Trial-and-Error, becauſe it proceeds by *trials* of falſe numbers, and thence finds out the true ones by a compariſon of the *errors*.

Poſition is either Single or Double.

SINGLE POSITION

SINGLE POSITION is that by which a queſtion is reſolved by means of one ſuppoſition only.

Queſtions which have their reſults proportional to their ſuppoſitions, belong to Single Poſition: ſuch as thoſe which require the multiplication or diviſion of the number ſought by any propoſed number; or when it is to be increaſed or diminiſhed by itſelf, or any parts of itſelf, a certain propoſed number of times.

RULE.

RULE

TAKE or aſſume any number for that which is required, and perform the ſame operations with it, as are deſcribed or performed in the queſtion.

Then ſay, As the reſult of the ſaid operation, is to the poſition, or number aſſumed; ſo is the reſult in the queſtion, to the number ſought *.

EXAMPLES.

1. A perſon, after ſpending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining 60l; what had he at firſt?

Suppoſe he had at firſt 120l.	Proof.
Now $\frac{1}{3}$ of 120 is 40	$\frac{1}{3}$ of 144 is 48
$\frac{1}{4}$ of it is 30	$\frac{1}{4}$ of 144 is 36
their ſum is 70	their ſum 84
which taken from 120	taken from 144
leaves 50	leaves 60 as per queſtion.

Then, 50 : 120 :: 60 : 144, the Anſwer.

2. What number is that, which multiplied by 7, and the product divided by 6, the quotient may be 14? Anſ. 12.

3. What number is that, which being increaſed by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, of itſelf, the ſum ſhall be 125? Anſ. 60.

4. A general, after ſending out a foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 700; what number had he in command? Anſ. 4200.

5. A gentleman diſtributed 78 pence among a number of poor people, conſiſting of men, women, and children; to each man he gave 6d, to each woman 4d, and to each child 2d: moreover there were twice as many women as men, and thrice as many children as women. How many were there of each? Anſ. 3 men, 6 women, and 18 children.

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* The reaſon of the Rule is evident, becauſe it is ſuppoſed that the reſults are proportional to the ſuppoſitions.

Thus, $nx : x :: na : a$,

or $\frac{x}{n} : x :: \frac{a}{n} : a$,

or $\frac{x}{n} \pm \frac{x}{m}$, &c $: x :: \frac{a}{n} \pm \frac{a}{m}$, &c $: a$,

and ſo on.

Ex. 6.

Ex. 6. One being aſked his age, ſaid, if $\frac{3}{5}$ of the years I have lived, be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the ſum will be 292. What was his age?

Anſ. 60 years.

DOUBLE POSITION.

DOUBLE POSITION is the method of reſolving certain queſtions by means of two ſuppoſitions of falſe numbers.

To the Double Rule of Poſition belong ſuch queſtions as have their reſults not proportional to their poſitions: ſuch are thoſe, in which the numbers ſought, or their parts or their multiples, are increaſed or diminiſhed by ſome given abſolute number, which is no known part of the number ſought.

RULE*.

TAKE or aſſume any two convenient numbers, and proceed with each of them ſeparately, according to the conditions of the queſtion, as in Single Poſition; and find how much each reſult is different from the reſult mentioned in the queſtion, noting alſo whether the reſults are too great or too little.

Then

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* *Demonſtr.* The Rule is founded on this ſuppoſition, namely, that the firſt error is to the ſecond, as the difference between the true and firſt ſuppoſed number, is to the difference between the true and ſecond ſuppoſed number; when that is not the caſe, the exact anſwer to the queſtion cannot be found by this Rule.—That the Rule is true, according to that ſuppoſition, may be thus proved.

Let a and b be the two ſuppoſitions, and A and B their reſults, produced by ſimilar operations; alſo r and s their errors, or the differences between the reſults A and B from the true reſult N; and let x denote the number ſought, anſwering to the true reſult N of the queſtion.

Then is $N - A = r$, and $N - B = s$. And, according to the ſuppoſition on which the Rule is founded, $r : s :: x - a : x - b$; hence, by multiplying extremes and means, $rx - rb = sx - sa$; then, by tranſpoſition, $rx - sx = rb - sa$; and, by diviſion, $x = \frac{rb - sa}{r - s}$ = the number ſought, which is the Rule when the reſults are both too little.

Then multiply each of the ſaid errors by the contrary ſuppoſition, namely, the firſt poſition by the ſecond error, and the ſecond poſition by the firſt error. Then,

If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the anſwer.

But if the errors are unlike, divide the ſum of the products by the ſum of the errors, for the anſwer.

Note, The errors are ſaid to be alike, when they are either both too great or both too little; and unlike, when one is too great and the other too little.

EXAMPLES.

1. What number is that, which being multiplied by 6, the product increaſed by 18, and the ſum divided by 9, the quotient ſhall be 20?

Suppoſe the two numbers 18 and 30. Then,

	First Poſition.		Second Poſition.		Proof,
	18	Suppoſe	30		27
	6	mult.	6		6
	108		180		162
	18	add	18		18
9)	126		9) 198		9) 180
	14	reſults	22		20
	20	true ref.	20		
	+ 6	errors unlike	— 2		
2d poſ.	30	mult.	18	1ſt poſ.	
Errors	2 180		36		
	6 36				
ſum	8) 216	ſum of products			
	27	Anſwer ſought.			

RULE

If the reſults be both too great, ſo that A and B are both greater than N; then N — A = — r, and N — B = — s, or r and s are both negative; hence — r : — s :: $x - a$: $x - b$, but — r : — s :: + r : + s, therefore r : s :: $x - a$: $x - b$, and the reſt will be exactly as in the former caſe.

But

RULE 2.*

FIND, by trial, two numbers, as near the true number as poſſible, and operate with them as in the queſtion; marking the errors which ariſe from each of them.

Multiply the difference of the two numbers, found by trial, by the leaſt error, and divide the product by the difference of the errors, when they are alike, but by their ſum when they are unlike.

Add the quotient, laſt found, to the number belonging to the leaſt error, when that number is too little, but ſubtract it when too great, and the reſult will give the true quantity ſought.

EXAMPLES.

2. A ſon aſking his father how old he was, received this Anſwer: Your age is now one-fourth of mine; but 5 years ago, your age was only one-fifth of mine. What then are their two ages? Anſ. 20 and 80.

3. A workman was hired for 30 days, at 2s 6d per day, for every day he worked; but with this condition, that for every day he played, he ſhould forfeit 1s. Now it ſo happened, that upon the whole he had 2l 14s to receive. How many of the days did he work? Anſ. 24.

4. A and B began to play together with equal ſums of money: A firſt won 20 guineas, but afterwards loſt back $\frac{2}{3}$ of what he then had; after which, B had 4 times as much as A. What ſum did each begin with? Anſ. 100 guineas.

5. Two perſons, A and B, have both the ſame income. A ſaves $\frac{1}{5}$ of his; but B, by ſpending 50l per annum more than A, at the end of 4 years finds himſelf 100l in debt. What doth each receive and ſpend per annum?

Anſ. They receive 125l per annum; alſo A ſpends 100l, and B ſpends 150l per annum.

But if one reſult A only be too little, and the other B too great, or one error r poſitive, and the other s negative, then the theorem becomes $x = \frac{rb + sa}{r + s}$, which is the Rule in this caſe, or when the errors are unlike.

* For ſince, by the ſuppoſition, $r : s :: x - a : x - b$, therefore by diviſion, $r - s : s :: b - a : x - b$, which is the 2d Rule.

PERMU-

PERMUTATIONS AND COMBINATIONS.

PERMUTATION is the altering, changing, or varying the position or order of things; or the shewing how many different ways they may be placed.—This is otherwise called Alternation, Changes, or Variation; and the only thing to be regarded here, is the order they stand in; for no two parcels are to have all their quantities placed in the same situation: as, how many changes may be rung on a number of bells, or how many different ways any number of persons may be placed, or how many several variations may be made of any number of letters, or any other things proposed to be varied.

COMBINATION is the shewing how often a less number of things can be taken out of a greater, and combined together, without considering their places, or the order they stand in. This is sometimes called Election or Choice; and here every parcel must be different from all the rest, and no two are to have precisely the same quantities or things.

Combinations of the same Form, are those in which there are the same number of quantities, and the same repetitions: thus, *aabc*, *bbcd*, *ccde*, are of the same form; *aabc*, *abbb*; *aabb*, are of different forms.

Composition of Quantities, is the taking a given number of quantities, out of as many equal rows of different quantities, one out of every row, and combining them together.

Some illustrations of these definitions are in the following Problems:

PROBLEM I.

To assign the Number of Permutations, or Changes, that can be made of any Given Number of Things, all different from each other.

RULE*.

MULTIPLY all the terms of the natural series of numbers, from 1 up to the given number, continually together, and the last product will be the answer required.

EXAM-

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* The reason of the Rule may be shewn thus: any one thing *a* is capable only of one position, as *a*.

Any two things *a* and *b*, are only capable of two variations; as *ab*, *ba*; whose number is expressed by 1×2.

If

EXAMPLES.

1. How many changes may be rung on 6 bells?

1
2
2
3
6
4
24
5
120
6
720 the Anſwer.

Or, $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$ the Anſwer.

2. How many days can 7 perſons be placed in a different poſition at dinner? Anſ. 5040 days.

3. How many changes may be rung on 12 bells, and what time would it require, ſuppoſing 10 changes to be rung in 1 minute, and the year to conſiſt of 365 days, 5 hours, and 49 minutes?

Anſ. 479001600 changes, and 91 years, 26 days, 22 hours, 41 minutes.

4. How many changes may be made of the words in the following verſe: *Tot tibi ſunt dotes, virgo, quot ſiderà cœlo?*

Anſ. 40320 changes.

If there be three things, *a*, *b*, and *c*; then any two of them, leaving out the 3d, will have 1×2 variations; and conſequently, when the 3d is taken in, there will be $1 \times 2 \times 3$ variations.

In the ſame manner, when there are 4 things, every three, leaving out the 4th, will have $1 \times 2 \times 3$ variations; conſequently by taking in ſucceſſively the 4 left out, there will be $1 \times 2 \times 3 \times 4$ variations. And ſo on as far as we pleaſe.

PROBLEM II.

Any Number of different Things being given; to find how many Changes can be made out of them, by taking a Given Number of Quantities at a Time.

RULE*.

TAKE a ſeries of numbers, beginning at the number of things given, and decreaſing by 1 to the number of quantities to be taken at a time, and the product of all the terms will be the anſwer required.

EXAMPLES.

1. How many changes may be rung with 3 bells out of 8?

8
7
56
6
336 the Anſwer.

Or, $8 \times 7 \times 6$ ($= 3$ terms) $= 336$ the Anſwer.

2. How many words can be made with 5 letters of the alphabet, admitting that a number of conſonants alone will not make a word? Anſ. 5100480.

PRO-

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* This Rule, expreſſed in algebraic terms, is as follows:

$m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$ &c, to n terms; where $m =$ the number of things given, and $n =$ the quantities to be taken at a time.

In order to demonſtrate the Rule, it will be proper to premiſe the following Lemma:

LEMMA. The number of changes of m things, taken n at a time, is equal to m changes of $m - 1$ things, taken $n - 1$ at a time.

Demonſtr. Let any five quantities, $a\ b\ c\ d\ e$ be given.

Firſt, leave out the a, and let $v =$ the number of all the variations of every two, bc, bd, &c, that can be taken out of the four remaining quantities $b\ c\ d\ e$.

Now, let a be put in the firſt place of each of them, a, b, c, a, b, d, &c, and the number of changes will ſtill remain the ſame; that is, $v =$ the number of variations of every 3 out of the 5, $a\ b\ c\ d\ e$, when a is firſt.

In

PROBLEM III.

Any Number of Things being given; of which there are ſeveral given Things of one Sort, and ſeveral of another, &c; To find how many Changes can be made out of them all.

RULE*.

TAKE the ſeries $1 \times 2 \times 3 \times 4$, &c, up to the number of things given, and find the product of all the terms.

Take the ſeries $1 \times 2 \times 3 \times 4$, &c, up to the number of given things of the firſt ſort, and the ſeries $1 \times 2 \times 3 \times 4$, &c, up to the number of given things of the ſecond ſort, &c.

Divide

In like manner, if b, c, d, e be ſucceſſively left out, the number of variations of all the two's will alſo be $= v$; and putting b, c, d, e reſpectively in the firſt place, to make 3 quantities out of 5, there will ſtill be $4v$ variations, as before.

But theſe are all the variations that can happen of 3 things out of 5, when a, b, c, d, e are ſucceſſively put firſt; and therefore the ſum of all theſe is the ſum of all the changes of 3 things out of 5.

But the ſum of theſe is ſo many times v, as is the number of things; that is $5v$, or mv, $=$ all the changes of 3 things out of 5.

And the ſame way of reaſoning may be applied to any numbers whatever.

Demon. of the Rule. Let any 7 things $a\ b\ c\ d\ e\ f\ g$ be given, and let 3 be the number of quantities to be taken.

Then $m = 7$, and $n = 3$.

Now, it is evident, that the number of changes that can be made by taking 1 by 1 out of 5 things, will be 5, which let $= v$.

Then, by the Lemma, when $m = 6$, and $n = 2$, the number of changes will be $= mv = 6 \times 5$; which let be $= v$ a ſecond time.

Again, by the Lemma, when $m = 7$ and $n = 3$, the number of changes is $mv = 7 \times 6 \times 5$; that is $mv = m \times (m - 1) \times (m - 2)$, continued to 3, or n terms.

And the ſame may be ſhewn for any other numbers.

* This Rule is expreſſed in terms thus:

$$\frac{1 \times 2 \times 3 \times 4 \times 5, \text{ \&c, to } m}{1 \times 2 \times 3, \text{ \&c. to } p \times 1 \times 2 \times 3, \text{ \&c. to } q \text{ \&c.}}$$

where $m =$ the number of things given, $p =$ the number of things of the firſt ſort, $q =$ the number of things of the ſecond ſort, &c.

The Demonſtration may be ſhewn as follows:

Any 2 quantities, $a\ b$, both different, admit of 2 changes; but if the quantities are the ſame, or $a\ b$ becomes $a\ a$, there will be only one poſition; which may be expreſſed by $\frac{1 \times 2}{1 \times 2} = 1$.

Any

Divide the product of all the terms of the firſt ſeries by the joint product of all the terms of the remaining ones, and the quotient will be the anſwer required.

EXAMPLES.

1. How many variations can be made of the letters in the word Bacchanalia?

1×2 (= number of c's) $= 2$

$1 \times 2 \times 3 \times 4$ (= number of a's) $= 24$

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11$ (= number of letters in the word) $= 39916800$

$2 \times 24 = 48$) 39916800 (831600 the Anſwer.
151
76
288

Any 3 quantities, $a\,b\,c$, all different from each other, afford 6 variations; but if the quantities be all alike, or $a\,b\,c$ becomes $a\,a\,a$, then the 6 variations will be reduced to 1; which may be expreſſed by $\frac{1 \times 2 \times 3}{1 \times 2 \times 3} = 1$. Again, if two of the quantities only are alike, or $a\,b\,c$ becomes $a\,a\,c$; then the 6 variations will be reduced to theſe 3, $a\,a\,c$, $c\,a\,a$, and $a\,c\,a$; which may be expreſſed by $\frac{1 \times 2 \times 3}{1 \times 2} = 3$.

Any 4 quantities, $a\,b\,c\,d$, all different from each other, will admit of 24 variations. But if the quantities be the ſame, or $a\,b\,c\,d$ becomes $a\,a\,a\,a$, the number of variations will be reduced to one; which is $= \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3 \times 4} = 1$.

Again, if three of the quantities only be the ſame, or $a\,b\,c\,d$ becomes $a\,a\,a\,b$, the number of variations will be reduced to theſe 4, $a\,a\,a\,b$, $a\,a\,b\,a$, $a\,b\,a\,a$, and $b\,a\,a\,a$; which is $= \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3} = 4$.

And thus it may be ſhewn, that if two of the quantities be alike, or the 4 quantities be $a\,a\,b\,c$, the number of variations will be reduced to 12; which may be expreſſed by $\frac{1 \times 2 \times 3 \times 4}{1 \times 2} = 12$.

And by reaſoning in the ſame manner, it will appear, that the number of changes which can be made of the quantities $a\,b\,b\,c\,c$ is equal to 60; which may be expreſſed by $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 3} = 60$. And ſo on for any other quantities whatever.

2. How many different numbers can be made of the following figures, 1220005555? Anſ. 12600.

3. How many varieties will take place in the ſucceſſion of the following muſical notes, fa, fa, fa, ſol, ſol, la, mi, fa? Anſ. 3360.

PROBLEM IV.

To find the Changes of any Given Number of Things, taking a Given Number at a Time; in which there are ſeveral Given Things of one Sort, ſeveral of another, &c.

RULE*.

FIND all the different forms of combination of all the given things, taken as many at a time as in the queſtion.

Find the number of changes in any form, and multiply it by the number of combinations in that form.

Do the ſame for every diſtinct form, and the ſum of all the products will give the whole number of changes required.

EXAMPLES.

1. How many alternations, or changes, can be made of every 4 letters out of theſe 8, *aaabbbcc*?

No. of forms.	No. of changes.
a^3b, a^3c, b^3a, b^3c	4
a^2b^2, a^2c^2, b^2c^2	6
a^2bc, b^2ac, c^2ab	12

The reaſon of this Rule is plain from what has been ſhewn before, and the nature of the problem.

A Rule for finding the Number of Forms.

1. PLACE the things ſo, that the greateſt indices may be firſt, and the reſt in order.

2. Begin with the firſt letter, and join it to the ſecond, third, fourth, &c, to the laſt.

3. Then take the ſecond letter, and join it to the third, fourth, &c, to the laſt. And ſo on, till they are entirely exhauſted, always remembering to reject ſuch combinations as have occurred before; and this will give the combinations of all the two's.

4. Join the firſt letter to every one of the two's, and the ſecond, third, &c, as before; and it will give the combinations of all the three's.

5. Proceed in the ſame manner to get the combinations of all the four's, &c, and you will at laſt get all the ſeveral forms of combination, and the number in each form.

Therefore

$$\text{Therefore} \begin{cases} 4 \times 4 = 16 \\ 3 \times 6 = 18 \\ 3 \times 12 = 36 \end{cases}$$

70 = number of changes required.

2. How many changes can be made of every 8 letters out of theſe 10; *aaaabbccde*? Anſ. 22260

3. How many different numbers can be made out of 1 unit, 2 two's, 3 three's, 4 four's, and 5 five's; taken 5 at a time? Anſ. 2111.

PROBLEM V.

To find the Number of Combinations of any Given Number of Things, all different from each other, taken any Given Number at a Time.

RULE*.

TAKE the ſeries 1, 2, 3, 4, &c, up to the number to be taken at a time, and find the product of all the terms.

Take

---

* This Rule, expreſſed algebraically, is,

$\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ &c, to n terms; where m is the number of given quantities, and n thoſe to be taken at a time.

Demonſtr. of the Rule. Let the number of things to be taken at a time be 2, and the things to be combined $= m$.

Now, when m, or the number of things to be combined, is only two, as a and b, it is evident that there can be but one combination, as ab; but if m be increaſed by one, or the letters to be combined be 3, as a, b, c; then it is plain that the number of combinations will be increaſed by 2, ſince with each of the former letters a and b, the new letter c may be joined. In this caſe therefore, it is evident that the whole number of combinations will be truly expreſſed by $1 + 2$.

Again, if m be increaſed by one letter more, or the whole number of letters be four, as a, b, c, d; then it will appear that the whole number of combinations muſt be increaſed by 3, ſince with each of the preceding letters the new letter d may be combined. The combinations, therefore, in this caſe, will be truly expreſſed by $1 + 2 + 3$.

And in the ſame manner it may be ſhewn, that the whole number of combinations of 2, in 5 things, will be $1 + 2 + 3 + 4$; of 2, in 6 things, $1 + 2 + 3 + 4 + 5$; and of 2, in 7 things, $1 + 2 + 3 + 4 + 5 + 6$, &c.: whence, univerſally, the number of combinations

Take a ſeries of as many terms, decreaſing by 1, from the given number, out of which the election is to be made, and find the product of all the terms.

Divide the laſt product by the former, and the quotient will be the number ſought.

EXAMPLES.

1. How many combinations can be made of 6 letters out of 10?

$1 \times 2 \times 3 \times 4 \times 5 \times 6$ (= the number to be taken at a time) $= 720$.

$10 \times 9 \times 8 \times 7 \times 6 \times 5$ (= ſame number from 10) $= 151200$.

---

of m things, taken 2 by 2, is $= 1 + 2 + 3 + 4 + 5 + 6$, &c, to $(m - 1)$ terms.

But the ſum of this ſeries is $= \frac{m}{1} \times \frac{m-1}{2}$; which is the ſame as the rule.

2. Let now the number of quantities in each combination be ſuppoſed to be three.

Then it is plain, that when $m = 3$, or the things to be combined are a, b, c, there can be only one combination. But if m be increaſed by 1, or the things to be combined are 4, as a, b, c, d, then will the number of combinations be increaſed by 3: ſince 3 is the number of combinations of 2 in all the preceding letters, a, b, c, and with each two of theſe the new letter d may be combined.

The number of combinations, therefore, in this caſe, is $1 + 3$.

Again, if m be increaſed by one more, or the number of letters be ſuppoſed 5; then the former number of combinations will be increaſed by 6, that is, by all the combinations of 2 in the 4 preceding letters, a, b, c, d; ſince, as before, with each two of theſe the new letter c may be combined.

The number of combinations, therefore, in this caſe, is $1 + 3 + 6$.

Whence, univerſally, the number of combinations of m things, taken 3 by 3, is $1 + 3 + 6 + 10$, &c, to $m - 2$ terms.

But the ſum of this ſeries is $= \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$; which is the ſame as the rule.

And the ſame thing will hold, let the number of things to be taken at a time be what it will; therefore the number of combinations of m things, taken n at a time, will be $=$

$\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c, to n terms. Q. E. D.

Then 720) 151200 (210 the Anſwer.
1440
720
720

2. How many combinations can be made of 2 letters out of the 24 letters of the alphabet? Anſ. 276.

3. A general, who had often been ſucceſsful in war, was aſked by his king what reward he ſhould confer upon him for his ſervices; the general only deſired a farthing for every file, of 10 men in a file, which he could make with a body of 100 men; what is the amount in pounds ſterling?

Anſ. 1803157235ol 9s 2d.

PROBLEM VI.

To find the Number of Combinations of any Given Number of Things, by taking any Given Number at a Time; in which there are ſeveral Things of one Sort, ſeveral of another, &c.

RULE.

FIND, by trial, the number of different forms which the things to be taken at a time will admit of, and the number of combinations there are in each.

Add all the combinations, thus found, together, and the ſum will be the number required.

EXAMPLES.

1. Let the things propoſed be $aaabbc$; it is required to find the number of combinations made of every 3 of theſe quantities?

Forms.	Combinations.
a^3	1
a^2b, a^2c, b^2a, b^2c	4
abc	1
Number of combinations required =	6

2. Let $aaabbbcc$ be propoſed; it is required to find the number of combinations of theſe quantities, taken 4 at a time? Anſ. 10.

3. How many combinations are there in $aaaabbccde$, taking 8 at a time? Anſ. 13.

4. How many combinations are there in *aaaaabbbbb ccccddddeeeefffg*, taking 10 at a time? Anſ. 2819.

PROBLEM VII.

To find the Compoſitions of any Number, in an equal Number of Sets, the Things themſelves being all different.

RULE*.

MULTIPLY the number of things in every ſet continually together, and the product will be the anſwer required.

EXAMPLES.

1. Suppoſe there are four companies, in each of which there are 9 men; it is required to find how many ways 9 men may be choſen, one out of each company?

$$\begin{array}{r} 9 \\ \times\ 9 \\ \hline 81 \\ \times\ 9 \\ \hline 729 \\ \times\ 9 \\ \hline 6561 \end{array}$$ the Anſwer.

Or, $9 \times 9 \times 9 \times 9 = 6561$ the Anſwer.

Ex. 2.

---

* *Demonſtr.* Suppoſe there are only two ſets; then, it is plain, that every quantity of the one ſet being combined with every quantity of the other, will make all the compoſitions of two things in theſe two ſets; and the number of theſe compoſitions is evidently the product of the number of quantities in one ſet by that in the other.

Again, ſuppoſe there are three ſets; then the compoſition of two, in any two of the ſets, being combined with every quantity of the third, will make all the compoſitions of three in the three ſets. That is, the compoſitions of two, in any two of the ſets, being multiplied by the number of quantities in the remaining ſet, will produce the compoſitions of three in the three ſets; which is evidently the continual product of all the three numbers in the three ſets.

And the ſame manner of reaſoning will hold, let the number of ſets be what it will. Q. E. D.

The

Ex. 2. Suppoſe there are 4 companies; in one of which there are 6 men, in another 8, and in each of the other two 9; what are the choices, by a compoſition of 4 men, one out of each company? Anſ. 3888.

3. How many changes are there in throwing 5 dice? Anſ. 7776.

PRACTICAL QUESTIONS IN ARITHMETIC.

Quest. 1. The ſwifteſt velocity of a cannon-ball, is about 2000 feet in a ſecond of time. Then in what time, at that rate, would ſuch a ball be in moving from the earth to the ſun; admitting the diſtance to be 100 millions of miles, and the year to contain 365 days 6 hours?

Anſ. $8\frac{4}{13}\frac{8}{14}\frac{0}{49}$ years.

Quest. 2. What is the ratio of the velocity of light to that of a cannon-ball, which iſſues from the gun with a velocity of 1500 feet per ſecond; light paſſing from the ſun to the earth in $7\frac{1}{2}$ minutes? Anſ. the ratio of $782222\frac{2}{9}$ to 1.

Quest. 3. The ſlow or parade-ſtep being 70 paces per minute, at 28 inches each pace, it is required to determine at what rate per hour that movement is? Anſ. $1\frac{1}{4}\frac{3}{2}$ miles.

Quest. 4. The quick-time or ſtep, in marching, being 2 paces per ſecond, or 120 per minute, at 28 inches each; then at what rate per hour does a troop march on a route, and how long will they be in arriving at a garriſon 20 miles diſtant, allowing a halt of one hour by the way to refreſh?

Anſ. { the rate is $3\frac{2}{11}$ miles an hour, and the time $7\frac{2}{7}$hr, or 7h $17\frac{1}{7}$min.

Quest. 5. Two perſons, A and B, being on oppoſite ſides of a wood, which is 268 yards about, they begin to go round it, both the ſame way, at the ſame inſtant of time; A

The doctrine of permutations, combinations, &c, is of very extenſive uſe in different parts of the Mathematics; particularly in the calculation of annuities and chances. The ſubject might have been purſued to a much greater length; but what is here done will be found ſufficient for moſt of the purpoſes to which things of this nature are applicable.

goes

goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes: the queſtion is, how many times will the wood be gone round before the quicker overtake the ſlower?

Anſ. 17 times.

QUEST. 6. To determine how far 500 millions of guineas will reach, when laid down in a ſtraight line touching one another; ſuppoſing each guinea to be an inch in diameter, as it is very nearly. Anſ. 7891 miles, 728 yds, 2 ft 8 in.

QUEST. 7. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall. It is required then to determine how many men muſt be added to the former, that the whole number of them may juſt finiſh the wall in the time propoſed, at the ſame rate of working. Anſ. 4 men to be added.

QUEST. 8. The hour and minute-hand of a clock are exactly together at 12 o'clock; when are they next together?

Anſ. At $1\frac{1}{11}$ hr, or 1 hr $5\frac{5}{11}$ min.

QUEST. 9. A perſon after ſpending 10l more than $\frac{1}{3}$ of his yearly income, had then remaining 15l more than the half of it; what was his income? Anſ. 150l.

QUEST. 10. A perſon who was poſſeſſed of a $\frac{3}{5}$ ſhare of a copper mine, ſold $\frac{3}{4}$ of his intereſt in it for 1710l: what was the reputed value of the whole at the ſame rate?

Anſ. 3800l.

QUEST. 11. A can do a piece of work alone in 10 days, and B alone in 13; in what time will they both together perform a like quantity of work? Anſ. $5\frac{15}{23}$ days.

QUEST. 12. A perſon bought 120 oranges at 2 a penny, and 120 more at 3 a penny; after which, ſelling them out again at 5 for 2 pence, whether did he gain or loſe by the bargain? Anſ. he loſt 4 pence.

QUEST. 13. If a gentleman whoſe annual income is 1200l, ſpend 20 guineas a week; whether will he ſave or run in debt, and how much in the year? Anſ. ſave 108l.

QUEST. 14. In the latitude of London, the diſtance round the earth, meaſured on the parallel of latitude, is about 15550 miles; now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from weſt to eaſt?

Anſ. $649\frac{2}{3}\frac{6}{5}\frac{9}{9}$ miles an hour.

QUEST. 15. If a quantity of proviſions ſerves 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how many

many men will the ſame proviſions maintain for 20 weeks, at the rate of 8 ounces a day for each man? Anſ. 2250 men.

QUEST. 16. If 1000 men beſieged in a town, with proviſions for 5 weeks, allowing each man 16 ounces a day, be reinforced with 500 men more; and ſuppoſing that they cannot be relieved till the end of 8 weeks, how many ounces a day muſt each man have, that the proviſion may laſt that time? Anſ. $6\frac{4}{5}$ ounces.

QUEST. 17. A father left his ſon a fortune, $\frac{1}{4}$ of which he ran through in 8 months; $\frac{3}{7}$ of the remainder laſted him 12 months longer; after which he had bare 410l left. What ſum did the father bequeath the ſon?

Anſ. 956l 13s, 4d.

QUEST. 18. A perſon, looking on his watch, was aſked what was the time of the day, who anſwered, It is between 4 and 5; but a more particular anſwer being required, he ſaid that the hour and minute hands were then exactly together: What was the time? Anſ. $21\frac{9}{11}$ min paſt 4.

QUEST. 19. If 20 men can perform a piece of work in 12 days, how many men will accompliſh another thrice as large in one-fifth of the time? Anſ. 300.

QUEST. 20. A younger brother received 6300l, which was juſt $\frac{7}{9}$ of his elder brother's fortune: What was the father worth at his death? Anſ. 14400l.

QUEST. 21. A perſon, making his will, gave to one child $\frac{13}{20}$ of his eſtate, and the reſt to another. When theſe legacies came to be paid, the one turned out 600l more than the other: What did the teſtator die worth? Anſ. 2000l.

QUEST. 22. A father deviſed $\frac{7}{18}$ of his eſtate to one of his ſons, and $\frac{7}{18}$ of the reſidue to another, and the ſurplus to his relict for life. The children's legacies were found to be 257l 3s 4d different: Pray what money did he leave the widow the uſe of? Anſ. 635l 0s $10\frac{8}{9}$d.

QUEST. 23. Two perſons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermiſſion; and B ſets out from London at 4 o'clock the ſame evening, and walks for Exeter at the rate of 4 miles an hour conſtantly. Now, ſuppoſing the diſtance between the two cities to be 130 miles, whereabouts on the road will they meet? Anſ. $69\frac{3}{7}$ miles from Exeter.

QUEST.

QUEST. 24. Two perſons, A and B, travel between London and Lincoln, diſtant 100 miles, A from London, and B from Lincoln, at the ſame inſtant. After 7 hours they meet on the road, when it appeared that A had rode $1\frac{1}{2}$ miles an hour more than B. At what rate per hour then did each of the travellers ride?

Anſ. A $7\frac{2}{2}\frac{5}{8}$, and B $6\frac{1}{2}\frac{1}{8}$ miles.

QUEST. 25. The clocks of Italy go on to 24 hours: Then how many ſtrokes do they ſtrike in one complete revolution of the index? Anſ. 300.

QUEST. 26. One hundred eggs being placed on the ground, in a ſtraight line, at the diſtance of a yard from each other: How far will a perſon travel who ſhall bring them one by one to a baſket, which is placed at one yard from the firſt egg? Anſ. 10100 yds, or 5 miles and 1300 yds.

QUEST. 27. One Seſſa, an Indian, having invented the game of cheſs, ſhewed it to his prince, who was ſo delighted with it, that he promiſed him any reward he ſhould aſk; on which Seſſa requeſted that he might be allowed one grain of wheat for the firſt ſquare on the cheſs board, 2 for the ſecond, 4 for the third, and ſo on, doubling continually, to 64, the whole number of ſquares. Now, ſuppoſing a pint to contain 7680 of theſe grains, and one quarter or 8 buſhels to be worth 27s 6d, it is required to compute the value of all the corn? Anſ. 6448148829l.

QUEST. 28. Two young gentlemen, without private fortune, obtain commiſſions at the ſame time, and at the age of 18. One thoughtleſsly ſpends 10l a year more than his pay; but, ſhocked at the idea of not paying his debts, gives his creditor a bond for the money, at the end of every year, and alſo inſures his life for the amount; each bond coſts him 30 ſhillings, beſides the lawful intereſt of 5 per cent, and to inſure his life coſts him 6 per cent.

The other, having a proper pride, is determined never to run in debt; and, that he may aſſiſt a friend in need, perſeveres in ſaving 10l every year, for which he obtains an intereſt of 5 per cent, which intereſt is every year added to his ſavings, and laid out, ſo as to anſwer the effect of compound intereſt.

Suppoſe theſe two officers to meet at the age of 50, when each receives from Government 400l. per annum; that the one, ſeeing his paſt errors, is reſolved in future to ſpend no more

more than he actually has, after paying the intereſt for what he owes, and the inſurance on his life.

The other, having now ſomething before hand, means in future to ſpend his full income, without increaſing his ſtock.

It is deſirable to know how much each has to ſpend per annum, and what money the latter has by him to aſſiſt the diſtreſſed, or leave to thoſe who deſerve it?

END OF ARITHMETIC,

OR OF

PART I. VOL. I.

LOGARITHMS

AND

ALGEBRA:

BEING

PART II. OF VOL. I.

CONTENTS OF PART II.

VOL. I.

LOGARITHMS.

	Page
Of Logarithms	145
To compute Logarithms	149
Deſcription and Uſe of Logarithms	153
Multiplication by Logarithms	157
Diviſion by Logarithms	158
Involution by Logarithms	159
Evolution by Logarithms	160

ALGEBRA.

Definitions and Notation	161
Addition	165
Subtraction	170
Multiplication	171
Diviſion	174
Fractions	178
Involution	189
Evolution	192
Surds	196
Infinite Series	203
Arithmetical Proportion	208
Arithmetical Progreſſion	210
Piles of Shot or Shells	213

Geo-

CONTENTS.

	Page
Geometrical Proportion - - - - - -	218
Simple Equations - - - - - - -	220
Quadratic Equations - - - - - - -	238
Cubic and Higher Powers - - - - - -	247
Simple Intereſt - - - - - - -	254
Compound Intereſt - - - - - - -	255
Annuities - - - - - - - -	258

LOGA-

LOGARITHMS

AND

ALGEBRA.

OF LOGARITHMS*.

LOGARITHMS are numbers ſo contrived, and adapted to other numbers, that the ſums and differences of the former ſhall correſpond to, and ſhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a ſeries of numbers in arithmetical progreſſion, anſwering to another ſeries of numbers in geometrical progreſſion.

Thus

* The invention of Logarithms is the undoubted right of Lord Napier, Baron of Merchiſton, in Scotland, and is properly conſidered as one of the moſt uſeful and excellent diſcoveries of modern times. A table of theſe numbers was firſt publiſhed by him at Edinburgh, in the year 1614, in a treatiſe entitled *Canon Mirificum Logarithmorum*; and as their great utility and extenſive application were ſufficiently apparent, they were immediately received by all the Learned throughout Europe. Mr. Henry Briggs, then profeſſor of geometry at Greſham College, on hearing of the diſcovery, ſet out upon a viſit to the noble inventor, and ſoon afterwards they jointly undertook the arduous taſk of computing new tables on this ſubject, and reducing them to a more convenient form than that which was at firſt thought of. But Lord Napier dying before they were finiſhed, the whole burden fell upon Mr. Briggs, who, with prodigious labour and great ſkill, made an entire Canon, according to the new form, for all numbers from 1 to 20000, and from 9000 to 10100, to 14 places of figures, and publiſhed it at London in the year 1624, in a treatiſe entitled Arithmetica Logarithmica, with directions for ſupplying the intermediate chiliads.

Thus { 0, 1, 2, 3, 4, 5, 6, Indices, or logarithms.
1, 2, 4, 8, 16, 32, 64, Geometric progreſſion.

Or { 0, 1, 2, 3, 4, 5, 6, Indices, or logarithms
1, 3, 9, 27, 81, 243, 729, Geometric progreſſion

Or { 0, 1, 2, 3, 4, 5, Indices, or logs.
1, 10, 100, 1000, 10000, 100000, Geom. progreſ.

Where it is evident, that the ſame indices ſerve equally for any geometric ſeries; and conſequently there may be an endleſs variety of ſyſtems of logarithms, to the ſame common numbers, by only changing the ſecond term, 2, 3, or 10, &c, of the geometrical ſeries of whole numbers; and

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This Canon was again publiſhed in Holland by Adrian Vlacq, anno 1628, together with the Logarithms of all the numbers which Mr. Briggs had omitted; but he contracted them down to 10 places of decimals. Mr. Briggs alſo computed the Logarithms of the ſines, tangents and ſecants, to every degree, and centeſm, or 100th part of a degree, of the whole quadrant; and ſubjoined them to the natural ſines, tangents, and ſecants, which he had before computed to 15 places of figures. Theſe Tables, together with their conſtruction and uſe, were firſt publiſhed in the year 1633, after Mr. Briggs's death, by Mr. Henry Gellibrand, under the title of Trigonometria Britannica.

Benjamin Urſinus alſo gave a Table of Logarithms to every 10 ſeconds. And Chr. Wolf, in his Mathematical Lexicon, ſays that one Van Loſer had computed them to every ſingle ſecond, but his untimely death prevented their publication.

A great number of other authors have treated on this ſubject; but as their numbers are frequently inaccurate and incommodiouſly diſpoſed, they are now generally neglected. The Tables in moſt repute at preſent, are thoſe of Gardiner in 4to, firſt publiſhed in the year 1742; my own Tables in 8vo, firſt printed in the year 1785, where the Logarithms of all numbers may be eaſily found from 1 to 1000000; and thoſe of the ſines, tangents, and ſecants, to any degree of accuracy required.

Alſo, Mr. Michael Taylor's Tables in large 4to, containing the common logarithms, and the logarithmic ſines and tangents to every ſecond of the quadrant. And, in France, the new book of logarithms by Callet; the 2d edition of which, in 1795, has the tables ſtill farther extended, and are printed with what they call ſtereotypes, the types in each page being ſoldered together into a ſolid maſs or block.

Dodſon's Antilogarithmic Canon is likewiſe a very ingenious work, and of great uſe for finding the numbers anſwering to any given logarithm.

by

by interpolation the whole ſyſtem of numbers may be made to enter the geometric ſeries, and receive their proportional logarithms, whether integers or decimals.

It is alſo apparent, from the nature of theſe ſeries, that if any two indices be added together, their ſum will be the index of that number which is equal to the product of the two terms, in the geometric progreſſion, to which thoſe indices belong.

Thus, the indices 2 and 3, being added together, make 5; and the numbers 4 and 8, or the terms correſponding to thoſe indices, being multiplied together, make 32, which is the number anſwering to the index 5.

In like manner, if any one index be ſubtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which thoſe indices belong.

Thus, the index 6, minus the index 4, is $= 2$; and the terms correſponding to thoſe indices are 64 and 16, whoſe quotient is $= 4$; which is the number anſwering to the index 2.

For the ſame reaſon, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus, the index or logarithm of 4, in the above ſeries, is 2; and if this number be multiplied by 3, the product will be $= 6$; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus, the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be $= 3$; which is the logarithm of 8, or the ſquare root of 64.

The logarithms moſt convenient for practice, are ſuch as are adapted to a geometric ſeries increaſing in a tenfold proportion, as in the laſt of the above forms; and are thoſe which are to be found, at preſent, in moſt of the common tables on this ſubject.

The diſtinguiſhing mark of this ſyſtem of logarithms is, that the index or logarithm of 10 is 1; that of 100 is 2; that of 1000 is 3, &c. And, in decimals, the logarithm of ·1 is —1; that of ·01 is —2; that of ·001 is —3; &c. The log. of 1 being 0 in every ſyſtem.

From whence it follows, that the logarithm of any number between 1 and 10, muſt be 0 and ſome fractional parts; and that of a number between 10 and 100, will be 1 and ſome fractional parts; and ſo on, for any other number whatever.

And ſince the integral part of a logarithm is always thus readily found, it is uſually called the Index, or Characteriſtic; and is commonly omitted in the tables; being left to be ſupplied by the operator himſelf, as occaſion requires.

Another definition of Logarithms is, that the logarithm of any number is the index of that power of ſome other number, which is equal to the given number. So if there be $N = r^n$, then n is the log. of N; where n may be either poſitive or negative, or nothing, and the root r any number whatever, according to the different ſyſtems of logarithms.

When n is $= 0$, then N is $= 1$, whatever the value of r is; which ſhews, that the log. of 1 is always 0, in every ſyſtem of logarithms.

When n is $= 1$, then N is $= r$; ſo that the radix r is always that number whoſe log. is 1, in every ſyſtem.

When the radix r is $= 2{\cdot}718281828459$ &c, the indices n are the hyperbolic or Napier's log. of the numbers N; ſo that n is always the hyp. log. of the number N or $\overline{2{\cdot}718, \&c.}^n$.

But when the radix r is $= 10$, then the index n becomes the common or Briggs's log. of the number N; ſo that the common log. of any number 10^n or N, is n the index of that power of 10 which is equal to the ſaid number. Thus, 100, being the ſecond power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm; hence alſo, if 50 be $= 10^{1{\cdot}69897}$, then is 1·69897 the common log. of 50. And, in general, the following decuple ſeries of terms,

viz. 10^4, 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4},
or 10000, 1000, 100, 10, 1, ·1, ·01, ·001, ·0001,
have 4, 3, 2, 1, 0, —1, —2, —3, —4,

for their logarithms, reſpectively. And from this ſcale of numbers and logarithms, the ſame properties eaſily follow, as before mentioned.

PRO-

PROBLEM.

To compute the Logarithm to any of the Natural Numbers 1, 2, 3, 4, 5, *&c.*

RULE I*.

1. TAKE the geometrical ſeries, 1, 10, 100, 1000, 10000, &c, and apply to it the arithmetical ſeries 1, 2, 3, 4, &c, as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the ſeries, betwixt which the number propoſed lies.

3. Between the mean, thus found, and the neareſt extreme, find another geometrical mean, in the ſame manner; and ſo on, till you are arrived within the propoſed limit of the number whoſe logarithm is ſought.

4. Find as many arithmetical means, in the ſame order as you found the geometrical ones, and the laſt of theſe will be the logarithm anſwering to the number required.

EXAMPLE.

Let it be required to find the logarithm of 9.

Here the propoſed number lies between 1 and 10.

Firſt, then, the log. of 10 is 1, and the log. of 1 is 0;
theref. $\overline{1+0} \div 2 = \frac{1}{2} = .5$ is the arithmetical mean,
and $\sqrt{10 \times 1} = \sqrt{10} = 3.1622777$ the geom. mean:
hence the log. of 3·1622777 is ·5.

Secondly, the log. of 10 is 1, and the log. of 3·1622777 is ·5;
theref. $\overline{1+.5} \div 2 = .75$ is the arithmetical mean,
and $\sqrt{10 \times 3.1622777} = 5.6234132$ is the geom. mean;
hence the log. of 5·6234132 is ·75.

Thirdly, the log. of 10 is 1, and the log. of 5·6234132 is ·75;
theref. $\overline{1+.75} \div 2 = .875$ is the arithmetical mean,
and $\sqrt{10 \times 5.6234132} = 7.4989422$ the geom. mean;
hence the log. of 7·4989422 is ·875.

Fourthly, the log. of 10 is 1, and the log. of 7·4989422 is ·875;
theref. $\overline{1+.875} \div 2 = .9375$ is the arithmetical mean,
and $\sqrt{10 \times 7.4989421} = 8.6596431$ the geom. mean;
hence the log. of 8·6596431 is ·9375.

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* The reader who wiſhes to inform himſelf more particularly concerning the hiſtory, nature, and conſtruction of Logarithms, may conſult the introduction to my Mathematical Tables, lately publiſhed, where he will find his curioſity amply gratified.

Fifthly,

Fifthly, the log. of 10 is 1, and the log. of 8·6596431 is ·9375; therefore $1 + \cdot 9375 \div 2 = \cdot 96875$ is the arithmetical mean, and $\sqrt{10 \times 8 \cdot 6596431} = 9 \cdot 3057204$ the geom. mean; hence the log. of 9·3057204 is ·96875.

Sixthly, the log. of 8·6596431 is ·9375, and the log. of 9·3057204 is ·96875;

therefore $\cdot 9375 + \cdot 96875 \div 2 = \cdot 953125$ is the arith. mean, and $\sqrt{8 \cdot 6596431 \times 9 \cdot 3057204} = 8 \cdot 9768713$ the geometric mean;

hence the log. of 8·9768713 is ·953125.

And proceeding in this manner, after 25 extractions, it will be found that the logarithm of 8·9999998 is ·9542425; which may be taken for the logarithm of 9, as it differs so little from it, that it is sufficiently exact for all practical purposes.

And in this manner were the logarithms of almost all the prime numbers at first found.

RULE II*.

Let b be the number whose logarithm is required to be found; and a the number next less than b, so that $b - a = 1$, the logarithm of a being known; and let s denote the sum of the two numbers $a + b$. Then

1. Divide the constant decimal ·8685889638 &c, by s, and reserve the quotient: divide the reserved quotient by the square of s, and reserve this quotient; divide this last quotient also by the square of s, and again reserve the quotient; and thus proceed, continually dividing the last quotient by the square of s, as long as division can be made.

2. Then write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers 1, 3, 5, 7, 9, &c, as long as division can be made; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, and so on.

3. Add all these last quotients together, and the sum will be the logarithm of $b \div a$; therefore to this logarithm add also the given logarithm of the said next less number a, so will the last sum be the logarithm of the number b proposed.

* For the demonstration of this rule, see the Mathematical Tables, 128, &c.

That

That is,

Log. of b is log. $a + \frac{n}{s} \times : 1 + \frac{1}{3s^2} + \frac{1}{5s^4} + \frac{1}{7s^6}$ + &c.

where n denotes the constant given decimal ·868588963 8 &c.

EXAMPLES.

Ex. 1. Let it be required to find the log. of the number 2. Here the given number b is 2, and the next less number a is 1, whose log. is 0; also the sum $2 + 1 = 3 = s$, and its square $s^2 = 9$. Then the operation will be as follows:

3)	·868588964	1)	·289529654 (	·289529654
9)	·289529654	3)	32169962 (	10723321
9)	32169962	5)	3574440 (	714888
9)	3574440	7)	397160 (	56737
9)	397160	9)	44129 (	4903
9)	44129	11)	4903 (	446
9)	4903	13)	545 (	42
9)	545	15)	61 (	4
9)	61			

log. of $\frac{2}{1}$ - ·301029995
add log. 1 - 000000000

log. of 2 - ·301029995

Ex. 2. To compute the logarithm of the number 3.
Here $b = 3$, the next less number $a = 2$, and the sum $a + b = 5 = s$, whose square s^2 is 25, to divide by which, always multiply by ·04. Then the operation is as follows:

5)	·868588964	1)	·173717793 (	·173717793
25)	173717793	3)	6948712 (	2316237
25)	6948712	5)	277948 (	55590
25)	277948	7)	11118 (	1588
25)	11118	9)	445 (	50
25)	445	11)	18 (	2
	18			

log. of $\frac{3}{2}$ - - ·176091260
log. of 2 add - ·301029995

log. of 3 fought ·477121255

Then, because the sum of the logarithms of numbers, gives the logarithm of their product; and the difference of the logarithms, gives the logarithm of the quotient of the numbers;

numbers; from the above two logarithms, and the logarithm of 10, which is 1, we may raiſe a great many logarithms, as in the following examples:

EXAMPLE 3.

Becauſe $2 \times 2 = 4$, therefore

to log. 2 -	·301029995$\frac{2}{3}$
add log. 2	·301029995$\frac{2}{3}$
ſum is log. 4	·602059991$\frac{1}{3}$

EXAMPLE 4.

Becauſe $2 \times 3 = 6$, therefore

to log 2 -	·301029995
add log. 3	477121255
ſum is log. 6	·778151250

EXAMPLE 5.

Becauſe $2^3 = 8$, therefore

log. 2 -	·301029995$\frac{2}{3}$
mult. by 3	3
gives log. 8	·903089987

EXAMPLE 6.

Becauſe $3^2 = 9$, therefore

log. 3 -	·477121254$\frac{7}{10}$
mult. by 2	2
gives log. 9	·954242509

EXAMPLE 7.

Becauſe $\frac{1}{2}^0 = 5$, therefore

from log. 10	1·000000000
take log. 2	·301029995$\frac{2}{3}$
leaves log. 5	·698970004$\frac{1}{3}$

EXAMPLE 8.

Becauſe $3 \times 4 = 12$, therefore

to log. 3 -	·477121255
add log. 4	·602059991
gives log. 12	1·079181246

And thus, computing, by this general rule, the logarithms to the other prime numbers 7, 11, 13, 17, 19, 23, &c, and then uſing compoſition and diviſion, we may eaſily find as many logarithms as we pleaſe, or may ſpeedily examine any logarithm in the table*.

* There are, beſides theſe, many other ingenious methods, which later writers have diſcovered for finding the logarithms of numbers in a much eaſier way than by the original inventor; but, as they cannot be underſtood without a knowledge of ſome of the higher branches of the mathematics, it is thought proper to omit them, and to refer the reader to thoſe works which are written expreſsly on the ſubject.

It would likewiſe much exceed the limits of this compendium, to point out all the peculiar artifices that are made uſe of for conſtructing an entire table of theſe numbers; ſuch as thoſe of Gardiner, Sherwin, and others, who have treated on this ſubject; but any information of this kind, which the learner may wiſh to obtain, may be found in the Tables, before mentioned.

De-

Description and Use of the TABLE *of* LOGARITHMS.

HAVING explained the method of making a table of the logarithms of numbers, greater than unity; the next thing to be done is, to shew how the logarithms of fractional quantities may be found. And, in order to this, it may be observed, that as in the former case a geometric series is supposed to increase towards the left, from unity, so in the latter case it is supposed to decrease towards the right hand, still beginning with unit; as exhibited in the general description, page 148, where the indices being made negative, still shew the logarithms to which they belong. Whence it appears, that as + 1 is the log. of 10, so — 1 is the log. of $\frac{1}{10}$ or ·1; and as + 2 is the log. of 100, so — 2 is the log. of $\frac{1}{100}$ or ·01; and so on.

Hence it appears in general, that all numbers which consist of the same figures, whether they be integral, or fractional, or mixed, will have the decimal parts of their logarithms the same, but differing only in the index, which will be more or less, and positive or negative, according to the place of the first figure of the number.

Thus, the logarithm of 2651 being 3·4234097, the log. of $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c, part of it, will be as follows:

Numbers.	Logarithms.
2 6 5 1	3·4 2 3 4 0 9 7
2 6 5·1	2·4 2 3 4 0 9 7
2 6·5 1	1·4 2 3 4 0 9 7
2·6 5 1	0·4 2 3 4 0 9 7
·2 6 5 1	—1·4 2 3 4 0 9 7
·0 2 6 5 1	—2·4 2 3 4 0 9 7
·0 0 2 6 5 1	—3·4 2 3 4 0 9 7

From this it also appears, that the index, or characteristic, of any logarithm, is always less by 1 than the number of integer figures which the natural number consists of; or it is equal to the distance of the first or left-hand figure, from the place of units, or first place of integers, whether on the left, or on the right of it: and this index is constantly to be placed on the left hand side of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a short line drawn before it, or else above it. Thus,

A number having 1, 2, 3, 4, 5, &c, integer places, the index of its log. is 0, 1, 2, 3, 4, &c, or 1 less than those places.

And

And a decimal fraction having its firſt figure in the
1ſt, 2d, 3d, 4th, &c, place of decimals, has always
— 1, — 2, — 3, — 4, &c, for the index of its logarithm.

It may alſo be obſerved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative.

I. TO FIND, IN THE TABLE, THE LOGARITHM TO ANY NUMBER.

1. *If the number do not exceed* 100000, the decimal part of the logarithm is found, by inſpection in the table, ſtanding againſt the given number, in this manner; viz. in moſt tables, the firſt four figures of the given number in the firſt column of the page, and the fifth figure one of thoſe along the top line of it; then in the angle of meeting are the laſt four figures of the logarithm, and the firſt 3 figures of *the* ſame at the beginning of the ſame line: to which is to be prefixed the proper index, which is always 1 leſs than the number of integer figures.

So the logarithm of 34·092 is 1·5326525, that is, the decimal 5326525 found in the table, with the index 1 prefixed, becauſe the given number contains two integers.

2. *But if the given number contain more than five figures;* take out the logarithm of the firſt five figures by inſpection in the table as before, as alſo the next greater logarithm, ſubtracting the one logarithm from the other, as alſo their correſponding numbers the one from the other. Then ſay,

As the difference between the two numbers,
Is to the difference of their logarithms,
So is the remaining part of the given number,
To the proportional part of the logarithm.

Which part being added to the leſs logarithm, before taken out, gives the whole logarithm ſought, very nearly.

EXAM-

EXAMPLE.

To find the logarithm of the number 34·09264.
The log. of 3409200, as before, is 5326525,
And log. of 3409309 — is 5326652,

The diffs. are 100 and 127

Then, as 100 : 127 :: 64 : 81, the proportional part.
This added to — — 5326525, the firſt log.

Gives, with the index, 1·5326606 for the log. of 34·09264.

Or, in the beſt tables, the proportional part may often be taken out by inſpection, by means of the ſmall tablets of proportional parts placed in the margin.

If the number conſiſt both of integers and fractions, or is entirely fractional; find the decimal part of the logarithm the ſame as if all its figures were integral; then this, being prefixed to the proper characteriſtic, will give the logarithm required.

And if the given number be a proper fraction; ſubtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm ſought; which, being that of a decimal fraction, muſt always have a negative index.

But if it be a mixed number; reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the ſame manner as before.

EXAMPLES.

1. To find the log. of $\frac{37}{97}$.

Log. of 37 — 1·5682017
Log. of 97 — 1·9731279
Dif. log. of $\frac{37}{97}$ — 1·5950738
Where the index 1 is negative.

2. To find the log. of $17\frac{14}{23}$.

Firſt, $17\frac{14}{23} = \frac{405}{23}$. Then,
Log. of 405 — 2·6074550
Log. of 23 — 1·3617278
Dif. log. of $17\frac{14}{23}$ 1·2457272

II. TO FIND THE NATURAL NUMBER TO ANY GIVEN LOGARITHM.

THIS is to be found in the tables by the reverſe method to the former, namely, by ſearching for the propoſed logarithm among thoſe in the table, and taking out the correſponding number by inſpection, in which the proper number of integers are to be pointed out, viz. 1 more than the index. For, in finding the number anſwering to any given logarithm, the index always ſhews how far the firſt figure muſt

muſt be removed from the place of units, viz. to the left hand, or integers, when the index is affirmative; but to the right hand, or decimals, when it is negative.

EXAMPLES.

So, the number to the log. 1·5326525 is 34·092.

And the number of the log. $\bar{1}$·5326525 is ·34092.

But if the logarithm cannot be exactly found in the table; take out the next greater and the next leſs, ſubtracting the one of theſe logarithms from the other, as alſo their natural numbers the one from the other, and the leſs logarithm from the logarithm propoſed. Then ſay,

As the difference of the firſt or tabular logarithms,
Is to the difference of their natural numbers,
So is the differ. of the given log. and the leaſt tabular log.
To their correſponding numeral difference.

Which being annexed to the leaſt natural number above taken, gives the natural number ſought, correſponding to the propoſed logarithm.

EXAMPLE.

So, to find the natural number anſwering to the given logarithm 1·5326606.

Here the next greater and next leſs tabular logarithms, with their correſponding numbers, &c, are as below:

Next greater	5326652	its num. 3409300;	given log. 5326606
Next leſs	5326525	its num. 3409200;	next leſs 5326525
Differences	127	100	81

Then, as 127 : 100 :: 81 : 64 nearly, the numeral differ.

Therefore 34·09264 is the number ſought, marking off two integers, becauſe the index of the given logarithm is 1.

Had the index been negative, thus $\bar{1}$·5326606, its correſponding number would have been ·3409264, wholly decimal.

Or the proportional numeral difference may be found, in the beſt tables, by inſpection, by means of the ſmall tablets of proportional parts placed in the margins of the pages.

III. MUTLIPLICATION BY LOGARITHMS.

RULE.

TAKE out the logarithms of the factors from the table, then add them together, and their ſum will be the logarithm of the product required. Then, by means of the table, take out the natural number anſwering to the ſum, for the product ſought.

Obſerving to add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or elſe ſubtracted from the negative.

Alſo, adding the indices together when they are of the ſame kind, both affirmative or both negative; but ſubtracting the leſs from the greater, when the one is affirmative and the other negative, and prefixing the ſign of the greater to the remainder.

EXAMPLES.

1. To multiply 23·14 by 5·062.

Numbers.	Logs.
23·14 -	1·3643634
5·062 -	0·7043221
Product 117·1347	2·0686855

2. To multiply 2·581926 by 3·457291.

Numbers.	Logs.
2·581926 -	0·4119438
3·457291 -	0·5387359
Prod. 8·92647 -	0·9506797

3. To mult. 3·902, and 597·16, and ·0314728 all together.

Numbers.	Logs.
3·902 -	0·5912873
597·16 -	2·7760907
·0314728 —	2·4979353
Prod. 73·33533 -	1·8653133

Here the —2 cancels the 2, and the 1 to carry from the decimals is ſet down.

4. To mult. 3·586, and 2·1046, and 0·8372, and 0·0294 all together.

Numbers.	Logs.
3·586 -	0·5546103
2·1046 -	0·3231696
0·8372 —	1·9228292
0·0294 —	2·4683473
Prod. 0·1857618 —	1·2689564

Here the 2 to carry cancels the —2, and there remains the —1 to ſet down.

Division by Logarithms.

Rule.

From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the number anſwering to the remainder will be the quotient required.

Obſerving to change the ſign of the index of the diviſor, from affirmative to negative, or from negative to affirmative; then take the ſum of the indices if they be of the ſame name, or their difference when of different ſigns, with the ſign of the greater, for the index to the logarithm of the quotient.

And alſo, when 1 is borrowed, in the left-hand place of the decimal part of the logarithm, add it to the index of the diviſor when that index is affimative, but ſubtract it when negative; then let the index ariſing from thence be changed, and worked with as before.

Examples.

1. To divide 24163 by 4567.

	Numbers.	Logs.
Dividend	24163	4·3831509
Diviſor	4567	3·6596310
Quot.	5·290782	0·7235199

2. To divide 37·149 by 523·76.

	Numbers.	Logs.
Dividend	37·149	1·5699471
Diviſor	523·76	2·7191323
Quot.	·07092752	—2·8508148

3. Divide ·06314 by ·007241.

	Numbers.	Logs.
Divid.	·06314	—2·8003046
Diviſor	·007241	—3·8597985
Quot.	8·719792	0·9405061

Here 1 carried from the decimals to the —3, makes it become —2, which taken from the other —2, leaves 0 remaining.

4. To divide ·7438 by 12·9476.

	Numbers.	Logs.
Divid.	·7438	—1·8714562
Diviſor	12·9476	1·1121893
Quot.	·05744694	—2·7592669

Here the 1 taken from the —1, makes it become —2, to ſet down.

Note. As to the Rule-of-Three, or Rule of Proportion, it is performed by adding the logarithms of the 2d and 3d terms, and ſubtracting that of the firſt term from their ſum.

INVOLUTION BY LOGARITHMS.

RULE.

TAKE out the logarithm of the given number from the table.

Multiply the log. thus found, by the index of the power propoſed.

Find the number anſwering to the product, and it will be the power required.

Note. In multiplying a logarithm with a negative index, by an affirmative number, the product will be negative.

But what is to be carried from the decimal part of the logarithm, will always be affirmative.

And therefore their difference will be the index of the product, and is always to be made of the ſame kind with the greater.

EXAMPLES.

1. To ſquare the number 2·5791.

Numb.	Log.
Root 2·5791 —	0.4114682
The index — —	2
Power 6·651756	0·8229364

2. To find the cube of 3·07146.

Numb.	Log.
Root 3·07146 —	0·4873449
The index — —	3
Power 28·97575	1·4620347

3. To raiſe ·09163 to the 4th power.

Numb.	Log.
Root ·09163 —	—2·9620377
The index —	4
Po. ·0000704938 —	—5·8481508

Here, 4 times the negative index being — 8, and 3 to carry, the difference — 5 is the index of the product.

4. To raiſe 1·0045 to the 365th power.

Numb.	Log.
Root 1·0045 —	0·0019499
The index —	365
	97495
	116994
	58497
Power 5·148888	·7117135

Evolution by Logarithms.

Rule.

Take the log. of the given number out of the table.

Divide the log. thus found, by the index of the root propoſed.

Find the number anſwering to the quotient, and it will be the root required.

Note. When the index of the logarithm, to be divided, is negative, and does not exactly contain the diviſor, without ſome remainder, increaſe the index by ſuch a number as will make it exactly diviſible by the index, carrying the units borrowed, as ſo many tens, to the left-hand place of the decimal, and then divide as in whole numbers.

Examples.

1. To find the ſquare root of 365.

	Numb.		Log.
Power	365	2)	2·5622929
Root	19·10498		1·2811465

2. To find the 3d root of 12345.

	Numb.		Log.
Power	12345	3)	4·0914911
Root	23·11162		1·3638304

3. To find the 10th root of 2.

	Numb.		Log.
Power	2 —	10)	0·3010300
Root	1·071773		0·0301030

4. To find the 365th root of 1·045.

	Numb.		Log.
Power	1·045	365)	0·0191163
Root	1·000121		0·0000524

5. To find the 2d root of ·093.

	Numb.		Log.
Power	·093	2)	—2·9684829
Root	·304959		—1·4842415

Here the diviſor 2 is contained exactly once in the negative index — 2, and therefore the index of the quotient is — 1.

6. To find the 3d root of ·00048.

	Numb.		Log.
Power	·00048	3)	—4·6812412
Ro.	·07829735		—2 8937471

Here the diviſor 3 not being exactly contained in —4, it is augmented by 2, to make up 6, in which the diviſor is contained juſt 2 times; then the 2, thus borrowed, being carried to the decimal figure 6, makes 26, which divided by 3, gives 8, &c.

ALGE-

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is the art of computing by ſymbols. It is ſometimes alſo called Analyſis; and is a general kind of arithmetic, or univerſal way of computation.

2. In this ſcience, quantities of all kinds are repreſented by the letters of the alphabet. And the operations to be performed with them, as addition or ſubtraction, &c, are denoted by certain ſimple characters, inſtead of being expreſſed by words at length.

3. In algebraical queſtions, ſome quantities are known or given, viz, thoſe whoſe values are known; and others unknown, or are to be found out, viz, thoſe whoſe values are not known. The former of theſe are repreſented by the leading letters of the alphabet, *a*, *b*, *c*, *d*, &c; and the latter, or unknown quantities, by the final letters, *z*, *y*, *x*, *u*, &c.

4. The characters uſed to denote the operations, are chiefly the following:

$+$ ſignifies addition, and is named *plus*.
$-$ ſignifies ſubtraction, and is named *minus*.
$\times$ ſignifies multiplication, and is named *into*.
$\div$ ſignifies diviſion, and is named *by*.
$\surd$ ſignifies the ſquare root; $\sqrt[3]{}$ the cube root; $\sqrt[4]{}$ the 4th root, &c; and $\sqrt[n]{}$ the *n*th root.
$:\ ::\ :$ ſignifies proportion.
$=$ ſignifies equality, and is named *equal to*.
And ſo on for other operations.

Thus $a + b$ ſhews, that the number repreſented by b is to be added to that repreſented by a.

$a - b$ ſhews, that the number repreſented by b is to be ſubtracted from that repreſented by a.

$a \sim b$ repreſents the difference of a and b, when it is not known which is the greater.

ab, or $a \times b$, or $a.b$, denotes the product, by multiplication, of the numbers repreſented by a and b.

$a \div b$, or $\frac{a}{b}$, ſhews that the number repreſented by a is to be divided by that which is repreſented by b.

$\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$, and $\frac{1}{a}$ the reciprocal of a.

$a : b :: c : d$ denotes that a is in the ſame proportion to b, as c is to d.

$x = a - b + c$ is an equation, ſhewing that x is equal to the difference of a and b, added to the quantity c.

$\sqrt{a}$, or $a^{\frac{1}{2}}$, is the ſquare root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, is the cube root of a; and $\sqrt[m]{a}$, or $a^{\frac{1}{m}}$, is the mth root of a.

a^2 is the ſquare of a; a^3 the cube of a; a^4 the fourth power of a; and a^m the mth power of a.

$\overline{a + b} \times c$, or $(a + b)\, c$, is the product of the compound quantity $a + b$ multiplied by the ſimple quantity c. Uſing the bar ——, or the parentheſis ( ) as a vinculum, to connect ſeveral quantities into one.

$\overline{a + b} \div \overline{a - b}$, or $\frac{a + b}{a - b}$, expreſſed like a fraction, is the quotient of $a + b$ divided by $a - b$.

$\sqrt{\overline{ab + cd}}$, or $(ab + cd)^{\frac{1}{2}}$, is the ſquare root of the compound quantity $ab + cd$. And $c\sqrt{\overline{ab + cd}}$ denotes the product of c into the ſquare root of the compound quantity $ab + cd$.

$\overline{a + b - c}^3$, or $(a + b - c)^3$, is the cube, or third power, of the quantity $a + b - c$.

$5a$ denotes that the quantity a is to be taken 5 times, and $7.(b + c)$ is 7 times $b + c$. And theſe numbers, 5 or 7, ſhewing how often the quantities are to be taken, or multiplied, are called Co-efficients.

5. Like quantities, are thoſe which conſiſt of the ſame letters, and powers. As a and $3a$; or $2ab$ and $4ab$; or $3a^2bc$ and $-5a^2bc$.

6. Unlike Quantities, are thoſe which conſiſt of different letters, or different powers. As a and b; or $2a$ and a^2; or $3ab^2$ and $3abc$.

7. Simple Quantities, are thoſe which conſiſt of one term only. As $3a$, or $5ab$, or $6abc^2$

8. Com-

8. Compound Quantities, are those which consist of two or more terms. As $a + b$, or $2a - 3c$, or $a + 2b - 3c$.

9. And when the compound quantity consists of two terms, it is called a Binomial, as $a + b$; when of three terms, it is a Trinomial, as $a + 2b - 3c$; when of four terms, a Quadrinomial, as $2a - 3b + c - 4d$; and so on. Also, a Multinomial or Polynomial, consists of many terms.

10. A Residual Quantity, is a binomial having one of the terms negative. As $a - 2b$.

11. Positive or Affirmative Quantities, are those which are to be added, or have the sign $+$. As a or $+a$, or ab: for when a quantity is found without a sign, it is understood to be positive, or to have the sign $+$ prefixed.

12. Negative Quantities, are those which are to be subtracted. As $-a$, or $-2ab$, or $-3ab^2$.

13. Like Signs, are either all positive ($+$), or all negative ($-$).

14. Unlike Signs, are when some are positive ($+$), and others negative ($-$).

15. The Co-efficient of any quantity, as shewn above, is the number prefixed to it. As 3, in the quantity $3ab$.

16. The Power of a quantity (a), is its square (a^2), or cube (a^3), or biquadrate (a^4), &c; called also, the 2d power, or 3d power, or 4th power, &c.

17. The Index or Exponent, is the number which denotes the power or root of a quantity. So 2 is the exponent of the square or second power a^2; and 3 is the index of the cube or 3d power; and $\frac{1}{2}$ is the index of the square root, $a^{\frac{1}{2}}$ or $\sqrt{a}$; and $\frac{1}{3}$ is the index of the cube root, $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$.

18. A Rational Quantity, is that which has no radical sign ($\sqrt{}$) or index annexed to it. As a, or $3ab$.

19. An Irrational Quantity, or Surd, is that which has not an exact root, or is expressed by means of the radical sign $\sqrt{}$. As $\sqrt{2}$, or $\sqrt{a}$, or $\sqrt[3]{a^2}$, or $ab^{\frac{1}{2}}$.

20. The Reciprocal of any quantity, is that quantity inverted, or unity divided by it. So, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

21. The letters by which any ſimple quantity is expreſſed, may be ranged according to any order at pleaſure. So the product of a and b, may be either expreſſed by ab, or ba; and the product of a, b, and c, by either abc, or acb, or bac, or bca, or cab, or cba; as it matters not which quantities are placed or multiplied firſt. But it will be ſometimes found convenient, in long operations, to place the ſeveral letters according to their order in the alphabet, as abc, which order alſo occurs moſt eaſily or naturally to the mind.

22. Likewiſe, the ſeveral members, or terms, of which a compound quantity is compoſed, may be diſpoſed in any order at pleaſure, without altering the value of the ſignification of the whole. Thus, $3a - 2ab + 4abc$ may alſo be written $3a + 4abc - 2ab$, or $4abc + 3a - 2ab$, or $-2ab + 3a + 4abc$, &c; for all theſe repreſent the ſame thing, namely, the quantity which remains, when the quantity or term $2ab$ is ſubtracted from the ſum of the terms or quantities $3a$ and $4abc$. But it is moſt uſual, and natural, to begin with a poſitive term, and with the firſt letters of the alphabet.

SOME EXAMPLES FOR PRACTICE,

In finding the numeral values of various expreſſions, or combinations, of quantities.

Suppoſing $a = 6$, and $b = 5$, and $c = 4$, and $d = 1$, and $e = 0$. Then

1. Will $a^2 + 3ab - c^2 = 36 + 90 - 16 = 110$.
2. And $2a^3 - 3a^2b + c^3 = 432 - 540 + 64 = -44$.
3. And $a^2 \times \overline{a + b} - 2abc = 36 \times 11 - 240 = 156$.
4. And $\frac{a^3}{a + 3c} + c^2 = \frac{216}{18} + 16 = 12 + 16 = 28$.
5. And $\sqrt{2ac + c^2}$ or $\overline{2ac + c^2}^{\frac{1}{2}} = \sqrt{64} = 8$.
6. And $\sqrt{c + \frac{2bc}{\sqrt{2ac + c^2}}} = 2 + \frac{40}{8} = 7$.
7. And $\frac{a^2 - \sqrt{b^2 - ac}}{2a - \sqrt{b^2 + ac}} = \frac{36 - 1}{12 - 7} = \frac{35}{5} = 7$.
8. And $\sqrt{b^2 - ac} + \sqrt{2ac + c^2} = 1 + 8 = 9$.
9. And $\sqrt{b^2 - ac + \sqrt{2ac + c^2}} = \sqrt{25 - 24 + 8} = 3$.
10. And $a^2b + c - d =$
11. And $5ab - 10b^2 + e =$

12. And

12. And $\frac{a^2b}{c} \times d =$

13. And $\frac{a+b}{c} \times \frac{b}{d} =$

14. And $\frac{a+b}{c} - \frac{a-b}{d} =$

15. And $\frac{a^2b}{c} + e =$

16. And $\frac{a^2b}{c} \times e =$

17. And $\overline{b-c} \times \overline{d-e} =$

18. And $\overline{a+b} - \overline{c-d} =$

19. And $\overline{a+b-c} - d =$

20. And $a^2c + d^3 =$

21. And $acd - d =$

22. And $a^2e + b^2e + d =$

23. And $\frac{b-e}{d-e} \times \frac{a+b}{c-d} =$

24. And $\sqrt{a^2+b^2} - \sqrt{a^2-b^2} =$

25. And $3ac^2 + \sqrt[3]{a^3-b^3} =$

ADDITION.

ADDITION, in Algebra, is the connecting the quantities together by their proper ſigns, and incorporating or uniting into one term or ſum, ſuch as are ſimilar, and can be united. As $3a + 2b - 2a = a + 2b$, the ſum.

The rule of addition in algebra, may be divided into three caſes; one, when the quantities are like, and their ſigns like alſo; a ſecond, when the quantities are like, but their ſigns unlike; and the third, when the quantities are unlike. Which are performed as follows*.

CASE

* The reaſons on which theſe operations are founded, will readily appear, by a little reflection on the nature of the quantities to be added,

CASE I.

When the Quantities are Like, and have Like Signs:

FIRST ſet down the common ſign; after which ſet the ſum of the co-efficients, found by adding them together; to which annex the common letter or letters of the like quantities.

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added, or collected together. For, with regard to the firſt example, where the quantities are $3a$ and $5a$, whatever a repreſents in the one term, it will repreſent the ſame thing in the other; ſo that 3 times any thing and 5 times the ſame thing, collected together, muſt needs make 8 times that thing. As if a denote a ſhilling; then $3a$ is 3 ſhillings, and $5a$ is 5 ſhillings, and their ſum 8 ſhillings. In like manner, $-2ab$ and $-7ab$, or -2 times any thing, and -7 times the ſame thing, make -9 times that thing.

As to the ſecond caſe, in which the quantities are like, but the ſigns unlike; the reaſon of its operation will eaſily appear, by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations denoted by their ſigns $+$ and $-$, or of addition and ſubtraction; which being of contrary or oppoſite natures, the one co-efficient muſt be ſubtracted from the other, to obtain the incorporated or united maſs.

As to the third caſe, where the quantities are unlike, it is plain that ſuch quantities cannot be united into one, or otherwiſe added, than by means of their ſigns: thus, for example, if a be ſuppoſed to repreſent a crown, and b a ſhilling; then the ſum of a and b can be neither $2a$ nor $2b$, that is, neither 2 crowns nor 2 ſhillings, but only one crown plus 1 ſhilling, that is $a+b$.

In this rule, the word *addition* is not very properly uſed; being much too ſcanty to expreſs the operation here performed. The buſineſs of this operation is to incorporate into one maſs, or algebraic expreſſion, different algebraic quantities; as far as an actual incorporation or union is poſſible; and to retain the algebraic marks for doing it, in caſes where the former is not poſſible. When we have ſeveral quantities, ſome affirmative and ſome negative; and the relation of theſe quantities can in the whole or in part be diſcovered; ſuch incorporation of two or more quantities into one, is plainly effected by the foregoing rules.

It may ſeem a paradox, that what is called addition in algebra, ſhould ſometimes mean addition, and ſometimes ſubtraction. But the paradox wholly ariſes from the ſcantineſs of the name given to the algebraic proceſs; from employing an old term in a new and more enlarged ſenſe. Inſtead of addition, call it incorporation, or union, or ſtriking a balance, or any name to which a more extenſive idea may be annexed, than that which is uſually implied by the word addition; and the paradox vaniſhes.

Thus,

Thus, $3a$ added to $5a$, makes $8a$.
And $-2ab$ added to $-7ab$, makes $-9ab$.
And $5a+7b$ added to $7a+3b$, makes $12a+10b$.

OTHER EXAMPLES FOR PRACTICE.

$5a$	$-6bx$	$8bxy$
$7a$	$-3bx$	$7bxy$
$8a$	$-2bx$	$3bxy$
$10a$	$-7bx$	$4bxy$
$2a$	$-bx$	$5bxy$
a	$-5bx$	bxy
$33a$	$-24bx$	$28bxy$

$2y$	$5x^2+5xy$	$7ax-4y$
$5y$	$3x^2+2xy$	$8ax-3y$
$7y$	x^2+3xy	$6ax-2y$
$4y$	$7x^2+8xy$	$4ax-3y$
$3y$	x^2+xy	$2ax-2y$
$21y$	$17x^2+19xy$	$27ax-14y$

$6xy$	$-2y^2$	$5a-4b$
$15xy$	$-8y^2$	$7a-6b$
$2xy$	$-7y^2$	$4a-3b$
$7xy$	$-y^2$	$2a-8b$
$1\frac{1}{2}xy$	$-6y^2$	$6a-b$
$\frac{1}{2}xy$	$-y^2$	$3a-2b$

$20-15x^{\frac{1}{2}}-2xy$	$7xy-5x+3ab$
$35-13x^{\frac{1}{2}}-4xy$	$3xy-x+2ab$
$18-12x^{\frac{1}{2}}-3xy$	$2xy-3x+2ab$
$12-14x^{\frac{1}{2}}-8xy$	$2xy-4x+8ab$
$10-28x^2-2xy$	$5xy-3x+ab$

CASE II.

When the Quantities are Like, but have Unlike Signs:

ADD all the affirmative co-efficients into one ſum, and all the negative ones into another, when there are ſeveral of a kind.

Subtract the leſs of theſe ſums from the greater, and to the difference prefix the ſign of the greater, and ſubjoin the common quantity.

So $+ 5a$ and $- 3a$, united, make $+ 2a$.
And $- 5a$ and $+ 3a$, united, make $- 2a$.

OTHER EXAMPLES FOR PRACTICE.

$-3a$	$+8ax^2$	$+6x^3+8y$
$+7a$	$+7ax^2$	$-3x^3+7y$
$+8a$	$-3ax^2$	$-13x^3+8y$
$-a$	$-4ax^2$	$+2x^3-3y$
$-2a$	$+4ax^2$	$+x^3-y$
$+9a$	$+12ax^2$	$-7x^3+19y$

$-2a^2$	$+8b^2y^3$	$-3ab+7$
$-3a^2$	$+6b^2y^3$	$+3ab-19$
$-8a^2$	$-10b^2y^3$	$+3ab-6$
$+10a^2$	$-20b^2y^3$	$-ab+2$
$+13a^2$	$-b^2y^3$	$-2ab+11$

$-2ax^{\frac{1}{2}}$	$-6\sqrt{ax}$	$-2y+2ax^{\frac{1}{2}}$
$+ax^{\frac{1}{2}}$	$+2\sqrt{ax}$	$+y+ax^{\frac{1}{2}}$
$-3ax^{\frac{1}{2}}$	$-6\sqrt{ax}$	$-7y-3ax^{\frac{1}{2}}$
$+7ax^{\frac{1}{2}}$	$+10\sqrt{ax}$	$+5y+3ax^{\frac{1}{2}}$

CASE III.

When the Quantities are Unlike.

HAVING collected together all the like quantities, as in the two foregoing cases, set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

$5xy$	$2xy - 10x^2$	$2ax - 150 + 2x^{\frac{1}{2}}$
$4ax$	$-3x^2 + xy$	$3x^2 + 2ax + 6x^2$
$-xy$	$-8x^2 - xy$	$5xy - 3x^{\frac{1}{2}} + 50$
$-4ax$	$-xy + 9x^2$	$\sqrt{x} + 100 - 5x^2$
$4xy$	$xy - 12x^2$	$4ax + 4x^2 + 5xy$
$6x^2y^2$	$12ax - x^2$	$6 + 10\sqrt{ax} - 3y$
$-4x^2y$	$4ax + xy$	$x + 4\sqrt{xy} + 3y$
$-2axy$	$3y^2 - ax$	$y - 2\sqrt{ax} - 3y$
$-3x^2y$	$2x^2 - 24$	$20 + 3\sqrt{ax} - 3y$
$3x^2y$	$2\sqrt{x} - 8y$	$a^2 - 8 + x^{\frac{1}{2}} - 2$
$-2xy^2$	$3\sqrt{xy} + 10x$	$a - 10 + a^2 - x$
$-3y^2x$	$2x + \sqrt{x} + y$	$x^2 - a^2 + 8 - 4$
$-8x^2y$	$-8 + \sqrt{xy}$	$10 - a - x^2 - y$

Add $a + b$ and $3a - 5b$ together.
Add $5a - 8x$ and $3a - 4x$ together.
Add $6x - 5b + a + 8$ to $-5a - 4x + 4b - 3$.
Add $a + 2b - 3c - 10$ to $3b - 4a + 5c + 10$ and $5b - c$.
Add $a + b$ and $a - b$ together.
Add $3a + b - 10$ to $c - d - a$ and $-4c + 2a - 3b - 7$.
Add $3a^2 + b^2 - c$ to $2ab - 3a^2 + bc - b$.
Add $a^3 + b^2c - b^2$ to $ab^2 - abc + b^2$.
Add $9a - 8b + 10x - 6d - 7c + 50$ to $2x - 3a - 5c + 4b + 6d - 10$.

SUB-

SUBTRACTION.

RULE*.

Set down in one line the firſt quantities from which the ſubtraction is to be made; and underneath them place all the other quantities compoſing the ſubtrahend; ranging the like quantities under each other, as in Addition.

Then change all the ſigns (+ and —) of the lower line, or conceive them to be changed; after which, collect all the terms together as in the caſes of Addition.

EXAMPLES.

$5a^2 - 2b$	$6x^2 - 8y + 3$	$5xy - 2 + 8x - y$
$2a^2 - 5b$	$2x^2 + 9y - 2$	$3xy - 8 - 8x - 3y$
$3a^2 + 3b$	$4x^2 - 17y + 5$	$2xy + 6 + 16x + 2y$
$3xy - 8$	$2y^2 - y - 1$	$-10 - 8x - 3xy$
$-xy + 8$	$y^2 + y + 1$	$xy - 7x + 3 - 4ay$
$4xy - 16$	$y^2 - 2y - 2$	$-13 - x - 4xy + 4ay$
$5x^2y - 8$	$4\sqrt{xy} - x\sqrt{xy}$	$5x^2 + \sqrt{x} - 8 - 4b$
$-3x^2y + 1$	$-2\sqrt{xy} + 2 + xy$	$6x^2 - 10 + 4b - x^{\frac{1}{2}}$

* This rule is founded on the conſideration, that addition and ſubtraction are oppoſite to each other in their nature and operation, as are the ſigns + and —, by which they are expreſſed and repreſented. So that, ſince to unite a negative quantity with a poſitive one of the ſame kind, has the effect of diminiſhing it, or ſubducting an equal poſitive one from it, therefore to ſubtract a poſitive (which is the oppoſite of uniting or adding) is to add the equal negative quantity. In like manner, to ſubtract a negative quantity, is the ſame in effect as to add or unite an equal poſitive one. So that, by changing the ſign of a quantity from + to —, or from — to +, changes its nature from a ſubductive quantity to an additive one; and any quantity is in effect ſubtracted by barely changing its ſign.

$3xy - 20$	$4x^3 - 3.(a+b)$	$xy^3 + 10a\sqrt{(xy+10)}$
$4xy - 30$	$3x^2 - 8.(a+b)$	$x^2y^2 + 2a\sqrt{(xy+10)}$

From $a + b$, take $a - b$,
From $4a - 4b$, take $b + a$.
From $4a - 4b$, take $5b + 3a$.
From $8a - 12x$, take $-4x + 3a$.
From $2x - 4a - b + 5$, take $8 - 5b + a + 6x$.
From $3a + b + c - d - 10$, take $c + a - d$.
From $3a + b + c - d - 10$, take $b - 10 + 3a$.
From $2ab + b^2 - c + bc - b$, take $3a^2 - c + b^2$.
From $a^3 + b^2c + ab^2 - abc$, take $b^2 + ab^2 - abc$.
From $12x + 6a - 4b + 40$, take $4b - 3a + 2x + 6d - 10$.
From $2x - 3a + 4b + 6c - 50$, take $9a + x + 8b - 6c - 40$.
From $6a - 4b - 12c + 12x$, take $2x - 3a + 4b - 5c$.

MULTIPLICATION.

CASE I.

When both the Factors are Simple Quantities.

RULE.

MULTIPLY the co-efficients of the two terms together, then to the product affix all the letters in thoſe terms, and the reſult will be the whole product required.

Note *. Like ſigns produce +, and unlike ſigns —.

EXAM-

* That like ſigns make +, and unlike ſigns —, in the product, may be ſhewn thus:

1. When $+ a$ is to be multiplied by $+ b$; this implies that $+ a$ is to be taken as many times as there are units in b; and ſince the ſum of any number of affirmative terms is affirmative, it follows that $+ a \times + b$ makes $+ ab$.

2. When

EXAMPLES.

$12a$	$-2a$	$5a$	$-9x$
$3b$	$+4b$	$-6x$	$-5b$
$36ab$	$-8ab$	$-30ax$	$+45bx$
$7ab$	$6a^2x$	$-x^2y$	$-7xy$
$-5ac$	$5x$	xy^2	$-xy$
$-35a^2bc$	$30a^2x^2$	$-x^3y^3$	$+7x^2y^2$
$-5ax$	$-ax$	$+5xy$	$-7xyz$
$3x$	$-7b$	-3	$-6ax$

CASE II.

When one of the Factors is a Compound Quantity.

RULE.

MULTIPLY every term of the multiplicand, feparately, by the multiplier, as in the former cafe; placing the products one after another, with the proper fignsa, and the refult will be the whole product required.

---

2. When two quantities are to be multiplied together, the refult will be exactly the fame, in whatever order they are placed; for a times b is the fame as b times a; and therefore, when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$: this is the fame thing as taking $-a$ as many times as there are units in $+b$; and fince the fum of any number of negative terms is negative, it follows that $-a \times +b$, or $+a \times -b$ make or produce $-ab$.

3. When $-a$ is to be multiplied by $-b$: here $-a$ is to be fubtracted as often as there are units in b: but fubtracting negatives is the fame thing as adding affirmatives, by the demonftration of the rule for fubtraction; confequently the product is b times a, or $+ab$.

Otherwife. Since $a - a = 0$, therefore $(a - a) \times -b$ is alfo $= 0$, becaufe 0 multiplied by any quantity, is ftill but 0; and fince the firft term of the product, or $a \times -b$ is $= -ab$, by the fecond cafe; therefore the laft term of the product, or $-a \times -b$, muft be $+ab$, to make the fum $= 0$, or $-ab + ab = 0$; that is, $-a \times -b = +ab$.

EXAM-

EXAMPLES.

$4a - 2b$	$6xy - 8$	$b^2 - 2x + 6$
$3a$	$2x$	xy
$12a^2 - 6ab$	$12x^2y - 16x$	$a^2xy - 2x^2y + 6xy$
$13x - ab$	$35x - 7a$	$3y - 8 + 2xy$
$12a$	$-x$	xy
$2x^2 + x$	$12x^2 - 4y^2$	$2y^2 - 8x^2 - 7x$
$2xy$	$-2x^2$	$3xy^2$

CASE III.

When both the Factors are Compound Quantities.

RULE

MULTIPLY every term of the multiplier into every term of the multiplicand, respectively; setting down the products one after another, with their proper signs; and add the several lines of products all together for the whole product required.

$x + y$	$5x + 4y$	$x^2 + xy - y^2$
$x + y$	$3x - 2y$	$x - y$
$x^2 + xy$	$15x^2 + 12xy$	$x^3 + x^2y - xy^2$
$+ xy + y^2$	$- 10xy - 8y^2$	$- x^2y - xy^2 + y^3$
$x^2 + 2xy + y^2$	$15x^2 + 2xy - 8y^2$	$x^3 \quad * \quad - 2xy^2 + y^3$
$x + y$	$x^2 + y$	$x^2 + xy + y^2$
$x - y$	$x^2 + y$	$x - y$
$x^2 + xy$	$x^4 + yx^2$	$x^3 + x^2y + xy^2$
$- xy - y^2$	$+ yx^2 + y^2$	$- x^2y - xy^2 - y^3$
$x^2 \quad * \quad - y^2$	$x^4 + 2yx^2 + y^2$	$x^3 \quad * \quad * \quad - y^3$

Note.

Note. In the multiplication of compound quantities, it is the beſt way to ſet them down in order, according to the powers and the letters of the alphabet. And in multiplying them, begin at the left hand ſide, and multiply from the left hand towards the right, in the manner that we write, which is contrary to the way of multiplying numbers. But in ſetting down the ſeveral products, as they ariſe, in the ſecond and following lines, range them under the like terms in the lines above, when there are ſuch like quantities; which is the eaſieſt way for adding them up together.

In many caſes, the multiplication of compound quantities is only to be performed by ſetting them down one after another, each within or under a vinculum, with a ſign of multiplication between them. As $(a + b) \times (a - b) \times 3ab$, or $\overline{a + b} \cdot \overline{a - b} \cdot 3ab$.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. Anſ. $36a^2x$.
2. Multiply $4x^2 - 2y$ by $2y$. Anſ. $8x^2y - 4y^2$.
3. Multiply $2x + 4y$ by $2x - 4y$. Anſ. $4x^2 - 16y^2$.
4. Multiply $x^2 - xy + y^2$ by $x + y$. Anſ. $x^3 + y^3$.
5. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$. Anſ. $x^4 - y^4$.
6. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
7. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 3$.
8. Multiply $2a^2 - 3ax + 4x^2$ by $5a^2 - 6ax - 2x^2$.
9. Multiply $3x^3 + 3x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 5y^3$.
10. Multiply $a^2 + ab + b^2$ by $a - b$.

DIVISION.

DIVISION in algebra, like that in numbers, is the converſe of multiplication; and it is performed like that of numbers alſo, by beginning at the left hand ſide, and dividing all the parts of the dividend by the diviſor, when they can be ſo divided; or elſe by ſetting them down like a fraction, the dividend over the diviſor, and then abbreviating the fraction as much as can be done. This will naturally divide into the following particular caſes.

CASE

CASE I.

When the Diviſor and Dividend are both Simple Quantities.

RULE.

SET the terms both down as in diviſion of numbers, either the diviſor before the dividend, or below it, like the denominator of a fraction. Then abbreviate theſe terms as much as can be done, by cancelling or ſtriking out all the letters that are common to both of them, and alſo dividing the one co-efficient by the other, or abbreviating them after the manner of a fraction, by dividing them by their common meaſure.

Note. Like ſigns in the two factors make + in the quotient; and unlike ſigns make —; the ſame as in multiplication *.

EXAMPLES.

1. To divide $8ab$ by $2a$.

Here $8ab \div 2a$, or $2a)\ 8ab$, or $\frac{8ab}{2a} = 4b$.

2. Alſo $a \div a = \frac{a}{a} = 1$; and $abc \div bcd = \frac{abc}{bcd} = \frac{a}{d}$.

3. Divide $16x^2$ by $8x$. Anſ. $2x$.

4. Divide $12a^2x^2$ by $-3a^2x$. Anſ. $-4x$.

5. Divide $-15ay^2$ by $3ay$. Anſ. $-5y$.

6. Divide $-18ax^2y$ by $-8axz$. Anſ. $\frac{9xy}{4z}$.

---

* Becauſe the diviſor multiplied by the quotient, muſt produce the dividend. Therefore,

1. When both the terms are +, the quotient muſt be +; becauſe + in the diviſor × + in the quotient, produces + in the dividend.

2. When the terms are both —, the quotient is alſo +; becauſe — in the diviſor × + in the quotient, produces — in the dividend.

3. When one term is + and the other —, the quotient muſt be —; becauſe + in the diviſor × — in the quotient produces — in the dividend, or — in the diviſor × + in the quotient gives — in the dividend.

So that the rule is general, like ſigns give +, and unlike ſigns give —, in the quotient.

CASE

CASE II.

When the Dividend is a Compound Quantity, and the Diviſor a Simple one.

RULE.

DIVIDE every term of the dividend by the diviſor, as in the former caſe.

EXAMPLES.

1. $(ab+b^2) \div 2b$, or $\frac{ab+b^2}{2b} = \frac{a+b}{2} = \frac{1}{2}a + \frac{1}{2}b$.
2. $(10ab+15ax) \div 5a$, or $\frac{10ab+15ax}{5a} = 2b + 3x$.
3. $(30az - 48z) \div z$, or $\frac{30az-48z}{z} = 30a - 48$.
4. Divide $4ab - 8ax + d$ by $2a$.
5. Divide $3x^2 - 15 + 6x + 3a$ by $3x$.
6. Divide $3abc + 12abx - 9a^2b$ by $3ab$.
7. Divide $10a^2x - 15x^2 - 5x$ by $5x$.
8. Divide $15a^2bc - 12acx^2 + 5ad^2$ by $-5ac$.
9. Divide $12a + 3ay - 18y^2$ by $21a$.
10. Divide $-40a^2b^2 + 60ab^3$ by $-6ab$.

CASE III.

When the Diviſor and Dividend are both Compound Quantities.

RULE.

1. SET them down as in common diviſion of numbers, the diviſor before the dividend, with a ſmall crooked line between them, and ranging the terms according to the powers of ſome one of the letters in both, the higher powers of it before the lower.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, as in the firſt caſe, and place the reſult in the quotient.

3. Multiply the whole diviſor by the term thus found, and ſubtract the reſult from the dividend.

4. To this remainder bring down as many terms of the dividend as are requiſite for the next operation, dividing as before; and ſo on to the end, as in common arithmetic.

Note.

Note. If the divisor be not exactly contained in the dividend, the quantity which remains after the operation is finished, may be placed over the divisor, like a vulgar fraction, and set down at the end of the quotient, as in common arithmetic.

EXAMPLES.

$$
\begin{array}{l}
x + y \,)\; x^2 + 2xy + y^2 \;(\; x + y \\
\qquad\quad\;\; \underline{x^2 + xy} \\
\qquad\qquad\qquad\quad xy + y^2 \\
\qquad\qquad\qquad\quad \underline{xy + y^2}
\end{array}
$$

$$
\begin{array}{l}
a + x \,)\; a^3 + 5a^2x + 5ax^2 + x^3 \;(\; a^2 + 4ax + x^2 \\
\qquad\quad\;\; \underline{a^3 + a^2x} \\
\qquad\qquad\quad 4a^2x + 5ax \\
\qquad\qquad\quad \underline{4a^2x + 4ax^2} \\
\qquad\qquad\qquad\qquad\quad ax^2 + x^3 \\
\qquad\qquad\qquad\qquad\quad \underline{ax^2 + x^3}
\end{array}
$$

$$
\begin{array}{l}
x - 3 \,)\; x^3 - 9x^2 + 27x - 27 \;(\; x^2 - 6x + 9 \\
\qquad\quad\;\; \underline{x^3 - 3x^2} \\
\qquad\qquad\quad - 6x^2 + 27x \\
\qquad\qquad\quad \underline{- 6x^2 + 18x} \\
\qquad\qquad\qquad\qquad\quad 9x - 27 \\
\qquad\qquad\qquad\qquad\quad \underline{9x - 27}
\end{array}
$$

$$
\begin{array}{l}
a - x \,)\; a^3 - x^3 \;(\; a^2 + ax + x \\
\qquad\quad\;\; \underline{a^3 - a^2x} \\
\qquad\qquad\quad a^2x - x^3 \\
\qquad\qquad\quad \underline{a^2x - ax^2} \\
\qquad\qquad\qquad\qquad\quad ax^2 - x^3 \\
\qquad\qquad\qquad\qquad\quad \underline{ax^2 - x^3}
\end{array}
$$

$$b - y \,)\, b^4 - 3y^4 \,(\, b^3 + b^2y + by^2 + y^3 - \frac{2y^4}{b-y}.$$

$$\begin{array}{r} b^4 - b^3y \\ \hline b^3y - 3y^4 \\ b^3y - b^2y^2 \\ \hline b^2y^2 - 3y^4 \\ b^2y^2 - by^3 \\ \hline by^3 - 3y^4 \\ by^3 - y^4 \\ \hline -2y^4 \end{array}$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2 + 2ax + x^2$ by $a + x$. Anf. $a + x$.
2. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
   Anf. $a^2 - 2ay + y^2$.
3. Divide 1 by $1 - x$. Anf. $1 + x + x^2 + x^3 +$ &c.
4. Divide $6x^4 - 96$ by $3x - 6$.
   Anf. $2x^3 + 4x^2 + 8x + 16$.
5. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$. Anf. $a^3 - 3a^2x + 3ax^2 - x^3$.
6. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
7. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$.
8. Divide $a^5 - x^5$ by $a - x$.
9. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
10. Divide $a^4 + 4a^2b^2 + 3b^4$ by $a + 2b$.

ALGEBRAIC FRACTIONS.

ALGEBRAIC FRACTIONS have the same names and rules of operation, as numeral fractions in common arithmetic; as appears in the following Problems:

PRO-

PROBLEM I.

To Reduce a Mixed Quantity to an Improper Fraction.

RULE.

MULTIPLY the integer by the denominator of the fraction, and to the product add the numerator; then the denominator being placed under this sum, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{5}{7}$, and $a - \frac{b}{c}$ to improper fractions.

First, $3\frac{5}{7} = \frac{3 \times 7 + 5}{7} = \frac{21 + 5}{7} = \frac{26}{7}$ the Anſ.

And, $a - \frac{b}{c} = \frac{a \times c - b}{c} = \frac{ac - b}{c}$ the Anſ.

2. Reduce $x + \frac{x^2}{a}$ and $x - \frac{a^2 - x^2}{x}$ to improper fractions.

First, $x + \frac{x^2}{a} = \frac{x \times a + x^2}{a} = \frac{ax + x^2}{a}$ the Anſ.

And, $x - \frac{a^2 - x^2}{x} = \frac{x^2 - a^2 + x^2}{x} = \frac{2x^2 - a^2}{x}$ Anſ.

3. Reduce $8\frac{6}{7}$ to an improper fraction. Anſ. $\frac{62}{7}$.

4. Reduce $1 - \frac{2x}{a}$ to an improper fraction. Anſ. $\frac{a - 2x}{a}$.

5. Reduce $x - \frac{ax + x^2}{2a}$ to an improper fraction.

6. Reduce $10 + \frac{2x - 8}{3x}$ to an improper fraction.

7. Reduce $a + \frac{1 - x - b}{b}$ to an improper fraction.

8. Reduce $1 + 2x - \frac{x - 3}{5x}$ to an improper fraction.

PROBLEM II.

To Reduce an Improper Fraction to a Whole or Mixed Quantity.

RULE.

DIVIDE the numerator by the denominator, for the integral part; and place the remainder, if any, over the denominator, for the fractional part; the two joined together will be the mixed quantity required.

EXAMPLES.

1. To reduce $\frac{17}{5}$, and $\frac{ax+a^2}{x}$ to mixed quantities.

First, $\frac{17}{5} = 17 \div 5 = 3\frac{2}{5}$, the Anſwer required.

And $\frac{ax+a^2}{x} = \overline{ax+a^2} \div x = a + \frac{a^2}{x}$ Anſ.

2. To reduce $\frac{ab-a^2}{b}$, and $\frac{ay+2y^2}{a+y}$ to mixed quantities.

Firſt, $\frac{ab-a^2}{b} = \overline{ab-a^2} \div b = a - \frac{a^2}{b}$ Anſ.

And, $\frac{ay+2y^2}{a+y} = \overline{ay+2y^2} \div \overline{a+y} = y + \frac{y^2}{a+y}$.

3. Let $\frac{35}{8}$, and $\frac{3ab-b^2}{a}$ be reduced to whole or mixed quantities. Anſ. $4\frac{3}{8}$, and $3b - \frac{b^2}{a}$.

4. Let $\frac{2x^2y}{2x}$, and $\frac{a^2+x^2}{a-x}$ be reduced to whole or mixed quantities.

5. Let $\frac{x^2-y^2}{x+y}$, and $\frac{x^3-y^3}{x-y}$ be reduced to whole or mixed quantities.

6. Reduce $\frac{10x^2-x+3}{5x}$ to a mixed quantity.

7. Reduce $\frac{12x^3+3x^2}{4x^3+x^2-4x-1}$ to a mixed quantity.

PROBLEM III.

To Reduce Fractions to a Common Denominator.

RULE.

MULTIPLY every numerator, ſeparately, into all the denominators except its own, for the new numerators; and all the denominators together, for the common denominator*.

* When the denominators have a common diviſor, it will be better, inſtead of multiplying by the whole denominators, to multiply only by thoſe parts which ariſe from dividing by the common diviſor.

EXAM-

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to a common denominator.

$$\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\}$$ the new numerators.

$b \times c = bc$ the common denominator.

therefore $\frac{a}{b}$ and $\frac{b}{c} = \frac{ac}{bc}$ and $\frac{b^2}{bc}$ the fractions required.

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$ to a common denominator.

$$\left.\begin{array}{l} a \times c \times d = acd \\ b \times b \times d = b^2 d \\ c \times b \times c = c^2 b \end{array}\right\}$$ the numerators.

$b \times c \times d = bcd$ the com. denom.

therefore $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d} = \frac{acd}{bcd}$, $\frac{b^2 d}{bcd}$ and $\frac{c^2 b}{bcd}$, the frac. required.

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. Anſ. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions having a common denominator. Anſ. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to fractions having a common denominator. Anſ. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{2x}{a}$, to fractions having a common denominator. Anſ. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2 + 24x}{12a}$.

7. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2 + x^2}{a + x}$ to a common denominator.

8. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$ to a common denominator.

PRO

PROBLEM IV

To find the Greateſt Common Meaſure of the Terms of a Fraction.

RULE*.

1. RANGE the quantities according to the dimenſions of ſome letters, as is ſhown in diviſion.

2. Divide the greater term by the leſs, and the laſt diviſor by the laſt remainder, and ſo on till nothing remains; then the diviſor laſt uſed will be the common meaſure required.

Note. All the letters or figures which are common to each term of the diviſors, muſt be thrown out of them, before they are uſed in the operation.

EXAMPLES.

1. To find the greateſt common meaſure of $\frac{cx + x^2}{ca^2 + a^2x}$.

$$\begin{array}{r} cx + x^2 \,)\, ca^2 + a^2x \\ \text{or}\quad c + x \,)\, ca^2 + a^2x \,(\, a^2 \\ ca^2 + a^2x \\ \hline \end{array}$$

Therefore the greateſt common meaſure is $c + x$.

2. To find the greateſt common meaſure of $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$.

$$\begin{array}{r} x^2 + 2bx + b^2 \,)\, x^3 - b^2x \,(\, x \\ x^3 + 2bx^2 + b^2x \\ \hline - 2bx^2 - 2b^2x \,)\, x^2 + 2bx + b^2 \\ \text{or}\quad x + b \,)\, x^2 + 2bx + b^2 \,(\, x + b \\ x^2 + bx \\ \hline bx + b^2 \\ bx + b^2 \\ \hline \end{array}$$

Therefore $x + b$ is the greateſt common diviſor.

3. To find the greateſt common diviſor of $\frac{x^2 - 1}{xy + y}$.

Anſ. $x + 1$.

* The ſimple diviſors, in this rule, may be eaſily found, by inſpection.

4. To

4. To find the greateſt common diviſor of $\frac{x^7 - b^4}{x^5 + b^2x^3}$.

Anſ. $x^2 + b^2$.

5. To find the greateſt common meaſure of $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^3b + 2a^2b^2 + 2ab^3 + b^4}$.

PROBLEM V.

To Reduce a Fraction to its Loweſt Terms.

RULE.

1. FIND the greateſt common meaſure, as in the laſt problem.

2. Divide both the terms of the fraction by the common meaſure thus found, and it will reduce it to its loweſt terms, as was required.

3. Or divide the terms by any quantity that it may appear will divide them both.

EXAMPLES.

1. Reduce $\frac{cx + x^2}{ca^2 + a^2x}$ to its loweſt terms.

$cx + x^2) ca^2 + a^2x$

or $c + x) ca^2 + a^2x (a^2$

$ca^2 + a^2x$

Here $cx + x^2$ is divided by x which is common to both terms.

Therefore $c + x$ is the greateſt common meaſure, and $c + x) \frac{cx + x^2}{ca^2 + a^2x} = \frac{x}{a^2}$ is the fraction required.

3. Having $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$ given, it is required to reduce it to its leaſt terms.

$x^2 + 2bx + b^2) x^3 - b^2x (x$

$x^3 + 2bx^2 + b^2x$

$- 2bx^2 - 2b^2x) x^2 + 2bx + b^2$

or $x + b) x^2 + 2bx + b^2 (x + b$

$x^2 + bx$

$bx + b^2$

$bx + b^2$

Therefore

Therefore $x + b$ is the greateſt common meaſure,

and $x + b) \frac{x^3 - b^2x}{x^2 + 2bx + b^2} = \frac{x^2 - bx}{x + b}$ is the fraction required.

3. Reduce $\frac{x^4 - b^4}{x^5 - b^2x^3}$ to its loweſt terms. Anſ. $\frac{x^2 + b^2}{x^3}$.

4. Reduce $\frac{x^2 - y^2}{x^4 - y^4}$ to its loweſt terms. Anſ. $\frac{1}{x^2 + y^2}$.

5. Reduce $\frac{a^4 - x^4}{a^3 - a^2x - ax^2 + x^3}$ to its loweſt terms.

6. Reduce $\frac{5a^5 + 10a^4x + 5a^3x^2}{a^3x + 2a^2x^2 + 2ax^3 + x^4}$ to its loweſt terms.

PROBLEM VI.

To Add Fractional Quantities together.

RULE.

1. If the fractions have a common denominator, add all the numerators together, then under their ſum write the common denominator, and it will give the ſum of the fractions required.

2. If they have not a common denominator, reduce them to a common denominator, and then add them as before.

EXAMPLES.

1. Having $\frac{x}{2}$ and $\frac{x}{3}$ given, to find their ſum.

Here $\left.\begin{matrix} x \times 3 = 3x \\ x \times 2 = 2x \end{matrix}\right\}$ the numerators.

and $2 \times 3 = 6$ the com. denom.

theref. $\frac{3x}{6} + \frac{2x}{6} = \frac{5x}{6}$ is the ſum required.

2. Having $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ given, to find their ſum.

Here $\left.\begin{matrix} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{matrix}\right\}$ the numerators.

and $b \times d \times f = bdf$ the com. denom.

theref. $\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$ the ſum required.

3. Let

3. Let $a - \frac{3x^2}{b}$ and $b + \frac{2ax}{c}$ be added together.

$3x^2 \times c = 3cx^2$
$2ax \times b = 2abx$ } the numerators.

and $b \times c = bc$ the common denominator.

Therefore $a - \frac{3}{b}^{2} + b + \frac{2ax}{c} = a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc} =$

$a + b + \frac{2abx - 3cx^2}{bc}$ the sum required.

4. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together. Anſ. $\frac{15x + 2bx}{10b}$.

5. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. Anſ. $x + \frac{x}{12}$ or $\frac{13}{12}x$.

6. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. Anſ. $\frac{19x - 14}{21}$.

7. Add $x + \frac{x-2}{3}$ to $3x + \frac{2x-3}{4}$. Anſ. $4x + \frac{10x - 17}{12}$

8. It is required to add $4x$, $\frac{5x^2}{2a}$ and $\frac{x+a}{3x}$ together.

9. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$ and $\frac{2x+1}{5}$ together.

10. It is required to add $4x$, $\frac{7x}{9}$ and $2 + \frac{x}{5}$ together.

11. It is required to add $3x + \frac{2x}{5}$ and $x - \frac{8x}{9}$ together.

PROBLEM VII.

To Subtract one Fractional Quantity from another.

RULE†.

1. REDUCE the fractions to a common denominator, as in addition, if they have not a common denominator.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

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* In the addition of mixed quantities, it is beſt to bring the fractional parts only to a common denominator, and to affix their ſum to the ſum of the integers, interpoſing the proper ſign.

† The ſame rule may be obſerved for mixed quantities, in ſubtraction, as in addition.

EXAMPLES.

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

Here $x \times 11 = 11x$
$2x \times 3 = 6x$ } the numerators.

and $3 \times 11 = 33$ the common denominator.

theref. $\frac{11x}{33} - \frac{6x}{33} = \frac{5x}{33}$ is the difference required.

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

Here $x - a \times 5c = 5cx - 5ac$
$2a - 4x \times 3b = 6ab - 12bx$ } the numerators.

and $3b \times 5c = 15bc$ the com. denom.

Then $\frac{5cx-5ac}{15bc} - \frac{6ab-12bx}{15bc} = \frac{5cx-5ac-6ab+12bx}{15bc}$ is the difference required.

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. Anf. $\frac{39x}{35}$.

4. Required the difference of $5y$ and $\frac{3y}{8}$. Anf. $\frac{37y}{8}$.

5. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$. Anf. $\frac{13x}{63}$.

6. Subtract $\frac{c}{d}$ from $\frac{x+a}{b}$. Anf. $\frac{dx+ad-bc}{bd}$.

7. Take $\frac{2x+7}{8}$ from $\frac{3x+a}{5b}$. Anf. $\frac{24x+8a-10bx-35b}{40b}$.

8. Take $x - \frac{x-a}{c}$ from $3x + \frac{x}{b}$. Anf. $2x + \frac{cx-bx+ab}{bc}$.

PROBLEM VIII.

To Multiply Fractional Quantities together.

RULE*.

MULTIPLY the numerators together, for a new numerator, and the denominators for a new denominator; and it will give the product required.

EXAM-

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by ſome quantity, which is common to both, the quotients may be uſed inſtead of them.

2. When a fraction is to be multiplied by an integer, the product is

EXAMPLES.

1. Required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$ the product required.

2. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

$\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product required.

3. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

Here $\frac{x \times (x+a)}{a \times (a+c)} = \frac{x^2+ax}{a^2+ac}$ the product required.

4. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. Anſ. $\frac{9ax}{2b}$

5. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. Anſ. $\frac{3x^3}{5a}$

6. To multiply $\frac{2x}{a}$, and $\frac{3ab}{c}$, and $\frac{3ac}{2b}$ together. Anſ. $9ax$.

7. Required the product of $b + \frac{bx}{a}$ and $\frac{a}{x}$. Anſ. $\frac{ab+bx}{x}$.

8. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

9. Required the product of x, and $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

PROBLEM IX.

To Divide one Fractional Quantity by another.

RULE*.

MULTIPLY the denominator of the diviſor by the numerator of the dividend, for a new numerator, and the numerator of the diviſor by the denominator of the dividend, for a new denominator.

O

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is found by multiplying the numerator by it; and if the integer be the ſame with the denominator, the numerator may be taken for the product.

3. When a fraction is to be multiplied by any quantity, it is the ſame thing whether the numerator be multiplied by it, or the denominator divided by it.

† 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the diviſor for the denominator.

2. When

Or, invert the terms of the divisor, and then multiply by it exactly as in multiplication.

EXAMPLES.

1. Required the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

Here $\frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\frac{1}{2}$ is the quot. required.

2. Required the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

Here $\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc}$ is the quotient required.

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$\frac{x+a}{2x-2b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{2x^2-2b^2}$ the quot. required.

4. Find the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

$\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2 \times (x+a)}{(a^3+x^3) \times x} = \frac{2x}{x^2-ax+a^2}$ is the quotient required.

5. Let $\frac{7x}{5}$ be divided by $\frac{1}{13}$. Anf. $\frac{91x}{60}$.

6. Let $\frac{4x^2}{7}$ be divided by $5x$. Anf. $\frac{4x}{35}$.

7. Let $\frac{x+1}{6}$ be divided by $\frac{2x}{3}$. Anf. $\frac{x+1}{4x}$.

8. Let $\frac{x}{x-1}$ be divided by $\frac{x}{2}$. Anf. $\frac{2}{x-1}$.

9. Let $\frac{5x}{3}$ be divided by $\frac{2a}{3b}$. Anf. $\frac{5bx}{2a}$.

10. Let $\frac{x-b}{8cd}$ be divided by $\frac{3cx}{4d}$. Anf. $\frac{x-b}{6c^2x}$.

11. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. Anf. $x+\frac{b^2}{x}$.

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2. When a fraction is to be divided by any quantity, it is the same thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions first proposed.

INVO-

INVOLUTION.

INVOLUTION is the raiſing of powers from any propoſed root; or the method of finding the ſquare, cube, biquadrate, &c, of any given quantity.

RULE *.

MULTIPLY the quantity into itſelf as many times as there are units in the index leſs one, and the laſt product will be the power required. Or,

Multiply the index of the quantity by the index of the power, and the reſult will be the ſame as before.

Note. When the ſign of the root is +, all the powers of it will be +; and when the ſign is —, all the even powers will be +, and all the odd powers —, as is evident from multiplication.

EXAMPLES.

a, root	a^2, root
a^2 = ſquare	a^4 = ſquare
a^3 = cube	a^6 = cube
a^4 = 4th power	a^8 = 4th power
a^5 = 5th power	a^{10} = 5th power
&c.	&c.
$-3a$, root	$-2ax^2$, root
$+9a^2$ = ſquare	$+4a^2x^4$ = ſquare
$-27a^3$ = cube	$-8a^3x^6$ = cube
$+81a^4$ = 4th power	$+16a^4x^8$ = 4th power
$-243a^5$ = 5th power	$-32a^5x^{10}$ = 5th power
$-\frac{2ax^2}{3b}$, root	$\frac{x}{a}$, root
$+\frac{4a^2x^4}{9b^2}$ = ſquare	$\frac{x^2}{a^2}$ = ſquare
$-\frac{8a^3x^6}{27b^3}$ = cube	$\frac{x^3}{a^3}$ = cube
$+\frac{16a^4x^8}{81b^4}$ = 4th power	$\frac{x^4}{a^4}$ = biquadrate

* Any power of the product of two or more quantities is equal to the ſame power of each of the factors, multiplied together.

And any power of a fraction is equal to the ſame power of the numerator, divided by the like power of the denominator.

Alſo, powers or roots of the ſame quantity, are multiplied by one another, by adding their exponents; or divided, by ſubtracting their exponents.

$x - a =$ root	$x + a =$ root
$x - a$	$x + a$
$x^2 - ax$	$x^2 + ax$
$\quad - ax + a^2$	$\quad + ax + a^2$
$x^2 - 2ax + a^2$ square	$x^2 + 2ax + a^2$
$x - a$	$x + a$
$x^3 - 2ax^2 + a^2x$	$x^3 + 2ax^2 + a^2x$
$\quad - ax^2 + 2a^2x - a^3$	$\quad + ax^2 + 2a^2x + a^3$
$x^3 - 3ax^2 + 3a^2x - a^3$	$x^3 + 3ax^2 + 3a^2x + a^3$

the cubes, or third powers of $x + a$ and $x - a$.

EXAMPLES FOR PRACTICE.

1. Required the cube or 3d power of $2a^2$. Anſ. $8a^6$.
2. Required the 4th power of $2a^2x$. Anſ. $16a^8x^4$.
3. Required the 3d power of $-8x^2y^3$. Anſ. $-512x^6y^9$.
4. To find the biquadrate of $-\frac{2a^2x}{3b^2}$. Anſ. $\frac{16a^8x^4}{81b^8}$.
5. Required the 5th power of $a - x$.
   Anſ. $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$.

Sir ISAAC NEWTON'S RULE *for raiſing a Binomial or Reſidual Quantity to any Power whatever**.

1. *To find the Terms without the Co-efficients.* The index of the firſt, or leading quantity, begins with that of the given power, and decreaſes continually by 1, in every term to the laſt; and in the following quantity, the indices of the terms are 0, 1, 2, 3, 4, &c.

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* This rule, expreſſed in general terms, is as follows:

$$\overline{a + b}^n = a^n + n.a^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 \text{ \&c.}$$

$$\overline{a - b}^n = a^n - n.a^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 - n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 \text{ \&c.}$$

Note. The ſum of the co-efficients, in every power, is equal to the number 2, raiſed to that power. Thus $1 + 1 = 2$ for the firſt power; $1 + 2 + 1 = 4 = 2^2$ for the ſquare; $1 + 3 + 3 + 1 = 8 = 2^3$ for the cube, or third power; and ſo on.

2. *To*

2. *To find the Unciæ, or Co-efficients.* The firſt is always 1, and the ſecond is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the ſecond term be —, all the odd terms will be +, and all the even terms —, which cauſes the terms to be + and — alternately.

EXAMPLES.

1. Let $a + x$ be involved to the 5th power.

The terms without the co-efficients will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the co-efficients will be

$$1,\ 5,\ \frac{5\times 4}{2},\ \frac{10\times 3}{3},\ \frac{10\times 2}{4},\ \frac{5\times 1}{5}$$

or 1, 5, 10, 10, 5, 1;

And therefore the 5th power altogether is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$$

2. Let $x - a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6,\ x^5a,\ x^4a^2,\ x^3a^3,\ x^2a^4,\ xa^5,\ a^6.$$

and the co-efficients will be

$$1,\ 6,\ \frac{6\times 5}{2},\ \frac{15\times 4}{3},\ \frac{20\times 3}{4},\ \frac{15\times 2}{5},\ \frac{6\times 1}{6},$$

or 1, 6, 15, 20, 15, 6, 1.

And therefore the 6th power of $x - a$ is

$$x^6 - 6x^5a + 15x^4a^2 - 20x^3a^3 + 15x^2a^4 - 6xa^5 + a^6.$$

3. Required the 4th power of $x - a$.

Anſ. $x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4$.

4. Required the 7th power of $x + a$.

Anſ. $x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7$.

EVO-

EVOLUTION.

EVOLUTION is the reverſe of Involution, being the method of finding the ſquare root, cube root, &c, of any given quantity, whether ſimple or compound.

CASE I.

To find the Roots of Simple Quantities.

RULE*.

EXTRACT the root of the co-efficient, for the numeral part; and divide the index of the letter, or letters, by the index of the power, and it will give the root of the literal part; then annex this to the former, for the whole root ſought.

EXAMPLES.

1. Required the ſquare root of $9x^2$; and the cube root of $8x^3$.

Anſ. $\sqrt{9x^2} = 3x^{\frac{2}{2}} = 3x$. And $\sqrt[3]{8x^3} = 2x^{\frac{3}{3}} = 2x$.

2. To find the ſq. root of $\frac{3x^2y^2}{4a^2}$, and cube root of $\frac{16x^4y^3}{27a^3}$.

Anſ. $\sqrt{\frac{3x^2y^2}{4a^2}} = \frac{xy}{2a}\sqrt{3}$. And $\sqrt[3]{\frac{16x^4y^3}{27a^3}} = \frac{2xy}{3a}\sqrt[3]{2x}$.

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* Any even root of an affirmative quantity, may be either + or —: thus the ſquare root of $+a^2$ is either $+a$, or $-a$; for $+a \times +a = +a^2$, and $-a \times -a = +a^2$ alſo.

And an odd root of any quantity will have the ſame ſign as the quantity itſelf: thus the cube root of $+a^3$ is $+a$, and the cube root of $-a^3$ is $-a$; for $+a \times +a \times +a = +a^3$, and $-a \times -a \times -a = -a^3$.

Any even root of a negative quantity is impoſſible; for neither $+a \times +a$, nor $-a \times -a$ can produce $-a^2$.

Any root of a product, is equal to the like root of each of the factors multiplied together.

And any root of a fraction, is equal to the like root of the numerator, divided by the ſame root of the denominator.

3. Re-

3. Required the ſquare root of $3a^2x^6$. Anſ. $ax^3\sqrt{3}$.
4. Required the cube root of $-125a^3x^6$. Anſ. $-5ax^2$.
5. Required the ſquare root of $\frac{9x^2y^2}{4a^3}$. Anſ. $\frac{3xy}{2a\sqrt{a}}$.
6. Required the 4th root of $256a^4x^8$. Anſ. $4ax^2$.
7. To find the 5th root of $-32x^5y^{10}$. Anſ. $-2xy^2$.

CASE II.

To find the Square Root of a Compound Quantity.

RULE.

THIS is performed like as in numbers, thus:

1. Range the quantities according to the dimenſions of ſome letter, and ſet the root of the firſt term in the quotient.

2. Subtract the ſquare of the root, thus found, from the firſt term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and ſet the reſult both in the quotient and diviſor.

4. Multiply the diviſor, thus increaſed, by the term laſt put in the quotient, and ſubtract the product from the dividend.

And ſo on, as in common arithmetic.

EXAMPLES.

1. Extract the ſquare root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

$x^4 - 4x^3 + 6x^2 - 4x + 1 \;(\; x^2 - 2x + 1 = \text{root}$
x^4

$2x^2 - 2x) - 4x^3 + 6x^2$
$\quad - 4x^3 + 4x^2$

$2x^2 - 4x + 1) \; 2x^2 - 4x + 1$
$\quad 2x^2 - 4x + 1$

2. Find the root of $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.

$4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4 \;(\; 2a^2 + 3ax + x^2$
$4a^4$

$4a^2 + 3ax) \; 12a^3x + 13a^3x^2$
$\quad 12a^3x + 9a^2x^2$

$4a^2 + 6ax + x^2) \; 4a^2x^2 + 6ax^3 + x^4$
$\quad 4a^2x^2 + 6ax^3 + x^4$

3. Required the ſquare root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$. Anſ. $a^2 + 2ax + x^2$.

4. Required the ſquare root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}$. Anſ. $x^2 - x + \frac{1}{4}$.

5. It is required to find the ſquare root of $a^2 + x^2$.

Anſ. $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ — &c.

CASE III.

To find the Roots of Powers in General.

RULE*.

1. FIND the root of the firſt term, and place it in the quotient.

2. Subtract its power from that term, and bring down the ſecond term for a dividend.

3. Involve the root, laſt found, to the next lower power, and multiply it by the index of the given power, for a diviſor.

4. Divide the dividend by the diviſor, and the quotient will be the next term of the root.

5. Involve the whole root, then ſubtract and divide as before; and ſo on till the whole is finiſhed.

EXAMPLES.

1. To find the ſq. root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \,(\, a^2 - ax + x^2$

a^4

$2a^2 \,)\, -2a^3x$

$a^4 - 2a^3x + a^2x^2$

$2a^2 \,)\, 2a^2x^2$

$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$

2. Find

* As this method, in high powers, is generally thought too laborious, it may not be improper to obſerve, that the roots of compound quantities may ſometimes be eaſily diſcovered, thus:

1. Ex-

2. Find the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$x^6 + 6x^5 - 40x^3 + 96x - 64 \; (x^2 + 2x - 4$

x^6

$3x^4) \; 6x^5$

$x^6 + 6x^5 + 12x^4 + 8x^3 = (x^2 + 2x)^3$

$3x^4) \; - 12x^4$

$x^6 + 6x^5 - 40x^3 + 96x - 64 = (x^2 + 2x - 4)^3$

3. Required the ſquare root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$. Anſ. $a + b + c$.

4. Required the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$. Anſ. $x^2 - 2x + 1$.

5. Required the biquadrate root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$. Anſ. $2a - 3x$.

6. Required the 5th root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$. Anſ. $2x - 1$.

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1. Extract the roots of ſome of the moſt ſimple terms, and connect them together by the ſign + or —, as may be judged moſt ſuitable for the purpoſe.

2. Involve the compound root, thus found, to the proper power; then, if it be the ſame with the given quantity, it is the root required.

3. But if it be found to differ only in ſome of the ſigns, change them from + to —, or from — to +, till its power agrees with the given one throughout.

Thus, in the 5th example, the root $2a - 3x$, is the difference of the roots of the firſt and laſt terms; and in the 3d example, the root $a + b + c$ is the ſum of the roots of the 1ſt, 4th, and 6th terms. The ſame may alſo be obſerved of the 6th example, where the root is found from the firſt and laſt terms.

SURDS.

SURDS are ſuch quantities as have no exact root, being uſually expreſſed by fractional indices, or by means of the radical ſign $\surd$.

Thus, $2^{\frac{1}{2}}$, or $\surd 2$, denotes the ſquare root of 2; and $3^{\frac{2}{3}}$ or $\sqrt[3]{3^2}$, the cube root of the ſquare of 3; where the numerator ſhews the power to which the quantity is to be raiſed, and the denominator its root.

PROBLEM I.

To Reduce a Rational Quantity to the Form of a Surd.

RULE.

RAISE the quantity to a power equivalent to that denoted by the index of the ſurd; then over this new quantity place the radical ſign, and it will be of the form required.

EXAMPLES.

1. To reduce 3 to the form of the ſquare root.
   Firſt, $3 \times 3 = 3^2 = 9$; then $\surd 9$ is the anſwer.
2. To reduce $2x^2$ to the form of the cube root.
   Firſt, $2x^2 \times 2x^2 \times 2x^2 = (2x^2)^3 = 8x^6$;
   then $\sqrt[3]{8x^6}$ or $(8x^6)^{\frac{1}{3}}$ is the anſwer.
3. Reduce 5 to the form of the cube root.
   Anſ. $(125)^{\frac{1}{3}}$ or $\sqrt[3]{125}$.
4. Reduce $\frac{1}{2}xy$ to the form of the ſquare root.
   Anſ. $\surd \frac{1}{4}x^2y^2$.
5. Reduce 2 to the form of the 5th root. Anſ. $(32)^{\frac{1}{5}}$.
6. Let $a^{\frac{1}{2}}$ be reduced to the form of the 6th root.
7. Reduce $a + b$ to the form of the ſquare root, and $a - b$ to the form of the cube root.

PROBLEM II.

To Reduce Quantities of Different Indices, to other equivalent ones, that ſhall have a Common Index.

RULE.

1. DIVIDE the indices of the quantities by the given index, and the quotients will be the new indices for thoſe quantities.

2. Over the ſaid quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

3. A common index may alſo be found by reducing the indices of the quantities to a common denominator, and involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{2}$.

$\frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2}$ the 1ſt index.

$\frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times \frac{2}{1} = \frac{2}{6} = \frac{1}{3}$ the 2d index.

Therefore $(15^{\frac{1}{2}})^{\frac{1}{2}}$ and $(9^{\frac{1}{3}})^{\frac{1}{2}}$ are the quantities required.

2. Reduce $a^{\frac{3}{4}}$ and $x^{\frac{1}{4}}$ to the ſame common index $\frac{1}{4}$.

$\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{3}{1} = \frac{6}{4}$ the 1ſt index.

$\frac{1}{4} \div \frac{1}{3} = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4}$ the 2d index.

Therefore $(a^6)^{\frac{1}{3}}$ and $(x^3)^{\frac{1}{3}}$ are the quantities required.

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Anſ. $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Anſ. $(a^4)^{\frac{1}{8}}$ and $(b^2)^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the ſame radical ſign.

Anſ. $\sqrt[mn]{a^m}$ and $\sqrt[mn]{b^n}$.

6. Let $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$ be reduced to a common index.

7. Let $(a+b)^{\frac{1}{3}}$ and $(a-b)^{\frac{1}{4}}$ be reduced to a common index.

PROBLEM III.

To Reduce Surds to their moſt Simple Terms.

RULE*.

FIND the greateſt power contained in the given ſurd, and ſet its root before the remaining quantities, with the proper radical ſign between them.

* When the given ſurd contains no exact power, it is already in its moſt ſimple terms.

EXAM-

EXAMPLES.

1. To reduce $\sqrt{48}$ to its moſt ſimple terms.

$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}$, the anſwer.

2. Required to reduce $\sqrt[3]{108}$ to its moſt ſimple terms.

$\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3 \times \sqrt[3]{4} = 3\sqrt[3]{4}$, the anſwer.

3. Reduce $\sqrt{125}$ to its ſimpleſt terms. Anſ. $5\sqrt{5}$.

4. Reduce $\sqrt{\frac{50}{147}}$ to its ſimpleſt terms. Anſ. $\frac{5}{21}\sqrt{6}$.

5. Reduce $\sqrt[3]{243}$ to its ſimpleſt terms. Anſ. $3\sqrt[3]{9}$.

6. Reduce $\sqrt[3]{\frac{16}{81}}$ to its ſimpleſt terms. Anſ. $\frac{2}{3}\sqrt[3]{18}$.

7. Reduce $\sqrt{98a^2x}$ to its ſimpleſt terms. Anſ. $7a\sqrt{2x}$.

8. Reduce $\sqrt{x^3 - a^2x^2}$ to its moſt ſimple terms.

9. Reduce $(a^3x + 3a^3x^2)^{\frac{1}{3}}$ to its moſt ſimple terms.

10. Reduce $(32a^6 - 96a^5x)^{\frac{1}{5}}$ to its moſt ſimple terms.

PROBLEM IV.

To Add Surd Quantities together.

RULE.

1. Reduce ſuch quantities as have unlike indices to other equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their ſimpleſt terms, as in the laſt problem.

3. Then, if the ſurd part be the ſame in them all, annex it to the ſum of the rational parts, with the ſign of multiplication, and it will give the total ſum required.

But if the ſurd part be not the ſame in all the quantities, they can only be added by the ſigns + and —.

EXAMPLES.

1. It is required to add $\sqrt{27}$ and $\sqrt{48}$ together.

Firſt, $\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$; and $\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$;
then, $3\sqrt{3} + 4\sqrt{3} = (3 + 4)\sqrt{3} = 7\sqrt{3}$ = ſum required.

2. It is required to add $\sqrt[3]{500}$, and $\sqrt[3]{108}$ together.

Firſt, $\sqrt[3]{500} = \sqrt[3]{125 \times 4} = 5\sqrt[3]{4}$; and $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}$;
then, $5\sqrt[3]{4} + 3\sqrt[3]{4} = (5 + 3)\sqrt[3]{4} = 8\sqrt[3]{4}$ = ſum required.

3. Re-

3. Required the sum of $\sqrt{72}$ and $\sqrt{128}$. Anſ. $14\sqrt{2}$.
4. Required the ſum of $\sqrt{27}$ and $\sqrt{147}$. Anſ. $10\sqrt{3}$.
5. Required the ſum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{2}{5}\frac{7}{0}}$. Anſ. $\frac{1}{3}\frac{0}{0}\sqrt{6}$.
6. Required the ſum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$. Anſ. $5\sqrt[3]{5}$.
7. Required the ſum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{3}\frac{1}{2}}$. Anſ. $\frac{1}{4}\sqrt[3]{2}$.
8. Required the ſum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.
9. Required the ſum of $9\sqrt{243}$ and $10\sqrt{363}$.
10. Required to find the ſum of $a^{\frac{1}{n}}$ and $a^{\frac{1}{m}}$.
11. Required the ſum of $\sqrt{27a^4x}$ and $\sqrt{3a^2x}$.

PROBLEM V.

To Subtract, or find the Difference of, Surd Quantities.

RULE.

PREPARE the quantities as in the laſt rule, and the difference of the rational parts annexed to the common ſurd, will give the difference of the ſurds required.

But if the quantities have no common ſurd, they can only be ſubtracted by means of the ſign —.

EXAMPLES.

1. Required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

Firſt, $\sqrt{448} = \sqrt{64 \times 7} = 8\sqrt{7}$; and $\sqrt{112} = \sqrt{16 \times 7} = 4\sqrt{7}$.
Then $8\sqrt{7} - 4\sqrt{7} = 4\sqrt{7}$ the difference required.

2. Required to find the difference of $192^{\frac{1}{3}}$ and $24^{\frac{1}{3}}$.

Firſt, $192^{\frac{1}{3}} = (64 \times 3)^{\frac{1}{3}} = 4.3^{\frac{1}{3}}$; and $24^{\frac{1}{3}} = (8 \times 3)^{\frac{1}{3}} = 2.3^{\frac{1}{3}}$.

Then $4.3^{\frac{1}{3}} - 2.3^{\frac{1}{3}} = 2.3^{\frac{1}{3}}$ the difference required.

3. Required the diff. of $2\sqrt{50}$ and $\sqrt{18}$. Anſ. $7\sqrt{2}$.
4. Required the diff. of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$. Anſ. $2.5^{\frac{1}{3}}$.
5. Required the diff. of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{2}\frac{5}{7}}$. Anſ. $\frac{4}{4}\frac{}{5}\sqrt{15}$.
6. Required the diff. of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{3}\frac{}{2}}$. Anſ. $\frac{1}{1}\frac{}{2}\sqrt[3]{18}$.
7. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.
Anſ. $(4a^2 - 2ax)\sqrt{5x}$.
8. Required the difference of $8\sqrt[3]{a^3b}$ and $\sqrt[3]{a^6b}$.
9. Required the difference of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PRO-

PROBLEM VI.

To Multiply Surd Quantities together.

RULE.

REDUCE the ſurds to the ſame index; next multiply the rational quantities together, and the ſurds together; then the one product annexed to the other will give the whole product required; which may be reduced to its moſt ſimple terms by Problem 3.

EXAMPLES.

1. Required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here, $3 \times 2 \times \sqrt{8} \times \sqrt{6} = 6\sqrt{8 \times 6} = 6\sqrt{48} = 6\sqrt{16 \times 3}$
$= 6 \times 4 \times \sqrt{3} = 24\sqrt{3}$, their product required.

2. Required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{8}}$.

Here $\frac{1}{2} \times \frac{3}{4}\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{5}{6}} = \frac{3}{8} \times \sqrt[3]{\frac{10}{18}} = \frac{3}{8} \times \sqrt[3]{\frac{5}{9}} = \frac{3}{8} \times \frac{1}{3} \times \sqrt[3]{15} = \frac{3}{24}\sqrt[3]{15} = \frac{1}{8}\sqrt[3]{15}$ the product required.

3. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$. Anſ. $30\sqrt{10}$.

4. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$. Anſ. $\sqrt[3]{4}$.

5. To find the product of $\frac{2}{3}\sqrt{\frac{1}{6}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$. Anſ. $\frac{1}{40}\sqrt{35}$.

6. Required the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$. Anſ. $10\sqrt[3]{9}$.

7. Required the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$. Anſ. $(a^3)^{\frac{1}{3}}$ or a.

8. Required the product of $(x+y)^{\frac{1}{2}}$ and $(x+y)^{\frac{1}{3}}$.

9. Required the product of $x + \sqrt{y}$ and $x - \sqrt{y}$.

10. Required the product of $(a + \sqrt{b})^{\frac{1}{2}}$, and $(a - \sqrt{b})^{\frac{1}{2}}$.

11. Required the product of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PROBLEM VII.

To Divide one Surd Quantity by another.

RULE.

REDUCE the ſurds to the ſame index; then take the quotient of the rational quantities, and annex it to the quotient of the ſurds, and it will give the whole quotient required; which may be reduced to its moſt ſimple terms as before in multiplication.

EXAM-

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$8 \div 2.\sqrt{(108 \div 6)} = 4\sqrt{18} = 4\sqrt{(9 \times 2)} = 4 \times 3\sqrt{2} = 12\sqrt{2}$, the quotient required.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

$8 \div 4 = 2$, and $512^{\frac{1}{3}} \div 2^{\frac{1}{3}} = 256^{\frac{1}{3}} = 4.4^{\frac{1}{3}}$;

therefore $2 \times 4 \times 4^{\frac{1}{3}} = 8.4^{\frac{1}{3}} = 8\sqrt[3]{4}$, is the quot. required.

3. Let $6\sqrt{100}$ be divided by $3\sqrt{2}$. Anſ. $10\sqrt{2}$.

4. Let $4\sqrt[3]{1000}$ be divided by $2\sqrt[3]{4}$. Anſ. $10\sqrt[3]{2}$.

5. Let $\frac{3}{4}\sqrt{1\frac{1}{35}}$ be divided by $\frac{2}{3}\sqrt{\frac{1}{5}}$. Anſ. $\frac{9}{8}\sqrt{3}$.

6. Let $\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ be divided by $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$. Anſ. $\frac{25}{21}\sqrt[3]{3}$.

7. Let $\frac{2}{5}\sqrt{a}$, or $\frac{2}{5}a^{\frac{1}{2}}$, be divided by $\frac{3}{4}a^{\frac{1}{3}}$. Anſ. $\frac{8}{15}a^{\frac{1}{6}}$.

8. Let the quantity $x^{\frac{1}{n}}$ be divided by the quantity $x^{\frac{1}{m}}$.

9. Let $x^2 - xd - b + d\sqrt{b}$ be divided by $x - \sqrt{b}$.

PROBLEM VIII.

To Involve or Raiſe Surd Quantities to any Power.

RULE.

MULTIPLY the index of the quantity by the index of the power to which it is to be raiſed, and to the reſult annex the power of the rational parts, and it will give the power required.

EXAMPLES.

1. It is required to find the ſquare of $\frac{2}{3}a^{\frac{1}{3}}$.

Firſt, $(\frac{2}{3})^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and $(a^{\frac{1}{3}})^2 = a^{\frac{1}{3} \times 2} = a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}$;

theref. $(\frac{2}{3}a^{\frac{1}{3}})^2 = \frac{4}{9}(a^2)^{\frac{1}{3}} = \frac{4}{9}\sqrt[3]{a^2}$, the ſquare required.

2. It is required to find the cube of $\frac{5}{7}\sqrt{7}$.

Firſt, $(\frac{5}{7})^3 = \frac{5}{7} \times \frac{5}{7} \times \frac{5}{7} = \frac{125}{343}$, and $(7^{\frac{1}{2}})^3 = 7^{\frac{3}{2}} = (7^3)^{\frac{1}{2}}$;

theref. $(\frac{5}{7}\sqrt{7})^3 = \frac{125}{343}(7^3)^{\frac{1}{2}} = \frac{125}{343}(343)^{\frac{1}{2}}$, the cube required.

3. Required the ſquare of $3\sqrt[3]{3}$. Anſ. $9\sqrt[3]{9}$.

4. Required the cube of $2^{\frac{1}{2}}$, or $\sqrt{2}$. Anſ. $2\sqrt{2}$.

5. Required the 4th power of $\frac{1}{6}\sqrt{6}$. Anſ. $\frac{1}{36}$.

6. It

6. It is required to find the nth power of $a^{\frac{r}{m}}$.
7. It is required to find the ſquare of $3 + \sqrt{5}$.
8. It is required to find the cube of $2x - 3\sqrt{y}$.

PROBLEM IX.

To Extract the Roots of Surd Quantities.

RULE*.

DIVIDE the index of the given quantity by the index of the root to be extracted; then to the reſult annex the root of the rational part, and it will give the root required.

EXAMPLES.

1. It is required to find the ſquare root of $9\sqrt[3]{3}$.

Firſt, $\sqrt{9} = 3$, and $(3^{\frac{1}{3}})^{\frac{1}{2}} = 3^{\frac{1}{3} \div 2} = 3^{\frac{1}{6}}$;
therefore $(9\sqrt[3]{3})^{\frac{1}{2}} = 3 \cdot 3^{\frac{1}{6}}$, is the ſquare root required.

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

Firſt, $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$, and $(\sqrt{2})^{\frac{1}{3}} = 2^{\frac{1}{2} \div 3} = 2^{\frac{1}{6}}$;
therefore $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}} = \frac{1}{2} \cdot 2^{\frac{1}{6}}$, is the cube root required.

3. Required the ſquare root of 10^3. Anſ. $10\sqrt{10}$.
4. Required the cube root of $\frac{3}{27}a^3$ Anſ. $\frac{2}{3}a$.
5. Required the 4th root of $3x^2$. Anſ. $3^{\frac{1}{4}} \cdot x^{\frac{1}{2}}$.
6. It is required to find the nth root of $x^{\frac{r}{m}}$.
7. Required the ſquare root of $x^2 - 4x\sqrt{a} + 4a$.

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* The ſquare root of a binomial or reſidual ſurd, $A + B$, or $A - B$, may be found thus: Take $\sqrt{A^2 - B^2} = D$;

then $\sqrt{A + B} = \sqrt{\frac{A + D}{2}} + \sqrt{\frac{A - D}{2}}$,

and $\sqrt{A - B} = \sqrt{\frac{A + D}{2}} - \sqrt{\frac{A - D}{2}}$.

Thus, the ſquare root of $8 + 2\sqrt{7} = 1 + \sqrt{7}$;
and the ſquare root of $3 - \sqrt{8} = \sqrt{2} - 1$:
but for the cube, or any higher root, no general rule is given.

INFINITE SERIES.

An Infinite Series is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a ſurd quantity; and is ſuch as, being continued, would run on infinitely, in the manner of a decimal fraction.

But, by obtaining a few of the firſt terms, the law of the progreſſion will be manifeſt, ſo that the ſeries may be continued, without actually performing the whole operation.

PROBLEM I.

To Reduce Fractional Quantities into Infinite Series.

RULE.

Divide the numerator by the denominator, as in common diviſion; and the operation continued, as far as may be thought neceſſary, will give the ſeries required.

EXAMPLES.

1. To change $\frac{ax}{a-x}$ into an infinite ſeries.

$$a-x \;)\; ax \;.\;.\; (\; x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \text{\&c.}$$

$$ax - x^2$$

$$x^2 - \frac{x^3}{a}$$

$$\frac{x^3}{a}$$

$$\frac{x^3}{a} - \frac{x^4}{a^2}$$

$$\frac{x^4}{a^2}$$

$$\frac{x^4}{a^2} - \frac{x^5}{a^3}$$

$$\frac{x^5}{a^3}, \text{ \&c.}$$

2. Let

2. Let $\frac{1}{1+x}$ be converted into an infinite series.

$$
\begin{array}{l}
1+x\,)\,1\,\ldots\ldots(\,1-x+x^2-x^3+x^4-\&c. \\
\quad\underline{1+x} \\
\quad\;\; -x \\
\quad\;\; \underline{-x-x^2} \\
\qquad\qquad x^2 \\
\qquad\qquad \underline{x^2-x^3} \\
\qquad\qquad\quad -x^3 \\
\qquad\qquad\quad \underline{-x^3-x^4} \\
\qquad\qquad\qquad\quad x^4 \\
\qquad\qquad\qquad\quad \underline{}
\end{array}
$$

3. Let $\frac{b}{a+x}$ be converted into an infinite series.

Anſ. $\frac{b}{a} \times (1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \&c.$

4. Let $\frac{b}{a-x}$ be converted into an infinite ſeries.

Anſ. $\frac{b}{a} \times (1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3}$ &c.)

5. Let $\frac{1+x}{1-x}$ be converted into an infinite ſeries:

Anſ. $1 + 2x + 2x^2 + 2x^3 + 2x^4$ &c.

6. Let $\frac{a^2}{(a+x)^2}$ be converted into an infinite ſeries.

Anſ. $1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3}$ &c.

7. Let $\frac{1}{2}$, or its equal $\frac{1}{1+1}$, be converted into an infinite ſeries.

PROBLEM II.

To Reduce a Compound Surd into an Infinite Series.

RULE*.

EXTRACT the root as in common arithmetic, and the operation, continued as far as may be thought neceſſary, will give the ſeries required.

* This rule is chiefly of uſe in extracting the ſquare root, the operation being too tedious for the higher powers.

EXAM-

EXAMPLES.

1. Extract the root of $a^2 + x^2$ in an infinite series.

$$a^2 + x^2 \quad \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \text{ &c.}\right.$$

$$\frac{a^2}{2a + \frac{x^2}{2a}) \quad x^2}$$

$$\frac{x^2 + \frac{x^4}{4a^2}}{2a + \frac{x^2}{a} - \frac{x^4}{8a^3}) \quad - \frac{x^4}{4a^2}}$$

$$\frac{- \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}}{2a + \frac{x^2}{a} - \frac{x^4}{8a^2}, \quad \text{&c.}) \quad \frac{x^6}{8a^4} - \frac{x^8}{64a^6}}$$

$$\frac{\frac{x^6}{8a^4} + \frac{x^8}{16a^6}, \text{ &c.}}{- \frac{5x^8}{64a^6}, \text{ &c.}}$$

2. Let $\sqrt{1+1}$ be converted into an infinite series.

Anf. $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$ &c.

3. Let $\sqrt{a^2 - x^2}$ be converted into an infinite series.

Anf. $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

4. Let $\sqrt[3]{1 - x^3}$ be converted into an infinite series.

Anf. $1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81}$ &c.

5. Let $\sqrt{a^2 + b}$ be converted into an infinite series.

PROBLEM III.

To Reduce a Binomial Surd into an Infinite Series; or to Extract any Root of a Binomial.

RULE*.

SUBSTITUTE the particular letters of the binomial, with their proper signs, in the following general form, and it will give the root required; obſerving that P is the firſt term, Q the ſecond term divided by the firſt, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c, the foregoing terms with their proper signs:

$$\overline{P + QP^n}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \&c.$$

EXAMPLES.

1. To extract the ſq. root of $r^2 - x^2$, in an infinite ſeries.

Here $P = r^2$, $Q = -\frac{x^2}{r^2}$, and $\frac{m}{n} = \frac{1}{2}$; therefore

$$P^{\frac{m}{n}} = (r^2)^{\frac{m}{n}} = (r^2)^{\frac{1}{2}} = r = A,$$

$$\frac{m}{n} AQ = \tfrac{1}{2} \times r \times -\frac{x^2}{r^2} = -\frac{x^2}{2r} = B,$$

$$\frac{m-n}{2n} BQ = \frac{1-2}{4} \times -\frac{x^2}{2r} \times -\frac{x^2}{r^2} = -\frac{x^4}{2.4r^3} = C,$$

$$\frac{m-2n}{3n} CQ = \frac{1-4}{6} \times -\frac{x^4}{2.4r^3} \times -\frac{x^2}{r^2} = -\frac{3x^6}{2.4.6r^5} = D,$$

Hence $r - \frac{x^2}{2r} - \frac{x^4}{2.4r^3} - \frac{3x^6}{2.4.6r^5} - \&c$, or

$r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}$ &c, is the ſeries required.

2. To find the value of $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$, in an infinite ſeries.

---

* Any ſurd may be taken from the denominator of a fraction and placed in the numerator, and vice verſa, by only changing the ſign of its index.

Here

Here $P = a$, $Q = \frac{b}{a} = a^{-1}b$, and $\frac{m}{n} = \frac{-2}{1} = -2$; therefore

$P^{\frac{m}{n}} = (a)^{-2} = a^{-2} = \frac{1}{a^2} = A$,

$\frac{m}{n} AQ = -2 \times \frac{1}{a^2} \times \frac{b}{a} = -\frac{2b}{a^3} = -2a^{-3}b = B$,

$\frac{m-n}{2n} BQ = -\frac{3}{2} \times -\frac{2b}{a^3} \times \frac{b}{a} = \frac{3b^2}{a^4} = 3a^{-4}b^2 = C$,

$\frac{m-2n}{3n} CQ = -\frac{4}{3} \times \frac{3b^2}{a^4} \times \frac{b}{a} = -\frac{4b^3}{a^5} = -4a^{-5}b^3 = D$,

Hence $a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 +$ &c, or

$\frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6}$ &c, is the series required.

3. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

Anf. $r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3}$, &c.

4. To find the value of $\frac{1}{(a^2-x^2)^{\frac{1}{2}}}$ in an infinite series.

Anf. $\frac{1}{a} + \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} + \frac{15x^6}{48a^7}$, &c.

5. To find the value of $\frac{a^2}{(a+x)^2}$ in an infinite series.

Anf. $1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4}$, &c.

6. To find the value of $(a^2 + b)^{\frac{1}{2}}$ in an infinite series.

Anf. $a + \frac{b}{2a} - \frac{b^2}{8a^3} + \frac{b^3}{16a^5} - \frac{5b^4}{128a^7}$, &c.

7. Find the value of $(a^2 - x^2)^{\frac{1}{5}}$ in an infinite series.

Anf. $a^{\frac{2}{5}} \times : 1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6}$, &c.

8. To find the value of $(a^3 - b)^{\frac{1}{3}}$ in an infinite series.

Anf. $a - \frac{b}{3a^2} - \frac{b^2}{9a^5} - \frac{5b^3}{81a^8} - \frac{10b^4}{243a^{11}}$, &c.

9. To find the square root of $\frac{a^2+x^2}{a^2-x^2}$ in an infinite series.

Anf. $1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^6}$, &c.

10. Find

10. Find the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite ſeries.

Anſ. $\frac{1}{a^{\frac{2}{3}}} \times : 1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}$, &c.

11. Find the value of $\frac{ax}{a^2 - ax + x^2}$ in an infinite ſeries.

Anſ. $\frac{x}{a} + \frac{x^2}{a^2} - \frac{x^4}{a^4} - \frac{x^5}{a^5}$, &c.

ARITHMETICAL PROPORTION.

ARITHMETICAL PROPORTION is the relation which two quantities, of the ſame kind, bear to each other, in reſpect to their difference.

Four quantities are ſaid to be in Arithmetical Proportion, when the difference between the firſt and ſecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, and a, $a + b$, c, $c + b$, are arithmetically proportional.

Arithmetical Progreſſion is when a ſeries of quantities either increaſe or decreaſe by the ſame common difference.

Thus, 1, 3, 5, 7, 9, 11, &c, and a, $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, $a + 5b$, &c, are ſeries in arithmetical progreſſion, whoſe common differences are 2 and b.

The moſt uſeful part of arithmetical proportion is contained in the following theorems:

1. When four quantities are in Arithmetical Proportion, the ſum of the two extremes is equal to the ſum of the two means.

Thus, if 2, 5, 7, 10, and a, b, c, d, be in arithmetical proportion; then is $2 + 10 = 5 + 7$, and $a + d = b + c$.

2. In any continued arithmetical progreſſion, the ſum of the two extremes is equal to the ſum of any two terms which are equally diſtant from them.

Thus,

Thus, in the ſeries 2, 4, 6, 8, 10, 12, &c.
Here $2 + 12 = 4 + 10 = 6 + 8 = 14$.

3. The laſt term of any arithmetical ſeries, is equal to the ſum, or difference, of the firſt term, and the product of the common difference multiplied by the number of terms leſs one; according as the ſeries is increaſing or decreaſing.

Thus, the 20th term of 2, 4, 6, 8, 10, 12, &c, is $= 2 + 2\,(20 - 1) = 2 + 2 \times 19 = 2 + 38 = 40$.

And the nth term of a, $a - x$, $a - 2x$, $a - 3x$, $a - 4x$, &c, is $= a - (n - 1) \times x = a - (n - 1)\,x$.

4. The ſum of any ſeries of quantities in arithmetical progreſſion, is equal to the ſum of the two extremes multiplied by half the number of terms.

Thus, the ſum of 1, 2, 3, 4, 5, 6, &c, continued to the 20th term, is $= \frac{(1 + 20) \times 20}{2} = \frac{21 \times 20}{2} = 21 \times 10 = 210$.

And the ſum of n terms of a, $a+x$, $a+2x$, $a+3x$, to $a+mx$, is $= (a + a + mx).\frac{n}{2} = (a + \frac{1}{2}mx).n = (a + \frac{n-1}{2}x)\,n$.

EXAMPLES.

1. The firſt term of an increaſing arithmetical ſeries is 3, the common difference 2, and the number of terms 20; required the ſum of the ſeries?

Firſt, $3 + 2 \times (20 - 1) = 3 + 2 \times 19 = 3 + 38 = 41$, the laſt term.

And $(3 + 41) \times \frac{20}{2} = 44 \times 10 = 440$, the ſum required.

2. The firſt term of a decreaſing arithmetical ſeries is 100, the common difference 3, and the number of terms 34; required the ſum of the ſeries?

Firſt, $100 - 3 \cdot (34 - 1) = 100 - 3 \cdot (33) = 100 - 99 = 1$, the laſt term.

And $(100 + 1) \times \frac{34}{2} = 101 \times 17 = 1717$, the ſum required.

3. Required the ſum of the neutral numbers 1, 2, 3, 4, 5, 6, &c, continued to 1000 terms? Anſ. 500500.

4. * Required the ſum of the odd numbers 1, 3, 5, 7, 9, &c, continued to 101 terms? Anſ. 10201.

5. How many ſtrokes do the clocks of Venice, which go on to 24 o'clock, ſtrike in the compaſs of a day? Anſ. 300.

6. The firſt term of a decreaſing arithmetical ſeries is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the ſum of the ſeries? Anſ. 140.

7. One hundred ſtones being placed on the ground, in a ſtraight line, at the diſtance of 2 yards from each-other; how far will a perſon travel, who ſhall bring them one by one to a baſket, which is placed 2 yards from the firſt ſtone? Anſ. 11 miles and 840 yards.

APPLICATION OF ARITHMETICAL PROGRESSION TO MILITARY AFFAIRS.

QUESTION I.

A TRIANGULAR † Battalion, conſiſting of thirty ranks, in which the firſt rank is formed of one man only, the ſecond

* The ſum of any number of terms (n) of the arithmetical ſeries of odd numbers 1, 3, 5, 7, 9, &c, is equal to the ſquare (n^2) of that number.

That is, if 1, 3, 5, 7, 9, &c, be the numbers, then will $1, 2^2, 3^2, 4^2, 5^2$, &c, be the ſums of 1, 2, 3, &c, terms.

For $0 + 1$, or the ſum of 1 term $= 1^2$, or 1
$1 + 3$, or the ſum of 2 terms $= 2^2$, or 4
$4 + 5$, or the ſum of 3 terms $= 3^2$, or 9
$9 + 7$, or the ſum of 4 terms $= 4^2$, or 16, &c.

Whence it is plain, that, let n be any number whatever, the ſum of n terms will be n^2.

See more on Arithmetical Proportion in the Arithmetic, p. 96.

† By triangular battalion, is to be underſtood, a body of troops ranged in the form of a triangle, in which the ranks exceed each other by an equal number of men: if the firſt rank conſiſt of one man only, and the difference between the ranks be alſo one, then its

ſecond of 3, the third of 5, and ſo on: What is the ſtrength of ſuch a triangular battalion?

Anſwer, 900 men.

QUESTION II.

400 men being given to be drawn up in a triangular battalion of an equilateral form, what is the number of ranks?

Anſwer, 27 ranks, and 22 men over.

QUESTION III.

A detachment having 12 ſucceſſive days to march, with orders to advance the firſt day only 2 leagues, the ſecond $3\frac{1}{2}$, and ſo on, increaſing $1\frac{1}{2}$ league each day's march: What is the length of the whole march, and what is the laſt day's march?

Anſwer, the laſt day's march is $18\frac{1}{2}$ leagues, and 123 leagues is the length of the whole march.

QUESTION IV.

A brigade * of ſappers, having carried on 15 yards of ſap the firſt night, the ſecond only 13 yards, and ſo on, decreaſing 2 yards every night, till at laſt they carried on in one night only 3 yards: What is the number of nights they were employed; and what is the whole length of the ſap?

Anſwer, they were employed 7 nights, and the length of the whole ſap was 63 yards.

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its form is that of an equilateral triangle; and when the difference between the ranks is more than one, its form may then be an iſoſceles or ſcalene triangle. The practice of forming troops in this order, which is now laid aſide, was formerly held in greater eſteem than forming them in a ſolid ſquare, as admitting of a greater front, eſpecially when the troops were to make ſimply a ſtand on all ſides.

* A brigade of ſappers, conſiſts generally of 8 men, divided equally into two parties; and whilſt one of theſe parties is advancing the ſap, the other is furniſhing the gabions, faſcines, and other neceſſary implements: and when the firſt party is tired, the ſecond takes its place, and ſo on, till each man in turn has been at the head of the ſap. A ſap is a ſmall ditch, between 3 and 4 feet in breadth and depth; and is diſtinguiſhed from the trench by its breadth only, the trench having between 10 and 15 feet breadth. As an encouragement to ſappers, the pay for all the work carried on by the whole brigade, is given to the ſurvivors.

QUESTION V.

A number of gabions * being given to be placed in ſix ranks, one above the other, in ſuch a manner as that each rank exceeding one another equally, the firſt may conſiſt of 4 gabions, and the laſt of 9: What is the number of gabions in the ſix ranks? and what is the difference between each rank?

Anſwer, the difference between the ranks will be one, and the number of gabions in the ſix ranks will be 39.

QUESTION VI.

Two detachments, diſtant from each other 37 leagues, and both deſigning to occupy an advantageous poſt equidiſtant from each others camp, ſet out at different times; the firſt detachment increaſing every day's march one league and a half, and the ſecond detachment increaſing each day's march 2 leagues: both the detachments arrive at the ſame time; the firſt after five days march, and the ſecond after four days march: What is the number of leagues marched by each detachment each day?

The progreſſion $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$, anſwers the conditions of the firſt detachment: and the progreſſion $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$, anſwers the conditions of the ſecond detachment.

QUESTION VII.

A deſerter, in his flight, travelling at the rate of 8 leagues a day; and a detachment of dragoons being ſent after him, with orders to march the firſt day only 2 leagues, the ſecond 5 leagues, the third 8 leagues, and ſo on: What is the number of days neceſſary for the detachment to overtake the deſerter, and what will be the number of leagues marched before he is overtaken?

Anſwer, 5 days are neceſſary to overtake him; and, conſequently, 40 leagues will be the extent of the march.

* Gabions are baſkets open at both ends, made of ozier twigs, and of a cylindrical form: thoſe made uſe of at the trenches are 2 feet wide, and about three feet high; which being filled with earth, ſerve as a ſhelter from the enemy's fire: and thoſe made uſe of to conſtruct batteries, are generally higher and broader. There is another ſort of gabion, made uſe of to raiſe a low parapet; its height is from 1 to 2 feet, and 1 foot wide at top, but ſomewhat leſs at bottom, to give room for placing the muzzle of a firelock between them: theſe gabions ſerve inſtead of ſand bags. A ſand bag is generally made to contain about a cubical foot of earth.

QUES-

QUESTION VIII.

A convoy * diſtant 35 leagues, having orders to join its camp, and to march at the rate of 5 leagues per day; its eſcort departing at the ſame time, with orders to march the firſt day only half a league, and the laſt day $9\frac{1}{2}$ leagues; and both the eſcort and convoy arriving at the ſame time: At what diſtance is the eſcort from the convoy at the end of each march?

OF COMPUTING SHOT OR SHELLS IN A FINISHED PILE.

SHOT and Shells are generally piled in three different forms, called triangular, ſquare, or oblong piles, according as their baſe is either a triangle, a ſquare, or a rectangle.

Fig. 1.

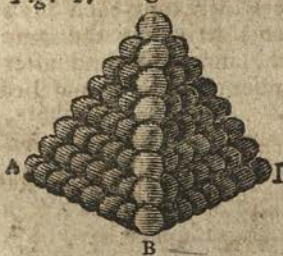
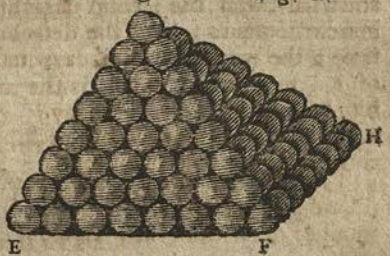
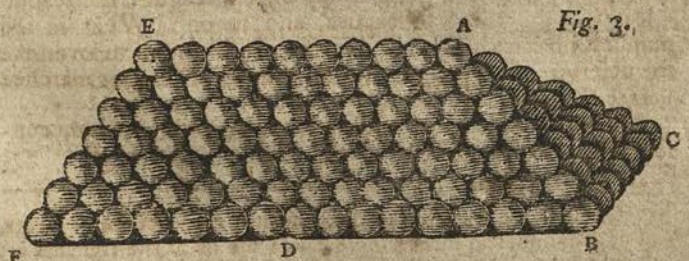


Fig. 2.



ABCD, fig. 1, is a triangular pile,
EFGH, fig. 2, is a ſquare pile.

Fig. 3.



ABCDEF, fig. 3, is an oblong pile.

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* By convoy is generally meant a ſupply of ammunition, or proviſions, conveyed to a town or army. The body of men that guard this ſupply, is called eſcort.

A tri-

A triangular pile is formed by the continual laying of triangular horizontal courſes of ſhot one above another, in ſuch a manner, as that the ſides of theſe courſes, called rows, decreaſe by unity from the bottom row to the top row, which ends always in 1 ſhot.

A ſquare pile is formed by the continual laying of ſquare horizontal courſes of ſhot one above another, in ſuch a manner, as that the ſides of theſe courſes decreaſe by unity from the bottom to the top row, which ends alſo in 1 ſhot.

In the triangular and the ſquare piles, the ſides or faces being equilateral triangles, the ſhot contained in thoſe faces form an arithmetical progreſſion, having for firſt term unity, and for laſt term and number of terms, the ſhot contained in the bottom row; for the number of horizontal rows, or the number counted on one of the angles from the bottom to the top, is always equal to thoſe counted on one ſide in the bottom: the ſides or faces in either the triangular or ſquare piles, are called arithmetical triangles; and the numbers contained in theſe, are called triangular numbers; ABC fig. 1, EFG fig. 2, are arithmetical triangles.

The oblong pile may be conceived as formed from the ſquare pile ABCD; to one ſide or face of which, as AD, a number of arithmetical triangles equal to the face have been added: and the number of arithmetical triangles added to the ſquare pile, by means of which the oblong pile is formed, is always equal to the ſhot in the top row leſs one; or, which is the ſame, the difference between the bottom row of the greater ſide and that of the leſſer.

QUESTION IX.

To find the ſhot in the triangular pile ABCD, fig. 1, the bottom row AB conſiſting of 8 ſhot.

SOLUTION.

The propoſed pile conſiſting of 8 horizontal courſes, each of which forms an equilateral triangle; that is, the ſhot contained in theſe being in an arithmetical progreſſion, of which the firſt and laſt term, as alſo the number of terms, are known; it follows, that the ſum of theſe particular courſes, or of the 8 progreſſions, will be the ſhot contained in the propoſed pile; then

The ſhot of the firſt or lower triangular courſe will be	$\overline{8+1} \times 4 = 36$
the ſecond - - -	$\overline{7+1} \times 3\frac{1}{2} = 28$
the third - - -	$\overline{6+1} \times 3 = 21$
the fourth - - -	$\overline{5+1} \times 2\frac{1}{2} = 15$
the fifth - - -	$\overline{4+1} \times 2 = 10$
the ſixth - - -	$\overline{3+1} \times 1\frac{1}{2} = 6$
the ſeventh - - -	$\overline{2+1} \times 1 = 3$
the eighth - - -	$\overline{1+1} \times \frac{1}{2} = 1$
Total -	120 ſhot in the pile propoſed.

QUESTION X.

To find the ſhot of the ſquare pile EFGH, fig. 2, the bottom row EF conſiſting of 8 ſhot.

SOLUTION.

The bottom row containing 8 ſhot, and the ſecond only 7; that is, the rows forming the progreſſion 8, 7, 6, 5, 4, 3, 2, 1, in which each of the terms being the ſquare root of the ſhot contained in each ſeparate ſquare courſe employed in forming the ſquare pile; it follows, that the ſum of the ſquares of theſe roots will be the ſhot required: and the ſum of the ſquares = 8. 7. 6. 5. 4. 3. 2. 1, being 204, expreſſes the ſhot in the propoſed pile.

QUESTION XI.

To find the ſhot of the oblong pile ABCDEF, fig. 3; in which BF = 16, and BC = 7.

SOLUTION.

The oblong pile propoſed, conſiſting of the ſquare pile ABCD, whoſe bottom row is 7 ſhot; beſides 9 arithmetical triangles or progreſſions, in which the firſt and laſt term, as alſo the number of terms, are known; it follows, that

if to the contents of the ſquare pile -	140
we add the ſum of the 9th progreſſion -	252
their total gives the contents required -	392 ſhot.

REMARK I.

The ſhot in the triangular and the ſquare piles, as alſo the ſhot in each horizontal courſe, may at once be aſcertained

tained by the following table: the vertical column A, contains the ſhot in the bottom row, from 1 to 20 incluſive; the column B contains the triangular numbers, or number of each courſe; the column C contains the ſum of the triangular numbers, that is, the ſhot contained in a triangular pile, commonly called pyramidal numbers; the column D contains the ſquare of the numbers of the column A, that is, the ſhot contained in each ſquare horizontal courſe; and the column F contains the ſum of theſe ſquares or ſhot in a ſquare pile.

C	B	A	D	F
Pyramidal numbers.	Triangular numbers.	Natural numbers.	Square of the natural numbers.	Sum of theſe ſquare numbers.
1	1	1	1	1
4	3	2	4	5
10	6	3	9	14
20	10	4	16	30
35	15	5	25	55
56	21	6	36	91
84	28	7	49	140
120	36	8	64	204
165	45	9	81	285
220	55	10	100	385
286	66	11	121	506
364	78	12	144	650
455	91	13	169	819
560	105	14	196	1015
680	120	15	225	1240
816	136	16	256	1496
969	153	17	289	1785
1140	171	18	324	2109
1330	190	19	361	2470
1540	210	20	400	2870

Thus, the bottom row in the triangular pile, conſiſting of 9 ſhot, the contents will be 165; and when of 9 in the ſquare pile, 285.—In the ſame manner, the contents either of a ſquare or triangular pile being given, the ſhot in the bottom row may be eaſily aſcertained.

The contents of any oblong pile by the preceding table may be alſo with little trouble aſcertained, the leſſer ſide not exceeding 20 ſhot, nor the difference between the leſſer and the greater ſide 20; thus, to find the ſhot in an oblong pile, the

the leſſer ſide being 15, and the greater 35, we are firſt to find the contents of the ſquare pile, by means of which the oblong pile may be conceived to be formed; that is, we are to find the contents of a ſquare pile, whoſe bottom row is 15 ſhot; which being 1240, we are, ſecondly, to add theſe 1240 to the product 2400 of the triangular number 120, anſwering to 15, the number expreſſing the bottom row of the arithmetical triangle, multiplied by 20, the number of thoſe triangles; and their ſum, being 3640, expreſſes the number of ſhot in the propoſed oblong pile.

REMARK II.

The following algebraical expreſſions, deduced from the inveſtigations of the ſums of the powers of numbers in arithmetical progreſſion, which are ſeen upon many gunners callipers*, ſerve to compute with eaſe and expedition the ſhot or ſhells in any pile.

That ſerving to compute any triangular pile, is repreſented by $\frac{\overline{n+2} \times \overline{n+1} \times n}{6}$.

That ſerving to compute any ſquare pile, is repreſented by $\frac{\overline{n+1} \times \overline{2n+1} \times n}{6}$.

In each of theſe, the letter n repreſents the number in the bottom row; hence in a triangular pile, the number in the bottom row being 30; then this pile will be $\overline{30+2} \times \overline{30+1} \times \frac{30}{6} = 4960$ ſhot or ſhells. In a ſquare pile, the number in the bottom row being alſo 30; then this pile will be $\overline{30+1} \times \overline{60+1} \times \frac{30}{6} = 9455$ ſhot or ſhells.

That ſerving to compute any oblong pile, is repreſented by $\frac{\overline{2n+1+3m} \times \overline{n+1} \times n}{6}$, in which the letter n denotes

* Callipers are large compaſſes, with bowed ſhanks, ſerving to take the diameters of convex and concave bodies. The gunners callipers conſiſt of two thin rules or plates, which are moveable quite round a joint, by the plates folding one over the other: the length of each rule or plate is ſix inches, the breadth about one inch. It is uſual to repreſent, on the plates, a variety of ſcales, tables, proportions, &c. ſuch as are eſteemed uſeful to be known by perſons employed about artillery; but, except the meaſuring of the caliber of ſhot and cannon, and the meaſuring of ſaliant and re-entering angles, none of the articles, with which the callipers are uſually filled, are eſſential to that inſtrument.

the

the number of courſes, and the letter m the number of ſhot, leſs one, in the top row: hence, in an oblong pile the number of courſes being 30, and the top row 31; this pile will be $\overline{60 + 1} \times \overline{90 \times 30 + 1} \times \frac{30}{6} = 23405$ ſhot or ſhells.

GEOMETRICAL PROPORTION.

GEOMETRICAL PROPORTION is that relation of two quantities of the ſame kind, which ariſes from conſidering what part the one is of the other, or how often it is contained in it.

When two quantities are compared together, the firſt is called the Antecedent, and the ſecond the Conſequent.

Ratio is the quotient which ariſes from dividing the antecedent by the conſequent, or the conſequent by the antecedent.

Four Quantities are ſaid to be proportional, when the firſt is the ſame part or multiple of the ſecond, as the third is of the fourth.

Thus, 2, 8, 3, 12, and a, ar, b, br, are geometrical proportionals.

Direct Proportion is when the ſame relation ſubſiſts between the firſt term and the ſecond, as between the third and the fourth.

Thus, 3, 6, 5, 10, and x, ax, y, ay, are in direct proportion.

Reciprocal, or Inverſe Proportion, is when one quantity increaſes in the ſame proportion as another diminiſhes.

Thus, 2, 6, 9, 3, and a, ar, br, b, are in inverſe proportion.

A Series of Quantities are ſaid to be in geometrical progreſſion, when the firſt has the ſame ratio to the ſecond as the ſecond to the third, and the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, &c, and a, ar, ar^2, ar^3, ar^4, ar^5, &c, are ſeries in geometrical progreſſion.

The

The moſt uſeful part of geometrical proportion, is contained in the following theorems:

1. If four quantities be in geometrical proportion, the product of the two means will be equal to the product of the two extremes.

Thus, if 2, 4, 6, 12, and a, ar, b, br, be geometrically proportional, then will $2 \times 12 = 4 \times 6$, and $a \times br = b \times ar$.

2. If four quantities be in geometrical proportion, the rectangle or product of the means divided by either of the extremes will give the other extreme.

Thus, if 3, 9, 5, 15, and x, ax, y, ay, are geometrically proportional, then will $\frac{9 \times 5}{3} = 15$, and $\frac{ax \times y}{ay} = x$

And this is the foundation of the Rule-of-Three.

3. In any continued geometrical progreſſion, the product of the two extremes, and that of any other two terms, equally diſtant from them, will be equal to each other.

Thus, in the ſeries 1, 3, 9, 27, 81, 243, &c,
it is $1 \times 243 = 3 \times 81 = 9 \times 27 = 243$.

4. In any continued geometrical ſeries, the laſt term, is equal to the firſt multiplied by ſuch a power of the ratio, as is denoted by the number of terms leſs one.

Thus, in the ſeries 2, 6, 18, 54, 162, &c; $2 \times 3^4 = 162$.

5. The ſum of any ſeries in geometrical progreſſion, is found by multiplying the laſt term by the ratio, and dividing the difference of this product and the firſt term by the ratio leſs one.

Thus, the ſum of 2, 4, 8, 16, 32, 64, 128, 256, 512, is $\frac{512 \times 2 - 2}{2 - 1} = 1024 - 2 = 1022$.

And the ſum of n terms of a, ar, ar^2, ar^3, ar^4, &c, to ar^{n-1}, is $\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{r^n - 1}{r - 1} a$.

6. If four quantities, a, b, c, d, or 2, 6, 5, 15, be proportional; then will any of the following forms of thoſe quantities be alſo proportional, viz.

1. Directly, $a : b :: c : d$ or $2 : 6 :: 5 : 15$.
2. Inverſely, $b : a :: d : c$ or $6 : 2 :: 15 : 5$.
3. Alternately, $a : c :: b : d$ or $2 : 6 :: 5 : 15$.

4. Com-

4. Compoundedly, $a : a + b :: c : c + d$ or $2 : 8 :: 5 : 20$.
5. Dividedly, $a : b - a :: c : d - c$ or $2 : 4 :: 5 : 10$.
6. Mixed, $b + a : b - a :: d + c : d - c$ or $8 : 4 :: 20 : 10$.
7. Multiplication, $ra : rb :: c : d$ or $2.3 : 6.3 :: 5 : 15$.
8. Diviſion, $a \div r : b \div r :: c : d$ or $1 : 3 :: 5 : 15$.
9. The numbers a, b, c, d, are in harmonical proportion, when $a : d :: a - b : c - d$; or when their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, are in arithmetical proportion.

EXAMPLES.

1. The firſt term of a geometrical ſeries is 1, the ratio 2, and the number of terms 10; what is the ſum of the ſeries?

Firſt, $1 \times 2^9 = 1 \times 512 =$ laſt term,

And, $\frac{512 \times 2 - 1}{2 - 1} = \frac{1024 - 1}{1} = 1023$, the ſum required.

2. The firſt term of a geometric ſeries is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the ſum of the ſeries?

Firſt, $\frac{1}{2} \times (\frac{1}{3})^4 = \frac{1}{2} \times \frac{1}{81} = \frac{1}{162} =$ the laſt term.

And, $(\frac{1}{2} - \frac{1}{162} \times \frac{1}{3}) \div (1 - \frac{1}{3}) = (\frac{1}{2} - \frac{1}{486}) \div \frac{2}{3} = \frac{242}{243} \times \frac{3}{2} = \frac{121}{81} \times \frac{1}{2} = \frac{121}{162}$, the ſum required.

3. Required the ſum of 1, 3, 9, 27, 81, &c. continued to 12 terms? Anſ. 265720.

4. Required the ſum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. continued to 12 terms? Anſ. $\frac{265720}{177147}$.

5. Required the ſum of 1, 2, 4, 8, 16, 32, &c. continued to 100 terms?

Anſ. 1267650600228229401496703205375.

See more of Geometrical Proportion in the Arithmetic.

SIMPLE EQUATIONS.

AN Equation, is when two equal quantities, differently expreſſed, are compared together by means of the ſign $=$ placed between them.

Thus, $12 - 5 = 7$ is an equation, expreſſing the equality of the quantities $12 - 5$ and 7.

A Simple

A Simple Equation, is that which contains only one power of the unknown quantity, without including different powers.

Thus, $x - a + b = c$ is a ſimple equation, containing only one power of the unknown quantitity x.

Reduction of Equations, is the method of finding the value of the unknown quantity.

It conſiſts in ordering the equation ſo, that the unknown letter or quantity may ſtand alone on one ſide of the equation, or of the mark of equality, without a co-efficient, and all the reſt, or the known quantities, on the other ſide.—In general, the unknown quantity is diſengaged from the known ones, by performing the reverſe operations. So, if they are connected with it by + or addition, they muſt be ſubtracted; if by minus (—), or ſubtraction, they muſt be added; if by multiplication, we muſt divide by them; if by diviſion, we muſt multiply; alſo, any power of the unknown quantity is taken away, by extracting the root; and any root is removed, by raiſing it to the power. As in the following rules *.

RULE I.

ANY quantity may be tranſpoſed from one ſide of the equation to the other by changing its ſign. And this rule is uſed to remove, or take away quantities, from the ſide of the unknown one, when they are connected with it, by the ſign + or —, or to remove the unknown quantity from them.

Thus, if $x + 3 = 7$; then will $x = 7 - 3 = 4$.

And, if $x - 4 + 6 = 8$; then will $x = 8 + 4 - 6 = 6$.

Alſo, if $x - a + b = c - d$; then will $x = a - b + c - d$.

And, in like manner, if $4x - 8 = 3x + 20$, then will $4x - 3x = 20 + 8$, or $x = 28$.

RULE II.

WHEN the unknown term is multiplied by any quantity; it is to be taken away by dividing all the terms of the equation by it.

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* Theſe are founded on the general principle of performing equal operations on equal quantities, when it is evident that the reſults muſt ſtill be equal; whether by equal additions, or ſubtractions, or multiplications, or diviſions, or roots, or powers.

Thus,

Thus, if $ax = ab - a$; then will $x = b - 1$.
And, if $2x+4=16$; then will $x+2=8$, and $x=8-2=6$.
In like manner, if $ax + 2ba = 3c^2$;
then will $x + 2b = \frac{3c^2}{a}$, and $x = \frac{3c^2}{a} - 2b$.

RULE III.

WHEN the unknown term is divided by any quantity; it may be taken away, by multiplying all the terms of the equation by it.

Thus, if $\frac{x}{2} = 5 + 3$; then will $x = 10 + 6 = 16$.

And, if $\frac{x}{a} = b + c - d$; then will $x = ab + ac - ad$.

In like manner, if $\frac{2x}{3} - 2 = 6 + 4$; then will $2x - 6 = 18 + 12$, and $2x = 18 + 12 + 6 = 36$, or $x = \frac{36}{2} = 18$.

RULE IV.

THE unknown quantity in any equation may be made free from furds, by tranſpoſing the reſt of the terms by Rule 1, and then involving each ſide to ſuch a power as is denoted by the index of the furd.

Thus, if $\sqrt{x} - 2 = 6$; then will $\sqrt{x} = 6 + 2 = 8$, and $x = 8^2 = 64$.

And, if $\sqrt{4x + 16} = 12$, then will $4x + 16 = 144$, or $4x = 144 - 16 = 128$; and if both ſides of the equation be divided by 4, x will be $= 32$.

In like manner, if $\sqrt[3]{2x + 3} + 4 = 8$; then will $\sqrt[3]{2x + 3} = 8 - 4 = 4$, and $2x + 3 = 4^3 = 64$, and $2x = 64 - 3 = 61$, or $x = \frac{61}{2} = 30\frac{1}{2}$.

RULE V.

IF that ſide of the equation which contains the unknown quantity be a complete power; it may be reduced, by extracting the root of the ſaid power on both ſides of the equation.

Thus, if $x^2 + 6x + 9 = 25$; then will $x + 3 = \sqrt{25} = 5$, or $x = 5 - 3 = 2$.

And,

And, if $3x^2 - 9 = 21 + 3$; then will $3x^2 = 21 + 3 + 9 = 33$, and $x^2 = \frac{33}{3} = 11$, or $x = \sqrt{11}$.

In like manner, if $\frac{2x^2}{3} + 10 = 20$; then will $2x^2 + 30 = 60$, and $x^2 + 15 = 30$, or $x^2 = 30 - 15 = 15$, or $x = \sqrt{15}$.

RULE VI.

ANY analogy or proportion may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x : 16 :: 5 : 10$; then will $3x \times 10 = 16 \times 5$, or $30x = 80$, or $x = \frac{80}{30} = \frac{8}{3} = 2\frac{2}{3}$.

And, if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3} = ab$, and $2cx = 3ab$, or $x = \frac{3ab}{2c}$.

In like manner, if $12 - x : \frac{x}{2} :: 4 : 1$; then will $12 - x = \frac{4x}{2} = 2x$, and $2x + x = 12$, or $x = \frac{12}{3} = 4$.

RULE VII.

IF the same quantity be found on both sides of an equation, with the same sign; it may be taken away from each; and if every term in an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $4x + a = b + a$; then will $4x = b$, and $x = \frac{b}{4}$.

And, if $3ax + 5ab = 8ax$; then will $3x + 5b = 8c$, and $x = \frac{8c - 5b}{3}$.

In like manner, if $\frac{2x}{3} - \frac{8}{3} = \frac{16}{3} - \frac{8}{3}$; then will $2x = 16$, and $x = 8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x - 15 = 2x + 6$; to find the value of x.

First, $5x - 2x = 6 + 15$
Or, $3x = 6 + 15 = 21$
Therefore $x = \frac{21}{3} = 7$

2. Given

2. Given $40 - 6x - 16 = 120 - 14x$; to find x.
First, $14x - 6x = 120 - 40 + 16$
Or $8x = 136 - 40 = 96$.
And therefore $x = \frac{96}{8} = 12$

3. Let $5ax - 3b = 2dx + c$ be given; to find x.
First, $5ax - 2dx = c + 3b$
Or $(5a - 2d) \times x = c + 3b$
And therefore $x = \frac{c + 3b}{5a - 2d}$

4. Let $3x^2 - 10x = 8x + x^2$ be given; to find x.
First, $3x - 10 = 8 + x$, by dividing by x
Or $3x - x = 8 + 10 = 18$
And therefore $2x = 18$, or $x = \frac{18}{2} = 9$.

5. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$; to find x.
First, dividing the whole by $3ax^2$,
we shall have $2x - 4b = x + 2$
Or $2x - x = 2 + 4b$
That is, $x = 2 + 4b$

6. Let $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$ be given, to find x.

First, $x - \frac{2x}{3} + \frac{2x}{4} = 20$

Alſo, $3x - 2x + \frac{6x}{4} = 60$

And $12x - 8x + 6x = 240$
Therefore $10x = 240$

And $x = \frac{240}{10} = 24$

7. Given $\frac{x - 3}{2} + \frac{x}{3} = 20 - \frac{x - 19}{2}$; to find x.

First, $x - 3 + \frac{2x}{3} = 40 - x + 19$

Alſo, $3x - 9 + 2x = 120 - 3x + 57$
Therefore, $3x + 2x + 3x = 120 + 57 + 9$
That is, $8x = 186$, or $x = \frac{186}{8} = 23\frac{1}{4}$

8. Let

8. Let $\sqrt{\frac{2x}{3}} + 5 = 7$, be given; to find x.

First, $\sqrt{\frac{2x}{3}} = 7 - 5 = 2$

Whence $\frac{2x}{3} = 2^2 = 4$

And $2x = 12$, or $x = \frac{12}{2} = 6$

9. Let $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$, be given; to find x.

First, $x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$

Whence $x\sqrt{a^2 + x^2} = a^2 - x^2$

And $x^2 \times \overline{a^2 + x^2} = \overline{a^2 - x^2}^2 = a^4 - 2a^2x^2 + x^4$

Or $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$

Or $a^2x^2 + 2a^2x^2 = a^4$, or $3a^2x^2 = a^4$

Conseq. $x^2 = \frac{a^4}{3a^2} = \frac{a^2}{3}$ and $x = \sqrt{\frac{a^2}{3}} = a\sqrt{\frac{1}{3}}$.

EXAMPLES FOR PRACTICE.

1. Given $3y - 2 + 24 = 31$; to find y. Anſ. $y = 3$.
2. Given $x + 18 = 3x - 5$; to find x. Anſ. $x = 11\frac{1}{2}$.
3. Given $6 - 2x + 10 = 20 - 3x - 2$; to find x. Anſ. $x = 2$.
4. Given $x + \frac{1}{2}x + \frac{1}{3}x = 11$; to find x. Anſ. $x = 6$.
5. Given $2x - \frac{1}{2}x + 1 = 5x - 2$; to find x. Anſ. $x = \frac{6}{7}$.
6. Given $3ax + \frac{a}{2} - 3 = bx - a$; to find x.

Anſ. $x = \frac{6 - 3a}{6a - 2b}$.

7. Given $\frac{1}{2}x + \frac{1}{3}x - \frac{1}{4}x = \frac{1}{2}$; to find x. Anſ. $x = \frac{6}{7}$.
8. Given $\sqrt{12 + x} = 2 + \sqrt{x}$; to find x. Anſ. $x = 4$.
9. Given $x + a = \frac{x^2}{a + x}$; to determine x. Anſ. $x = -\frac{a}{2}$.
10. Given $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$; to find x.

Anſ. $x = \sqrt{\frac{b^4 - a^4}{2a^2}}$.

11. Given $\sqrt{x} + \sqrt{a + x} = \frac{2a}{\sqrt{a + x}}$; to find x.

Anſ. $x = \frac{a}{3}$.

12. Given $\frac{a}{1+x} + \frac{a}{1-x} = b$; to find x.

Anſ. $x = \sqrt{\frac{b-2a}{b}}$.

13. Given $a + x = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$; to find x.

Anſ. $x = \frac{b^2}{4a} - a$.

OF REDUCING DOUBLE, TRIPLE, &c, EQUATIONS, CONTAINING TWO, THREE, OR MORE UNKNOWN QUANTITIES.

PROBLEM I.

To Exterminate Two Unknown Quantities; Or, to Reduce the Two Simple Equations containing them, to a Single one.

RULE I.

1. OBSERVE which of the unknown quantities is the leaſt involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values, thus found, be made equal to each other, and there will ariſe a new equation with only one unknown quantity in it, whoſe value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} 2x + 3y = 23 \\ 5x - 2y = 10 \end{matrix}\right\}$; to find x and y.

From the firſt equation $x = \frac{23 - 3y}{2}$

And from the ſecond $x = \frac{10 + 2y}{5}$

Conſequently $\frac{23 - 3y}{2} = \frac{10 + 2y}{5}$

Or $115 - 15y = 20 + 4y$

Or $19y + 115 - 20 = 95$

That is $y = \frac{95}{19} = 5$

And $x = \frac{23 - 15}{2} = 4$.

2. Given

2. Given $\left\{\begin{matrix} x+y=a \\ x-y=b \end{matrix}\right\}$; to find x and y.

From the first equation, $x = a - y$

And from the second $x = b + y$

Therefore $a - y = b + y$, or $2y = a - b$

Conseq. $y = \frac{a-b}{2}$, and $x = a - y$

Or $x = a - \frac{a-b}{2} = \frac{a+b}{2}$.

3. Given $\left\{\begin{matrix} \frac{1}{2}x + \frac{1}{3}y = 7 \\ \frac{1}{3}x + \frac{1}{2}y = 8 \end{matrix}\right\}$; to find x and y

From the first equation $x = 14 - \frac{2y}{3}$

And from the second $x = 24 - \frac{3y}{2}$

Therefore $14 - \frac{2y}{3} = 24 - \frac{3y}{2}$

And $42 - 2y = 72 - \frac{9y}{2}$

Or $84 - 4y = 144 - 9y$

Whence $5y = 144 - 84 = 60$

Therefore $y = \frac{60}{5} = 12$

And $x = 14 - \frac{2y}{3} = 14 - \frac{24}{3} = 6$.

4. Given $4x + y = 34$, and $4y + x = 16$; to find x and y.

Anf. $x = 8$, and $y = 2$.

5. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$; to find x and y.

Anf. $x = \frac{1}{2}$, and $y = \frac{1}{3}$.

6. Given $x + y = s$, and $x^2 - y^2 = d$; to find x and y.

Anf. $x = \frac{s^2+d}{2s}$, and $y = \frac{s^2-d}{2s}$.

7. Given $x - y = d$, and $x : y :: a : m$; to find x and y.

RULE II.

1. CONSIDER which of the unknown quantities you would first exterminate, and let its value be found in that equation where it is leaft involved.

2. Subftitute the value, thus found, for its equal in the other equation, and there will arife a new equation, with

only one unknown quantity, whose value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} x + 2y = 17 \\ 3x - y = 2 \end{matrix}\right\}$; to find x and y.

From the first equation $x = 17 - 2y$.
This value substituted for x in the second,
Gives $(17 - 2y) \times 3 - y = 2$
Or $51 - 6y - y = 2$, or $51 - 7y = 2$
That is $7y = 51 - 2 = 49$
Hence $y = \frac{49}{7} = 7$, and $x = 17 - 2y = 17 - 14 = 3$.

2. Given $\left\{\begin{matrix} x + y = 13 \\ x - y = 3 \end{matrix}\right\}$; to find x and y.

From the first equation $x = 13 - y$
This value being substituted for x in the 2d,
Gives $13 - y - y = 3$, or $13 - 2y = 3$
That is $2y = 13 - 3 = 10$
Hence $y = \frac{10}{2} = 5$, and $x = 13 - y = 13 - 5 = 8$.

3. Given $\left\{\begin{matrix} a : b :: x : y \\ x^2 + y^2 = c \end{matrix}\right\}$; to find x and y.

The first analogy turned into an equation,

gives $bx = ay$, or $x = \frac{ay}{b}$.

And this value of x being substituted in the 2d, gives

$$(\tfrac{ay}{b})^2 + y^2 = c, \text{ or } \tfrac{a^2y^2}{b^2} + y^2 = c$$

Theref. $a^2y^2 + b^2y^2 = b^2c$, or $y^2 = \frac{b^2c}{a^2 + b^2}$

Hence $y = \sqrt{\frac{b^2c}{a^2 + b^2}}$, and $x = \sqrt{\frac{a^2c}{a^2 + b^2}}$.

4. Given $2x + 3y = 16$, and $3x - 2y = 11$; to find x and y. Anſ. $x = 5$, and $y = 2$.

5. Given $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$; to find x and y.
Anſ. $x = 7$, and $y = 14$.

6. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$, and $\frac{x + y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27$; to find x and y. Anſ. $x = 60$, and $y = 40$.

7. Given

7. Given $a : b :: x : y$, and $x^3 - y^3 = a$; to find x and y.

Anſ. $x = \sqrt[3]{\frac{ba^3}{a^3 - b^3}}$, and $y = \sqrt[3]{\frac{ab^3}{a^3 - b^3}}$.

RULE III.

1. LET the given equations be multiplied, or divided, by ſuch numbers or quantities as will make the term which contains one of the unknown quantities the ſame in both equations.

2. Then, by adding or ſubtracting the equations, according as the caſe may require, there will ariſe a new equation, with only one unknown quantity, as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} 3x + 5y = 40 \\ x + 2y = 14 \end{matrix}\right\}$; to find x and y.

Firſt, multiply the 2d equation by 3,
and it will give $3x + 6y = 42$.
Then ſubtract the firſt from the laſt equation,
and it will give $6y - 5y = 42 - 40$, or $y = 2$,
and therefore $x = 14 - 2y = 14 - 4 = 10$.

2. Given $\left\{\begin{matrix} 5x - 3y = 9 \\ 2x + 5y = 16 \end{matrix}\right\}$; to find x and y.

Let the firſt equation be multiplied by 2, and the 2d by 5, and we ſhall have $10x - 6y = 18$
and $10x + 25y = 80$

Then if the former of theſe be ſubtracted from the latter it will give $31y = 62$, or $y = \frac{62}{31} = 2$

Conſeq. $x = \frac{9 + 3y}{5}$, by the firſt equation

Or $x = \frac{9 + 6}{5} = \frac{15}{5} = 3$.

Another Method.

Multiply the firſt equation by 5, and the ſecond by 3,
and we ſhall have $\left\{\begin{matrix} 25x - 15y = 45 \\ 6x + 15y = 48 \end{matrix}\right.$

Now, let theſe two equations be added together,
and the ſum will be $31x = 93$, or $x = \frac{93}{31} = 3$.

Conſeq.

Conſeq. $y = \frac{16 - 2x}{5}$, by the ſecond equation,

Or $y = \frac{16 - 6}{5} = \frac{10}{5} = 2$, as before.

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x + 2}{3} + 8y = 31$, and $\frac{y + 5}{4} + 10x = 192$; to find x and y. Anſ. $x = 19$, and $y = 3$.

2. Given $\frac{2x - y}{2} + 14 = 18$, and $\frac{2y + x}{3} + 16 = 19$; to find x and y. Anſ. $x = 5$, and $y = 2$.

3. Given $\frac{2x + 3y}{6} + \frac{x}{3} = 8$, and $\frac{7y - 3x}{2} - y = 11$; to find x and y. Anſ. $x = 6$, and $y = 8$.

4. Given $ax + by = c$, and $dx + ey = f$; to find x and y.

Anſ. $x = \frac{ce - bf}{ae - bd}$, and $y = \frac{af - dc}{ae - bd}$.

PROBLEM II.

To Exterminate Three Unknown Quantities; Or, to Reduce the Three Simple Equations, containing them, to a Single one.

RULE.

1. LET x, y, and z, be the three unknown quantities, to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the firſt value of x with the ſecond, and an equation will ariſe involving only y and z.

4. In like manner, compare the firſt value of x with the third, and another equation will ariſe involving only y and z.

5. Find the values of y and z from theſe two equations, according to the former rules; and x, y, and z, will be exterminated as required.

Note. Much in the ſame manner may any number of unknown quantities be exterminated. But there are often ſhorter methods for performing the operations which will be beſt learnt from practice.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10 \end{array}\right\}$; to find x, y, and z.

From the first, $x = 29 - y - z$
From the second, $x = 62 - 2y - 3z$
From the third, $x = 20 - \frac{2}{3}y - \frac{1}{2}z$
Whence $29 - y - z = 62 - 2y - 3z$
And $29 - y - z = 20 - \frac{2}{3}y - \frac{1}{2}z$

Alſo from the firſt of theſe, $y = 33 - 2z$
And from the ſecond, $y = 27 - \frac{3}{2}z$
Theref. $33 - 2z = 27 - \frac{3}{2}z$, or $z = 12$
Whence alſo $y = 33 - 2z = 9$
And $x = 29 - y - z = 8$.

2. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 62 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 47 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 38 \end{array}\right\}$; to find x, y, and z.

Firſt, the given equations, cleared of fractions, become

$$\begin{array}{l} 12x + 8y + 6z = 1488 \\ 20x + 15y + 12z = 2820 \\ 30x + 24y + 20z = 4560 \end{array}$$

Then, if the ſecond of theſe equations be ſubtracted from double the firſt, and 3 times the third from 5 times the ſecond, we ſhall have

$$\begin{array}{l} 4x + y = 156 \\ 10x + 3y = 420 \end{array}$$

And again, if the ſecond of theſe be ſubtracted from 3 times the firſt, it will give

$$12x - 10x = 468 - 420, \text{ or } x = \frac{48}{2} = 24,$$

Theref. $y = 156 - 4x = 60$, and $z = \frac{1488 - 8y - 12x}{6} = 120$.

3. Given $x + y + z = 53$, and $x + 2y + 3z = 105$, and $x + 3y + 4z = 134$; to find x, y, and z.

Anſ. $x = 24$, $y = 6$, and $z = 23$.

4. Given $x + y = a$, $x + z = b$, and $y + z = c$; to find x, y, and z.

5. Given $\left\{\begin{array}{l} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{array}\right\}$; to find x, y, and z.

A COL-

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, ſuch, that their ſum ſhall be 40, and their difference 16.

Let x denote the leaſt of the two numbers required,
Then will $x + 16 =$ to the greater,
And $x + x + 16 = 40$ by the queſtion,
That is, $2x = 40 - 16 = 24$
Or $x = \frac{24}{2} = 12 =$ leaſt number,
And $x + 16 = 12 + 16 = 28 =$ greater number required.

2. What number is that, whoſe $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x =$ number required,
Then will its $\frac{1}{3}$ part be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$;
Therefore $\frac{1}{3}x - \frac{1}{4}x = 16$ by the queſtion,
That is, $x - \frac{1}{4}x = 48$, or $4x - 3x = 192$;
Hence $x = 192$, the number required.

3. Divide 1000l. among A, B, and C, ſo that A ſhall have 72l. more than B, and C 100l. more than A.

Let $x =$ B's ſhare of the given ſum,
Then will $x + 72 =$ A's ſhare,
And $x + 172 =$ C's ſhare,
The ſum of all their ſhares is $x + x + 72 + x + 172$,
Or $3x + 244 = 1000$ by the queſtion,
That is, $3x = 1000 - 244 = 756$,
Or $x = \frac{756}{3} = 252$l. $=$ B's ſhare;
Hence $x + 72 = 252 + 72 = 324$l. $=$ A's ſhare,
And $x + 172 = 252 + 172 = 424$l. $=$ C's ſhare,

B's ſhare,	252l.
A's ſhare,	324l.
C's ſhare,	424l.
Sum of all,	1000l. the proof.

4. A prize of 1000l. is to be divided between two perſons, whoſe ſhares of it are in the proportion of 7 to 9; required the ſhare of each?

Let $x =$ firſt perſon's ſhare,
Then will $1000 - x =$ ſecond perſon's ſhare,
And $x : 1000 - x :: 7 : 9$, by the queſtion,

That

That is, $9x = (1000 - x) \times 7 = 7000 - 7x$,
Or $9x + 7x = 16x = 7000$,
Hence $x = \frac{7000}{16} = 437$l. 10s. = 1ſt ſhare,
And $1000 - x = 1000 - 437$l. 10s. = 562l. 10s. 2d ſhare.

5. The paving of a ſquare at 2s. a yard, coſt as much as the incloſing it at 5s. a yard: required the ſide of the ſquare?

Let x = ſide of the ſquare ſought,
Then $4x$ = yards of incloſure,
And x^2 = yards of pavement;
Hence $4x \times 5 = 20x$ = price of incloſing,
And $x^2 \times 2 = 2x^2$ = price of paving,
But $2x^2 = 20x$ by the queſtion,
Therefore $2x = 20$, and $x = 10$ = length of the ſide required.

6. A labourer engaged to ſerve for 40 days, on theſe conditions; that for every day he worked, he was to receive 20d. but for every day he played, or was abſent, he was to forfeit 8d. Now at the end of the time he had to receive 1l. 11s. 8d. It is required to find how many days he worked, and how many he was idle?

Let x be the number of days he worked,
Then will $40 - x$ be the number of days he was idle,
Alſo $x \times 20 = 20x$ = the ſum earned,
And $(40 - x) \times 8 = 320 - 8x$ = ſum forfeited,
Hence $20x - (320 - 8x) = 380$d. = (1l 11s. 8d.) by the queſtion; that is, $20x - 320 + 8x = 380$,
Or $28x = 380 + 320 = 700$,
Hence $x = \frac{700}{28} = 25$ = number of days he worked,
And $40 - x = 40 - 25 = 15$ = number of days he was idle,

7. Out of a caſk of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full; how much did it hold?

Let it be ſuppoſed to have held x gallons,
Then it would have leaked $\frac{1}{3}x$ gallons,
Conſeq. there had been taken away $21 + \frac{1}{3}x$ gallons.
But $21 + \frac{1}{3}x = \frac{1}{2}x$ by the queſtion,
That is, $63 + x = \frac{3}{2}x$
Or $126 + 2x = 3x$
Hence $3x - 2x = 126$
Or $x = 126$ = number of gallons required.

8. What

8. What fraction is that, to the numerator of which, if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be represented by $\frac{x}{y}$,

Then will $\frac{x+1}{y} = \frac{1}{3}$,

And $\frac{x}{y+1} = \frac{1}{4}$,

Hence $3x + 3 = y$,

And $4x = y + 1$,

Theref. $4x - 3x - 3 = y + 1 - y$,

That is $x - 3 = 1$,

Or $x = 4$; and hence $y = 3x + 3 = 12 + 3 = 15$.

So that $\frac{4}{15}$ is the fraction required.

9. A market woman bought in a certain number of eggs at 2 a penny, and as many more at 3 a penny, and sold them all out again at the rate of 5 for two-pence, and by so doing lost 4d. what number of eggs had she?

Let x = number of eggs of each sort,

Then will $\frac{1}{2}x$ = price of the first sort,

And $\frac{1}{3}x$ = price of the second sort;

But $5 : 2 :: 2x$ (the whole number of eggs) $: \frac{4}{5}x$;

Whence $\frac{4}{5}x$ = price of both sorts, at 5 for 2 pence,

And $\frac{1}{2}x + \frac{1}{3}x - \frac{4}{5}x = 4$ by the question;

That is $x + \frac{2}{3}x - \frac{8}{5}x = 8$;

Or $3x + 2x - \frac{24}{5}x = 24$;

Or $15x + 10x - 24x = 120$.

Or $x = 120$ = number of eggs of each sort.

10. If A can do a piece of work alone in 10 days, and B in 13; set them both about it together, in what time will it be finished?

Let the time sought be denoted by x.

Then 10 days : 1 work :: x days : $\frac{1}{10}x$,

And 13 days : 1 work :: x days : $\frac{1}{13}x$:

Hence $\frac{1}{10}x$ = part done by A in x days,

And $\frac{1}{13}x$ = part done by B in x days;

Consequently $\frac{1}{10}x + \frac{1}{13}x = 1$;

That is $\frac{23}{130}x + x = 13$, or $13x + 10x = 130$;

Or $23x = 130$, and $x = \frac{130}{23} = 5\frac{15}{23}$ days, the time required.

11. If

11. If one agent A, alone, can produce an effect e, in the time a; and another agent B, alone, in the time b; in what time will they both together produce the ſame effect?

Let the time ſought be denoted by x.

Then $a : e :: x : \frac{ex}{a}$ = part of the effect produced by A,

And $b : e :: x : \frac{ex}{b}$ = part of the effect produced by B,

Hence $\frac{ex}{a} + \frac{ex}{b} = e$ by the queſtion;

Or $\frac{x}{a} + \frac{x}{b} = 1$;

Therefore $x + \frac{ax}{b} = a$;

And $bx + ax = ab$;

Conſeq. $x = \frac{ab}{a+b}$ = time required.

QUESTIONS FOR PRACTICE.

1. WHAT two numbers are thoſe whoſe difference is 7, and ſum 33. Anſ. 13 and 20.

2. To divide the number 75 into two ſuch parts, that three times the greater may exceed ſeven times the leſs by 15. Anſ. 54 and 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder; how many gallons were there of each? Anſ. 85 of wine, and 35 of cyder.

4. A bill of 120l. was paid in guineas and moidores, and the number of pieces of both ſorts that were uſed was juſt 100; how many were there of each? Anſ. 50 of each.

5. Two travellers ſet out at the ſame time from London and York, whoſe diſtance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? Anſ. In 10 days.

6. At a certain election 375 perſons voted, and the candidate choſen had a majority of 91; how many voted for each? Anſ. 233 for one, and 142 for the other.

7. What number is that from which, if 5 be ſubtracted, $\frac{2}{3}$ of the remainder will be 40? Anſ. 65.

8. A poſt is $\frac{1}{8}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is the whole length? Anſ. 24 feet.

9. There is a fiſh whoſe tail weighs 9lb. his head weighs as much as his tail and half his body, and his body weighs as much as his head and tail; what is the whole weight of the fiſh? Anſ. 72lb.

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purſe; what was in it at firſt?

Anſ. 120 guineas.

11. A's age is double of B's, and B's is triple of C's, and the ſum of all their ages is 140; what is the age of each?

Anſ. A's = 84, B's = 42, and C's = 14.

12. Two perſons, A and B, lay out equal ſums of money in trade; A gains 126l. and B loſes 87l. and A's money is now double of B's; what did each lay out? Anſ. 300l.

13. A perſon bought a chaiſe, horſe, and harneſs, for 60l. the horſe came to twice the price of the harneſs, and the chaiſe to twice the price of the horſe and harneſs: what did he give for each?

Anſ. 13l. 6s. 8d. for the horſe, 6l. 13s. 4d. for the harneſs, and 40l. for the chaiſe.

14. Two perſons, A and B, have both the ſame income; A ſaves $\frac{1}{5}$ of his yearly, but B, by ſpending 50l. *per annum* more than A, at the end of 4 years finds himſelf 100l. in debt; what is their income? Anſ. 125l.

15. A perſon has two horſes, and a ſaddle worth 50l. now if the ſaddle be put on the back of the firſt horſe, it will make his value double that of the ſecond; but if it be put on the back of the ſecond, it will make his value triple that of the firſt; what is the value of each horſe?

Anſ. One 30l. and the other 40l.

16. To divide the number 36 into three ſuch parts, that $\frac{1}{2}$ of the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, may be all equal to each other? Anſ. The parts are 8, 12, and 16.

17. A footman agreed to ſerve his maſter for 8l. a year and a livery; but was turned away at the end of 7 months, and received only 2l. 13s. 4d. and his livery; what was its value? Anſ. 4l. 16s.

18. A perſon was deſirous of giving 3d. a-piece to ſome beggars; but found that he had not money enough in his pocket by 8d.; he therefore gave them each 2d, and had then 3d. remaining; required the number of beggars?

Anſ. 11.

19. A

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but two of the greyhound's leaps are as much as three of the hare's: how many leaps muſt the greyhound take to catch the hare? Anſ. 300.

20. A perſon in play loſt $\frac{1}{4}$ of his money, and then won 3 ſhillings; after which he loſt $\frac{1}{3}$ of what he then had, and then won 2 ſhillings; laſtly he loſt $\frac{1}{7}$ of what he then had: and, this done, found he had but 12s. remaining; what had he at firſt? Anſ. 20s.

21. To divide the number 90 into 4 ſuch parts, that if the firſt be increaſed by 2, the ſecond diminiſhed by 2, the third multiplied by 2, and the fourth divided by 2; the ſum, difference, product, and quotient ſhall be all equal to each other. Anſ. The parts are 18, 22, 10, and 40, reſpectively.

22. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together?
Anſ. 1 hour, $5\frac{5}{11}$ minutes.

23. There is an iſland 73 miles in circumference, and three footmen all ſtart together to travel the ſame way about it: A goes 5 miles a day, B 8, and C 10: when will they all come together again? Anſ. 73 days.

24. How much foreign brandy at 8s. per gallon, and Britiſh ſpirits at 3s. per gallon, muſt be mixed together, ſo that in ſelling the compound at 9s. per gallon, the diſtiller may clear 30 per cent.?
Anſ. 51 gallons of brandy, and 14 of ſpirits.

25. A man and his wife uſually drank out a caſk of beer in 12 days; but when the man was from home, it laſted the woman 30 days; how many days would the man alone be in drinking it? Anſ. 20 days.

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days; how many days will it take each perſon to perform the ſame work alone?
Anſ. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{51}$.

27. If three agents, A, B, and C, can produce the effects a, b, c, in the times e, f, g, reſpectively; in what time would they jointly produce the effect d?
Anſ. $d \div (\frac{a}{e} + \frac{b}{f} + \frac{c}{g})$ time.

QUA-

QUADRATIC EQUATIONS.

A SIMPLE Quadratic Equation, is that which involves the ſquare of the unknown quantity only.

An Affected Quadratic Equation, is that which involves the ſquare of the unknown quantity in one term, and the firſt power in another term.

Thus, $ax^2 = b$, is a ſimple quadratic equation;
And $ax^2 + bx = c$, is an affected quadratic equation.

The rule for a ſimple quadratic equation has been given already.

All affected quadratic equations fall under the three following forms:

1. $x^2 + ax = b$
2. $x^2 - ax = b$
3. $x^2 - ax = -b$.

The rule for finding the value of x, in each of theſe equations, is as follows:

RULE*.

1. TRANSPOSE all the terms which involve the unknown quantity to one ſide of the equation, and the known terms to the other, and let them be ranged according to their dimenſions, as in the forms above.

2. When

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* The ſquare root of any quantity may be either + or —, and therefore all quadratic equations admit of two ſolutions. Thus the ſquare root of $+ n^2$ is either $+ n$ or $- n$; for $+ n \times + n$ and $- n \times - n$ are each equal to $+ n^2$. But the ſquare root of $- n^2$, or $\sqrt{-n^2}$, is imaginary or impoſſible, as neither $+ n$ nor $- n$, when ſquared, give $- n^2$.

So, in the firſt form, $x^2 + ax = b$, where $x + \frac{1}{2}a$ is found $= \sqrt{b + \frac{1}{4}a^2}$, the root may be either $+ \sqrt{b + \frac{1}{4}a^2}$, or $-\sqrt{b + \frac{1}{4}a^2}$, ſince either of them being multiplied by itſelf will produce $b + \frac{1}{4}a^2$.

And

2. When the ſquare of the unknown quantity has any co-efficient prefixed to it, let all the reſt of the terms be divided by that co-efficient; which brings the equation to one of the three forms above.

3. Then complete the unknown ſide to a ſquare in this manner, viz. Take half the co-efficient of the ſecond term and ſquare it, which ſquare add to both ſides of the equation, then that ſide which involves the unknown quantity will be a complete ſquare.

4. Extract the ſquare root of both ſides of the equation, and the value of the unknown quantity will be determined, as was required, making the root of the known ſide either + or —, which will give two roots of the equation, or two values of the unknown quantity.

Note,

And this ambiguity is expreſſed by writing the uncertain ſign ± before $\sqrt{b + \frac{1}{4}a^2}$; thus $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$.

In this form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, the firſt value of x, viz. $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, is always affirmative; for ſince $\frac{1}{4}a^2 + b$ is greater than $\frac{1}{4}a^2$, the greateſt ſquare muſt neceſſarily have the greateſt ſquare root; therefore $\sqrt{b + \frac{1}{4}a^2}$ will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and conſequently $+\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be affirmative.

The ſecond value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be negative, becauſe it is compoſed of two negative terms. Therefore when $x^2 + ax = b$, we ſhall have $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ for the affirmative value of x, and $x = -\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ for the negative value of x.

In the ſecond form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ the firſt value, viz. $x = +\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always affirmative, ſince it is compoſed of two affirmative terms. The ſecond value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$, will always be negative; for ſince $b + \frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, $\sqrt{b + \frac{1}{4}a^2}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and conſequently $-\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always a negative quantity.

Therefore, when $x^2 - ax = b$, we ſhall have $x = +\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the affirmative value of x, and $x = -\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the negative value of x: ſo that in both the firſt and ſecond forms,

Note, 1. The ſquare root of the firſt ſide of the equation is always equal to the unknown quantity, with half the coefficient of the ſecond term ſubjoined to it.

2. All equations, in which there are two terms involving the unknown quantity, and which have the index of the one juſt double that of the other, are reſolved like quadratics, by completing the ſquare, as above.

Thus, $x^4 + ax^2 = b$, or $x^{2n} + ax^n = b$, or $x + ax^{\frac{1}{2}} = b$, are the ſame as quadratics, and the value of the unknown quantity may be determined accordingly.

EXAMPLES.

1. Given $x^2 + 4x = 140$; to find x.

First, $x^2 + 4x + 4 = 140 + 4 = 144$, by completing the ſquare.

Then $\sqrt{x^2 + 4x + 4} = \sqrt{144}$, by extracting the roots;

Or, which is the ſame thing, $x + 2 = \pm 12$,

Therefore $x = \pm 12 - 2 = 10$ or -14, the two roots.

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forms, the unknown quantity has always two values, one of which is poſitive, and the other negative.

In the third form, where $x = \pm\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, both the values of x will be poſitive, ſuppoſing $\frac{1}{4}a^2$ is greater than b. For the firſt value, viz. $x = +\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will then be affirmative, being compoſed of two affirmative terms.

The ſecond value, viz. $x = -\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ is affirmative alſo; for ſince $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2 - b$, $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{\frac{1}{4}a^2 - b}$; and conſequently $-\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will always be an affirmative quantity. Therefore, when $x^2 - ax = -b$, we ſhall have $x = +\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, and alſo $x = -\sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ for the values of x, both affirmative.

But in this third form, if b be greater than $\frac{1}{4}a^2$, the ſolution of the propoſed queſtion will be impoſſible. For ſince the ſquare of any quantity (whether that quantity be affirmative or negative) is always affirmative, the ſquare root of a negative quantity is impoſſible, and cannot be aſſigned. But if b be greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2 - b$ is a negative quantity; and therefore $\sqrt{\frac{1}{4}a^2 - b}$ is impoſſible, or imaginary; conſequently, in that caſe, $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b}$, or the two roots or values of x, are both impoſſible, or imaginary.

2. Given

2. Given $x^2 - 6x + 8 = 80$; to find x.

First, $x^2 - 6x = 80 - 8 = 72$, by tranſpoſition;
Then $x^2 - 6x + 9 = 72 + 9 = 81$, by completing the ſq.
And $x - 3 = \sqrt{81} = \pm 9$, by extracting the root;
Theref. $x = \pm 9 + 3 = 12$ or -6.

3. Given $2x^2 + 8x - 20 = 70$; to find x.

First, $2x^2 + 8x = 70 + 20 = 90$ by tranſpoſition;
Then $x^2 + 4x = 45$ by dividing by 2;
And $x^2 + 4x + 4 = 49$ by completing the ſquare;
Hence $x + 2 = \sqrt{49} = \pm 7$ by extracting the root;
Conſequently $x = \pm 7 - 2 = 5$ or -9.

4. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$; to find x.

Here, $x^2 - x + 2 = 1\frac{7}{9}$ by dividing by 3,
And $x^2 - x = 1\frac{7}{9} - 2$ by tranſpoſition;
Alſo $x^2 - x + \frac{1}{4} = 1\frac{7}{9} - 2 + \frac{1}{4} = \frac{1}{36}$ by compl. the ſq.
Hence $x - \frac{1}{2} = \sqrt{\frac{1}{36}} = \pm \frac{1}{6}$ by evolution;
Therefore $x = \pm \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$ or $\frac{1}{3}$.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$; to find x.

Here, $\frac{1}{2}x^2 - \frac{1}{3}x = 42\frac{2}{3} - 20\frac{1}{2} = 22\frac{1}{6}$ by tranſpoſition;
And $x^2 - \frac{2}{3}x = 44\frac{1}{3}$ by multiplying by 2.
Then $x^2 - \frac{2}{3}x + \frac{1}{9} = 44\frac{1}{3} + \frac{1}{9} = 44\frac{4}{9}$ by compl. ſqu.
Hence $x - \frac{1}{3} = \sqrt{44\frac{4}{9}} = \pm 6\frac{2}{3}$ by evolution;
Therefore $x = \pm 6\frac{2}{3} + \frac{1}{3} = 7$ or $-6\frac{1}{3}$.

6. Given $ax^2 + bx = c$; to find x.

First, $x^2 + \frac{b}{a}x = \frac{c}{a}$ by diviſion;

Then $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$ by compl. ſquare.

And $x + \frac{b}{2a} = \sqrt{(\frac{c}{a} + \frac{b^2}{4a^2})} = \pm\sqrt{\frac{4ac + b^2}{4a^2}}$ by evolut.

Therefore $x = \pm\sqrt{(\frac{4ac + b^2}{4x^2})} - \frac{b}{2a}$.

7. Given $ax^2 - bx + c = d$; to find x.

Here, $ax^2 - bx = d - c$ by tranſpoſition;

And $x^2 - \frac{b}{a}x = \frac{d - c}{a}$ by diviſion;

Alſo $x^2 - \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{d-c}{a} + \frac{b^2}{4a^2}$ by compl. the ſq.

Hence $x - \frac{b}{2a} = \pm\sqrt{(\frac{d-c}{a} + \frac{b^2}{4a^2})}$ by evolution;

Therefore $x = \frac{b}{2a} \pm \sqrt{(\frac{d-c}{a} + \frac{b^2}{4a^2})}$.

8. Given $x^4 + 2ax^2 = b$; to find x.
Here, $x^4 + 2ax^2 + a^2 = b + a^2$ by compl. the ſq.
And $x^2 + a = \pm\sqrt{b + a^2}$ by evolution;
Hence $x^2 = \pm\sqrt{b + a^2} - a$, by tranſpoſition;
And conſequently $x = \pm\sqrt{-a \pm \sqrt{b + a^2}}$.

9. Given $ax^n - bx^{\frac{n}{2}} - c = -d$; to find x.
Firſt, $ax^n - bx^{\frac{n}{2}} = c - d$ by tranſpoſition;
And $x^n - \frac{b}{a}x^{\frac{n}{2}} = \frac{c-d}{a}$ by diviſion;
Alſo $x^n - \frac{b}{a}x^{\frac{n}{2}} + \frac{b^2}{4a^2} = \frac{c-d}{a} + \frac{b^2}{4a^2}$ by compl. the ſq.
Hence $x^{\frac{n}{2}} - \frac{b}{2a} = \pm\sqrt{(\frac{c-d}{a} + \frac{b^2}{4a^2})}$ by evolution;
Therefore $x^{\frac{n}{2}} = \frac{b}{2a} \pm \sqrt{(\frac{c-d}{a} + \frac{b^2}{4a^2})}$;
And conſeq. $x = (\frac{b}{2a} \pm \sqrt{\frac{4ac - 4ad + b^2}{4a^2}})^{\frac{2}{n}}$.

EXAMPLES FOR PRACTICE.

1. Given $x^2 - 8x + 10 = 19$; to find x. Anſ. $x = 9$.
2. Given $x^2 - x - 40 = 170$; to find x. Anſ. $x = 15$.
3. Given $3x^2 + 2x - 9 = 76$; to find x. Anſ. $x = 5$.
4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$; to find x. Anſ. $x = 1\frac{1}{2}$.
5. Given $2x^4 - x^2 = 496$; to find x. Anſ. $x = 4$.
6. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$; to find x. Anſ. $x = 49$.
7. Given $\frac{2}{3}x^2 + \frac{3}{4}x = \frac{4}{5}$; to find x. Anſ. .6689.
8. Given $x^6 + 6x^3 = 2$; to find x. Anſ. $x = \sqrt[3]{-3 \pm \sqrt{11}}$.
9. Given $x^2 + x = a$; to find x. Anſ. $x = \sqrt{a + \frac{1}{4}} - \frac{1}{2}$.

10. Given

10. Given $x - \sqrt{x} = a$; to find x. Anf. $x = (\frac{1}{2} \pm \sqrt{a + \frac{1}{4}})^2$.

11. Given $3x^{2n} - 2x^n = 25$; to find x.

Anf. $x = (\frac{1}{3}\sqrt{76} + \frac{1}{3})^{\frac{1}{n}}$.

12. Given $\sqrt{1 + x} - 2\sqrt[4]{1 + x} = 4$; to find x.

Anf. $x = (1 + \sqrt{5})^4 - 1$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whoſe difference is 8, and product 240.

Let $x =$ to the leaſt number,
Then will $x + 8 =$ to the greater,
And $x \times (x + 8) = x^2 + 8x = 240$ by the queſtion;
Thence $x^2 + 8x + 16 = 240 + 16 = 256$ by completing the ſquare;
And hence $x + 4 = \sqrt{256} = 16$ by evolution;
Theref. $x = 16 - 4 = 12$ the leſs number,
And $12 + 8 = 20$ the greater.

2. To divide the number 60 into two ſuch parts, that their product may be 864.

Let $x =$ the greater part,
Then will $60 - x =$ the leſs,
And $x \times (60 - x) = 60x - x^2 = 864$ by the queſtion;
That is $x^2 - 60x = -864$;
Then $x^2 - 60x + 900 = -864 + 900 = 36$ by completing the ſquare;
Hence $x - 30 = \pm\sqrt{36} = \pm 6$ by extracting the root;
Theref. $x = \pm 6 + 30 = 36$ or 24 the two parts.

3. Given the ſum of two numbers $= 10$ (a), and the ſum of their ſquares $= 58$ (b); to find thoſe numbers.

Let $x =$ the greater of the two numbers,
Then will $a - x =$ the leſs;
And $x^2 + (a - x)^2 = 2x^2 + a^2 - 2ax = b$ by the queſtion,
Or $x^2 + \frac{1}{2}a^2 - ax = \frac{1}{2}b$ by diviſion.
Or $x^2 - ax = \frac{1}{2}b - \frac{1}{2}a^2$ by tranſpoſition.
Then $x^2 - ax + \frac{1}{4}a^2 = \frac{1}{2}b - \frac{1}{4}a^2$ by compl. the ſqu.
Hence $x - \frac{1}{2}a = \pm\frac{1}{2}\sqrt{2b - a^2}$ by extracting the root;
Theref. $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{2b - a^2}$ the two numbers.

That is, $x = \frac{1}{2}a + \frac{1}{2}\sqrt{2b - a^2}$ the greater number,
And $x = \frac{1}{2}a - \frac{1}{2}\sqrt{2b - a^2}$ the leſs number.

Hence theſe two theorems, being put into numbers, give 7 and 3, for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth coſt me; what was the price of the cloth?

Let x = pounds the cloth coſt,
Then $24 - x$ = whole gain,
But $100 : x :: x : 24 - x$ by the queſtion,
Or $x^2 = 100 \times 24 - x = 2400 - 100x$,
That is, $x^2 + 100x = 2400$;
Then $x^2 + 100x + 2500 = 4900$ by compl. the ſq.
And $x + 50 = \sqrt{4900} = 70$ by extraction of roots,
Conſequently $x = 70 - 50 = 20$l. = price of the cloth.

5. A perſon bought a number of oxen for 80l. and if he had bought 4 more for the ſame money, he would have paid 1l. leſs for each: how many did he buy?

Let the number of oxen be repreſented by x,
Then will $\frac{80}{x}$ be the price of each,
And $\frac{80}{x+4}$ = price of each, if $x + 4$ had coſt 80.
But $\frac{80}{x} = \frac{80}{x+4} + 1$ by the queſtion,
Or $80 = \frac{80x}{x+4} + x$, by multiplication,
Or $80x + 320 = 80x + x^2 + 4x$, by the ſame,
That is, $x^2 + 4x = 320$;
Then $x^2 + 4x + 4 = 324$ by completing the ſquare,
And $x + 2 = \sqrt{324} = 18$ by evolution,
Conſeq. $x = 18 - 2 = 16$, the numb. of oxen required.

6. What two numbers are thoſe, whoſe ſum, product, and difference of their ſquares, are all equal to each other?

Let x = the greater number,
And y = the leſs,
Then $\left\{\begin{matrix} x + y = xy \\ x + y = x^2 - y^2 \end{matrix}\right\}$ by the queſtion;
Hence $1 = \frac{x^2 - y^2}{x + y} = x - y$, or $x = y + 1$ from the 2d equation,

Alſo

Alſo $\overline{y+1} + y = \overline{y+1} \times y$ from the firſt equation,
Or $2y + 1 = y^2 + y$,
That is, $y^2 - y = 1$;
Then $y^2 - y + \frac{1}{4} = 1\frac{1}{4}$ by completing the ſquare,
Alſo $y - \frac{1}{2} = \sqrt{1\frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$ by evolution,
Conſequently $y = \frac{1}{2}\sqrt{5} + \frac{1}{2}$,
And $x = y + 1 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$.

And if theſe expreſſions be turned into numbers, we ſhall have $x = 2{\cdot}6180 +$
and $y = 1{\cdot}6180 +$

7. There are four numbers in arithmetical progreſſion, of which the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let x = the leſs extreme,
and y = the common difference;

Then x, $x + y$, $x + 2y$, $x + 3y$, will be the four numbers,
Hence $x \times \overline{x + 3y} = x^2 + 3xy = 45$,
And $\overline{x + y} \times \overline{x + 2y} = x^2 + 3xy + 2y^2 = 77$ } by the queſtion,
Hence $2y^2 = 77 - 45 = 32$ by ſubtraction.
And $y^2 = \frac{32}{2} = 16$ by diviſion,
Or $y = \sqrt{16} = 4$ by evolution;
Therefore $x^2 + 3xy = x^2 + 12x = 45$ by the 1ſt equation,
And $x^2 + 12x + 36 = 45 + 36 = 81$ by completing the ſq.
Alſo $x + 6 = \sqrt{81} = 9$ by the extraction of roots;
Conſeq. $x = 9 - 6 = 3$,
And the numbers are 3, 7, 11, and 15.

8. To find 3 numbers in geometrical progreſſion, whoſe ſum ſhall be 14, and the ſum of their ſquares 84.

Let x, y, and z be the three numbers ſought,
Then $xz = y^2$ by the nature of proportion,

And $\left\{\begin{matrix} x + y + z = 14 \\ x^2 + y^2 + z^2 = 84 \end{matrix}\right\}$ by the queſtion;

Hence $x + z = 14 - y$ by the 2d equation,
And $x^2 + 2xz + z^2 = 196 - 28y + y^2$ by ſquaring,
Or $x^2 + z^2 + 2y^2 = 196 - 28y + y^2$ by putting $2y^2$ for its equal $2xz$,
That is, $x^2 + z^2 + y^2 = 196 - 28y$ by ſubtraction,
Or $196 - 28y = 84$ by equality,
Hence $y = \frac{196 - 84}{28} = 4$ by tranſpoſition and diviſion.

Again,

Again, $xz = y^2 = 16$, or $x = \frac{16}{z}$ by the 1ſt equation,

And $x + y + z = \frac{16}{z} + 4 + z = 14$ by the 2d equation,

Or $16 + 4z + z^2 = 14z$, or $z^2 - 10z = -16$;
Then $z^2 - 10z + 25 = 25 - 16 = 9$ by compl. ſq.
And $z - 5 = \pm\sqrt{9} = \pm 3$ by extracting the roots;
Hence $z = 5 \pm 3 = 8$ and 2, the two other numbers,
That is, $x = 2$, and $z = 8$.
And the three numbers are 2, 4, 8.

QUESTIONS FOR PRACTICE.

1. WHAT two numbers are thoſe, whoſe ſum is 20, and their product 36? Anſ. 2 and 18.

2. To divide the number 60 into two ſuch parts, that their product may be to the ſum of their ſquares, in the ratio of 2 to 5. Anſ. 20 and 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117; what are thoſe numbers? Anſ. 2 and 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was ſettled, two of them left the room, and then thoſe who remained had 10s. a-piece more to pay than before: how many were there in company? Anſ. 7.

5. A grazier bought as many ſheep as coſt him 60l. and, after reſerving 15 out of the number, he ſold the remainder for 54l. and gained 2s. a head by them; how many ſheep did he buy? Anſ. 75.

6. There are two numbers whoſe difference is 15, and half their product is equal to the cube of the leſſer number; what are thoſe numbers? Anſ. 3 and 18.

7. A perſon bought cloth for 33l. 15s. which he ſold again at 2l. 8s. *per* piece, and gained by the bargain as much as one piece coſt him; required the number of pieces? Anſ. 15.

8. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? Anſ. 24.

9. What two numbers are thoſe, whoſe ſum multiplied by the greater is equal to 77; and whoſe difference multiplied by the leſſer is equal to 12? Anſ. 4 and 7.

10. To

10. To find a number ſuch, that if you ſubtract it from 10, and multiply the remainder by the number itſelf, the product ſhall be 21. Anſ. 7 or 3.

11. To divide 100 into two ſuch parts, that the ſum of their ſquare roots may be 14. Anſ. 64 and 36.

12. It is required to divide the number 24 into two ſuch parts, that their product may be equal to 35 times their difference. Anſ. 10 and 14.

13. The ſum of two numbers is 8, and the ſum of their cubes is 152; what are the numbers? Anſ. 3 and 5.

14. The ſum of two numbers is 7, and the ſum of their 4th powers is 641; what are the numbers? Anſ. 2 and 5.

15. The ſum of two numbers is 6, and the ſum of their 5th powers is 1056; what are the numbers? Anſ. 2 and 4.

16. The ſum of four numbers in arithmetical progreſſion is 56, and the ſum of their ſquares is 864; what are the numbers? Anſ. 8, 12, 16, and 20.

17. To find four numbers in geometrical progreſſion, whoſe ſum is 15, and the ſum of their ſquares 85?
Anſ. 1, 2, 4, and 8.

18. It is required to find four numbers in arithmetical progreſſion, ſuch that their common difference may be 4, and their continued product 176985.
Anſ. 15, 19, 23, and 27.

19. Two partners, A and B, gained 140l. by trade; A's money was 3 months in trade, and his gain was 60l. leſs than his ſtock; and B's money, which was 50l. more than A's, was in trade 5 months; what was A's ſtock?
Anſ. 100l.

RESOLUTION OF CUBIC AND HIGHER EQUATIONS.

A Cubic Equation, or Equation of the 3d degree or power, is one that contains the third power of the unknown quantity. As $x^3 - ax^2 + bx = c$.

A *Biquadratic*, or Double Quadratic, is an equation that contains the 4th power of the unknown quantity. As $x^4 - ax^3 + bx^2 - cx = d$.

An

An Equation of the 5th Power or Degree, is one that contains the 5th power of the unknown quantity. As $x^5 - ax^4 + bx^3 - cx^2 + dx = e$.

An Equation of the 6th Power or Degree, is one that contains the 6th power of the unknown quantity. As $x^6 - ax^5 + bx^4 - cx^3 + dx^2 - ex = f$.

And ſo on, for all other higher powers. Where it is to be noted however, that all the powers or terms, in the equation, are ſuppoſed to be freed from ſurds or fractional exponents.

There are many particular, and prolix rules, uſually given for the reſolution of ſome of the above-mentioned powers or equations. But they may be all eaſily reſolved by the following eaſy rule of Double Poſition, ſometimes called Trial-and-Error.

RULE.

1. FIND, by trial, two numbers, as near the true root as you can, and ſubſtitute them ſeparately in the given equation, inſtead of the unknown quantity; marking the errors which ariſe from each of them.

2. Multiply the difference of the two numbers, found by trial, by the leaſt error, and divide the product by the difference of the errors, when they are alike, but by their ſum when they are unlike. Or ſay, As the difference or ſum of the errors is to the difference of the two numbers, ſo is the leaſt error to the correction of its ſuppoſed number.

3. Add the quotient, laſt found, to the number belonging to the leaſt error, when that number is too little, but ſubtract it when too great, and the reſult will give the true root *nearly*.

4. Take this root and the neareſt of the two former, or any other that may be found nearer; and, by proceeding in like manner as above, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

Note 1. It is beſt to employ always two aſſumed numbers that ſhall differ from each other only by unity in the laſt figure on the right hand; becauſe then the difference, or multiplier, is only 1.

Note 2. The above rule is the 2d rule for Double Poſition, given in the Arithmetic, pag. 124, (where it is demonſtrated). It is the eaſieſt to be uſed here, and each new operation commonly doubles the number of true figures in the root.

EXAM-

EXAMPLES.

Ex. 1. To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of x in it.

Here it is ſoon found that x lies between 4 and 5. Aſſume therefore theſe two numbers, and the operation will be as follows:

1ſt. Sup.		2d Sup.
4	x	5
16	x^2	25
64	x^3	125
84	ſums	155
−16	errors	+55

the ſum of which is 71.
Then as $71 : 1 :: 16 : 225$.
Hence $x = 4.225$ nearly.

Again, ſuppoſe 4·2 and 4·3, and repeat the work as follows;

1ſt Sup.		2d Sup.
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	ſums	102.297
−4·072	errors	+2·297

the ſum of which is 6·369.
As $6.369 : 1 :: 2.297 : 0.036$
This taken from 4·300
leaves x nearly $= 4.264$

Again, ſuppoſe 4·264, and 4·265, and work as follows:

4·264	x	4·265
18·181696	x^2	18·190225
77·526752	x^3	77·581310
99·972448	ſums	100·036535
−0·027552	errors	+0·036535

the ſum of which is ·064087.
Then as $·064087 : ·001 :: 4.264 : 0.0004299$
To this adding 4·264
gives x very nearly $= 4.2644299$

The work of the example above might have been much ſhortened, by the uſe of the table of powers at p. 90, &c, which would have given two or three figures by inſpection. But the example has been worked out ſo particularly as it is, the better to ſhew the method.

Ex. 2. To find the root of the equation $x^3 - 15x^2 + 63x = 50$, or the value of x in it.

Here it ſoon appears that x is very little above 1.

Sup-

Suppoſe therefore 1·0 and 1·1, and work as follows:

1·0	- x -	1·1
63·0	- $63x$ -	69·3
—15	— $15x^2$ -	—18·15
1	- x^3 -	1·1
49	- ſums -	52·25
— 1	- errors -	+2·25

3·25 ſum of the errors.

As 3·25 : ·1 :: 1 : ·03 correct.

1·06

Hence x = 1·03 nearly.

Again, ſuppoſe the two numbers 1·03 and 1·02, &c, as follows:

1·03	- x -	1·02
64·89	- $63x$	64·26
—15·9135	— $15x^2$	—15·6060
1·092727	- x^3	1·061208
50·069227	ſums	49·715208
+·069227	errors	—·284792

·284792

As ·354019 : ·01 :: ·069227 : ·0019555

This taken from 1·03

leaves x nearly = 1·02804

Note 3. Every equation has as many roots as it contains dimenſions, or as there are units in the index of its higheſt power. That is, a ſimple equation has only one value of the root; but a quadratic equation has two values or roots, a cubic equation has three roots, a biquadratic equation has four roots, and ſo on.

And when one of the roots of an equation has been found by approximation, as above, the reſt may be found as follows.—Take for a dividend, the given equation, with the known term tranſpoſed, with its ſign changed, to the unknown ſide of the equation; and for a diviſor, take x minus the root juſt found. Divide the ſaid dividend by the diviſor, and the quotient will be the equation depreſſed a degree lower than the given one.

Find a root of this new equation by approximation, as before, and it will be a ſecond root of the original equation. Then, by means of this root, depreſs the ſecond equation one degree lower, and from thence find a third root, and ſo on, till the equation be reduced to a quadratic; then the two roots of this being found, by the method of completing the ſquare, they will make up the remainder of the roots. Thus, in the foregoing equation, having found one root to be 1·02804, connect it by minus with x for a diviſor, and the equation for a dividend, &c, as follows:

$x - 1{\cdot}02804)\ x^3 - 15x^2 + 63x - 50\ (x^2 - 13{\cdot}97196x + 48{\cdot}63627 = 0.$

Then the two roots of this quadratic equation, or $x^2 - 13{\cdot}97196x = -48{\cdot}63627$, by completing the ſquare, are 6·57653 and 7·39543, which are alſo the other two roots of the given cubic equation. So that all the three roots of that equation, viz. $x^3 - 15x^2 + 63x = 50$,

are 1 02804,
and 6·57653,
and 7·39543,
ſum 15·00000

and the ſum of all the roots is found to be 15, being equal to the coefficient of the 2d term of the equation, which the ſum of the roots always ought to be, when they are right.

Note 4. It is alſo a particular advantage of the foregoing rule, that it is not neceſſary to prepare the equation, as for other rules, by reducing it to the uſual final form and ſtate of equations. Becauſe the rule may be applied at once to an unreduced equation, though it be ever ſo much embarraſſed by ſurd and compound quantities. As in the following example:

Ex. 3. Let it be required to find the root x of the equation $\sqrt{144x^2 - (x^2 + 20)^2} + \sqrt{196x^2 - (x^2 + 24)^2} = 114$, or the value of x in it.

By a few trials, it is ſoon found that the value of x is but little above 7. Suppoſe, therefore, firſt, that $x = 7$, and then $x = 8$.

First, when $x = 7$.			Second, when $x = 8$.
47·906	-	$\sqrt{144x^2 - (x^2 + 20)^2}$	- 46·476
65·384	-	$\sqrt{196x^2 - (x^2 + 24)^2}$	- 69·283
113·290	-	the ſums of theſe	- 115 759
114 000	-	the true number	- 114·000
−0·710	-	the two errors	- +1·759
+1·759			

As 2·469 : 1 :: 0·710 : 0·2 nearly.
7·0

$x = 7{\cdot}2$ nearly.

Sup-

Suppoſe again $x = 7{\cdot}2$, and then, becauſe it turns out too great, ſuppoſe x alſo $= 7{\cdot}1$, &c. as follows:

Supp. $x = 7{\cdot}2$.			Supp. $x = 7{\cdot}1$
47·990	-	$\sqrt{144x^2 - (x^2 + 20)^2}$	- 47·973
66·402	-	$\sqrt{196x^2 - (x^2 + 20)^2}$	- 65·904
114·392	-	the ſums of theſe -	- 113·877
114·000	-	the true number -	- 114·000
+0·392	-	the errors -	- −0·123
0·123			

·515 : ·123 :: ·1 : ·024 the correction,
7·100
Therefore $x = 7{\cdot}124$ nearly the root required.

Note 5. The ſame rule alſo, among other more difficult forms of equations, ſucceeds very well in what are called exponential ones, or thoſe which have an unknown quantity in the exponent of the power; as in the following example:

Ex. 4. To find the value of x in the exponential equation $x^x = 100$.

For more eaſily reſolving ſuch kind of equations, it is convenient to take the logarithms of them, and then compute the terms by means of a table of logarithms. Thus, the logarithms of the two ſides of the preſent equation are, $x \times$ log. of $x = 2$ the log. of 100. Then, by a few trials it is ſoon perceived that the value of x is ſomewhere between the two numbers 3 and 4, and indeed nearly in the middle between them, but rather nearer the latter than the former. Taking therefore firſt $x = 3{\cdot}5$, and then $= 3{\cdot}6$, and working with the logarithms, the operation will be as follows:

Firſt Supp. $x = 3{\cdot}5$.	Second Supp. $x = 3{\cdot}6$.
Log. of $3{\cdot}5 = 0{\cdot}5440680$	Log. of $3{\cdot}6 = 0{\cdot}5563025$
then $3{\cdot}5 \times$ log. $3{\cdot}5 = 1{\cdot}904238$	then $3{\cdot}6 \times$ log. $3{\cdot}6 = 2{\cdot}002689$
the true number $2{\cdot}000000$	the true number $2{\cdot}000000$
error, too little, −·095762	error, too great, +·002689
·002689	

·098451 ſum of the errors. Then,

As

As ·098451 : ·1 :: ·002689 : 0·00273 the correction taken from 3·60000

leaves - 3·59727 = x nearly.

On trial, this is found to be a very small matter too little. Take therefore again, $x = 3{\cdot}59727$, and next $= 3{\cdot}59728$, and repeat the operation as follows:

First, Supp. $x = 3{\cdot}59727$.	Second, Supp. $x = 3{\cdot}59728$.
Log. of 3·59727 is 0·5559731	Log. of 3·59728 is 0·5559743
3·59727 × log. of 3·59727 = 1·9999854	3·59728 × log. of 3·59728 = 1·9999953
the true number 2·0000000	the true number 2·0000000
error, too little, —0·0000146	error, too little, —0·0000047
0·0000047	
0·0000099 differ. of the errors. Then,	

As ·0000099 : ·00001 :: ·0000047 : 0·00000474747 the cor. added to - 3·59728000000

gives nearly the value of $x = 3{\cdot}59728474747$

Ex. 5. To find the value of x in the equation $x^3 + 10x^2 + 5x = 2600$. Anf. $x = 11{\cdot}0c673$.

Ex. 6. To find the value of x in the equation $x^3 - 2x = 5$. Anf. 2·0045551.

Ex. 7. To find the value of x in the equation $x^3 + 2x^2 - 23x = 70$. Anf. $x = 5{\cdot}1349$.

Ex. 8. To find the value of x in the equation $x^3 - 17x^2 + 54x = 350$. Anf. $x = 14{\cdot}95407$.

Ex. 9. To find the value of x in the equation $x^4 - 3x^2 - 75x = 10000$. Anf. $x = 10{\cdot}2615$.

Ex. 10. To find the value of x in the equation $2x^4 - 16x^3 + 40x^2 - 30x = -1$. Anf. $x = 1{\cdot}284724$.

Ex. 11. To find the value of x in the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$. Anf. $x = 8{\cdot}414455$.

Ex. 12. To find the value of x in the equation $x^5 = 123456789$. Anf. $x = 8{\cdot}6400268$.

OF

OF SIMPLE INTEREST.

As the intereſt of any ſum, for any time, is directly proportional to the principal ſum, and to the time; therefore the intereſt of 1 pound, for 1 year, being multiplied by any given principal ſum, and by the time of its forbearance, in years and parts, will give its intereſt for that time. That is, if there be put

r = the rate of intereſt of 1 pound per annum,
p = any principal ſum lent,
t = the time it is lent for, and
a = the amount, or ſum of principal and intereſt; then is prt = the intereſt of the ſum p, for the time t, and conſeq. $p + prt$ or $p \times (1 + rt) = a$, the amount for that time.

From this expreſſion, other theorems can eaſily be deduced, for finding any of the quantities above mentioned; which theorems, collected altogether, will be as below:

1ſt, $a = p + prt$, the amount,

2d, $p = \frac{a}{1 + rt}$, the principal,

3d, $r = \frac{a - p}{pt}$, the rate,

4th, $t = \frac{a - p}{pr}$, the time.

For Example. Let it be required to find, in what time any principal ſum will double itſelf, at any rate of ſimple intereſt.

In this caſe, we muſt uſe the firſt theorem, $a = p + prt$, in which the amount a muſt be made $= 2p$, or double the principal, that is, $p + prt = 2p$, or $prt = p$, or $rt = 1$; and hence $t = \frac{1}{r}$.

Here, r being the intereſt of 1l. for 1 year, it follows, that the doubling at ſimple intereſt, is equal to the quotient of any ſum divided by its intereſt for 1 year. So, if the rate of intereſt be 5 per cent. then $100 \div 5 = 20$, is the time of doubling at that rate.

Or, the 4th theorem gives at once

$= \frac{a - p}{pr} = \frac{2p - p}{pr} = \frac{2 - 1}{r} = \frac{1}{r}$, the ſame as before.

COM-

COMPOUND INTEREST.

BESIDE the quantities concerned in Simple Intereſt, namely,

p = the principal ſum,
r = the rate, or intereſt of 1l. for 1 year,
a = the whole amount of the principal and intereſt,
t = the time,

there is another quantity employed in Compound Intereſt, viz. the ratio of the rate of intereſt, which is the amount of 1l. for 1 time of payment, and which here let be denoted by R, viz.

$R = 1 + r$, the amount of 1l. for 1 time.

Then, the particular amounts for the ſeveral times may be thus computed, viz. As 1l. is to its amount for any time, ſo is any propoſed principal ſum, to its amount for the ſame time; that is, as

1l : R :: p : pR, the 1ſt year's amount,
1l : R :: pR : pR^2, the 2d year's amount,
1l : R :: pR^2 : pR^3, the 3d year's amount,
and ſo on.

Therefore, in general, $pR^t = a$ is the amount for the t year, or t time of payment. From whence the following general theorems are deduced:

1ſt, $a = pR^t$, the amount,

2d, $p = \frac{a}{R^t}$, the principal,

3d, $R = \sqrt[t]{\frac{a}{p}}$, the ratio,

4th, $t = \frac{\text{log. of } a - \text{log. of } p}{\text{log. of } R}$, the time.

From which, any one of the quantities may be found, when the reſt are given.

As to the whole intereſt, it is found by barely ſubtracting the principal p from the amount a.

Example. Suppoſe it be required to find, in how many years any principal ſum will double itſelf, at any propoſed rate of compound intereſt.

In

In this case the 4th theorem must be employed, making $a = 2p$; and then it is

$$t = \frac{\text{l. } a^t - \text{l. } p}{\log. R} = \frac{\text{l. } 2p - \text{l. } p.}{\log. R} = \frac{\log. 2}{\log. R}.$$

So, if the rate of interest be 5 per cent. per annum; then $R = 1 + .05 = 1.05$; and hence

$$t = \frac{\log. 2}{\log. 1.05} = \frac{.3010300}{.0211893} = 14.2067 \text{ nearly};$$

that is, any sum doubles itself in $14\frac{1}{5}$ years nearly, at the rate of 5 per cent. per annum compound interest.

From hence, and from the like question in Simple Interest, above given, are deduced the times in which any sum doubles itself, at several rates of interest, both simple and compound; viz.

At		At Simp. Int.	At Comp. Int.
		Years.	Years.
2		50	35.0028
$2\frac{1}{2}$		40	28.0701
3	per cent per annum	$33\frac{1}{3}$	23.4498
$3\frac{1}{2}$	interest, 11, or any	$28\frac{4}{7}$	20.1488
4	other sum, will	25	17.6730
$4\frac{1}{2}$	double itself in the	$22\frac{2}{9}$	15.7473
5	following years.	20	14.2067
6		$16\frac{2}{3}$	11.8957
7		$14\frac{2}{7}$	10.2448
8		$12\frac{1}{2}$	9.0065
9		$11\frac{1}{9}$	8.0432
10		10	7.2725

The following Table will very much facilitate the calculation of the compound interest of any sum, for any number of years, at various rates of interest.

The

The Amounts of 1l. in any Number of Years.						
Yrs.	3	3½	4	4½	5	6
1	1·0300	1·0350	1·0400	1·0450	1·0500	1·0600
2	1·0609	1·0712	1·0816	1·0920	1·1025	1·1236
3	1·0927	1·1087	1·1249	1·1412	1·1576	1·1910
4	1·1255	1·1475	1·1699	1·1925	1·2155	1·2625
5	1·1593	1·1877	1·2167	1·2462	1·2763	1·3382
6	1·1941	1·2293	1·2653	1·3023	1·3401	1·4185
7	1·2299	1·2723	1·3159	1·3609	1·4071	1·5036
8	1·2668	1·3168	1·3686	1·4221	1·4775	1·5939
9	1·3048	1·3629	1·4233	1·4861	1·5513	1·6895
10	1·3439	1·4106	1·4802	1·5530	1·6289	1·7909
11	1·3842	1·4600	1·5395	1·6229	1·7103	1·8983
12	1·4258	1·5111	1·6010	1·6959	1·7959	2·0122
13	1·4685	1·5640	1·6651	1·7722	1·8856	2·1329
14	1·5126	1·6187	1·7317	1·8519	1·9799	2·2609
15	1·5580	1·6753	1·8009	1·9353	2·0789	2·3966
16	1·6047	1·7340	1·8730	2·0224	2·1829	2·5404
17	1·6528	1·7947	1·9479	2·1134	2·2920	2·6928
18	1·7024	1·8575	2·0258	2·2085	2·4066	2·8543
19	1·7535	1·9225	2·1068	2·3079	2·5270	3·0256
20	1·8061	1·9898	2·1911	2·4117	2·6533	3·2071

The uſe of this Table, which contains all the powers R^t, to the 20th power, or the amounts of 1l, is chiefly to calculate the intereſt, or the amount, of any principal ſum, for any time, not more than 20 years.

For example, let it be required to find, to how much 523l. will amount in 15 years, at the rate of 5 per cent. per annum compound intereſt.

In the table, on the line 15, and in the column 5 per cent.

is the amount of 1l, viz. - -	2·0789
this multiplied by the principal	523
gives the amount - -	1087·2647
or - - -	1087l. 5s. 3¼d,
and therefore the intereſt is	564l. 5s. 3¼d.

Note 1. When the rate of intereſt is to be determined to any other time than a year; as ſuppoſe to ½ a year, or ¼ a year, &c; the rules are ſtill the ſame; but then t will expreſs that time, and R muſt be taken the amount for that time.

Note 2. When the compound intereſt, or amount, of any ſum, is required for the parts of a year, it may be determined in the following manner:

1*ſt*, For any time which is ſome aliquot part of a year:—Find the amount of 1l. for 1 year, as before; then that root of it, which is denoted by the aliquot part, will be the amount of 1l. This amount being multiplied by the principal ſum, will produce the amount of the given ſum as required.

2*d*, When the time is not an aliquot part of a year:—Reduce the time into days, and take the 365th root of the amount of 1l. for 1 year, which will give the amount of the ſame for 1 day. Then raiſe this amount to that power whoſe index is equal to the number of days, and it will be the amount for that time. Which amount being multiplied by the principal ſum, will produce the amount of that ſum as before.

And in theſe calculations, the operation by logarithms will be very uſeful.

OF ANNUITIES.

ANNUITY is a term uſed for any periodical income, ariſing from money lent, or from houſes, lands, ſalaries, penſions, &c, payable from time to time, but moſtly by annual payments.

Annuities are divided into thoſe that are in Poſſeſſion, and thoſe in Reverſion: the former meaning ſuch as have commenced; and the latter ſuch as will not begin till ſome particular event has happened, or till after ſome certain time has elapſed.

When an annuity is forborn for ſome years, or the payments not made for that time, the annuity is ſaid to be in Arrears.

An annuity may alſo be for a certain number of years; or it may be without any limit, and then it is called a Perpetuity.

The Amount of an annuity, forborn for any number of years, is the ſum ariſing from the addition of all the annuities for that number of years, together with the intereſt due upon each after it becomes due.

The

The Preſent Worth or Value of an annuity, is the price or ſum which ought to be given for it, ſuppoſing it to be bought off, or paid all at once.

Let a = the annuity, penſion, or yearly rent.
n = the number of years forborn, or lent for.
R = the amount of 1l. for 1 year.
m = the amount of the annuity.
v = its value, or its preſent worth.

Now, 1 being the preſent value of the ſum R, by proportion the preſent value of any other ſum a, is thus found: as $R : 1 :: a : \frac{a}{R}$, the preſent value of a due 1 year hence. In like manner $\frac{a}{R^2}$ is the preſent value of a due 2 years hence; for $R : 1 :: \frac{a}{R} : \frac{a}{R^2}$. So alſo $\frac{a}{R^3}$, $\frac{a}{R^4}$, $\frac{a}{R^5}$, &c, will be the preſent values of a, due at the end of 3, 4, 5, &c, years, reſpectively. Conſequently the ſum of all theſe, or $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4} +$ &c, continued to n terms, will be the preſent value of all the n years annuities. And the value of the perpetuity, is the ſum of the ſeries to infinity.

But this ſeries, it is evident, is a geometrical progreſſion, having $\frac{a}{R}$ both for its firſt term and common ratio, and the number of its terms n; therefore the ſum v of all the terms, or the preſent value of all the annual payments, will be

$$v = \frac{a}{R-1} - \frac{a}{R-1} \times \frac{1}{R^n} \text{ or } = \frac{R^n - 1}{R-1} \times \frac{a}{R^n}.$$

When the annuity is a perpetuity, n being infinite, R^n is alſo infinite, and therefore the laſt quantity $\frac{1}{R^n}$ becomes $= 0$, therefore $\frac{a}{R-1} \times \frac{1}{R^n}$ alſo $= 0$; conſequently the expreſſion becomes barely $v = \frac{a}{R-1}$; that is, any annuity divided by the intereſt of 1l. for 1 year, gives the value of the perpetuity. So, if the rate of intereſt be 5 per cent,

Then $100a \div 5 = 20a$ is the value of the perpetuity at 5 per cent:

Alſo

Alſo $100a \div 4 = 25a$ is the value of the perpetuity at 4 per cent:

And $100a \div 3 = 33\frac{1}{3}a$ is the value of the perpetuity at 3 per cent: and ſo on.

Again, becauſe the amount of 1l. in n years, is R^n, its increaſe in that time will be $R^n - 1$; but its intereſt for one ſingle year, or the annuity anſwering to that increaſe, is $R - 1$; therefore as $R - 1$ is to $R^n - 1$, ſo is a to m: that is $m = \frac{R^n - 1}{R - 1} \times a$.

Hence, the ſeveral caſes relating to Annuities in Arrear, will be reſolved by the following equations:

$$m = \frac{R^n - 1}{R - 1} \times a = vR^n$$

$$v = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n} = \frac{m}{R^n}$$

$$a = \frac{R - 1}{R^n - 1} \times m = \frac{R - 1}{R^n - 1} \times vR^n$$

$$n = \frac{\log. m - \log. v}{\log. R}$$

$$\text{Log. } R = \frac{\log. m - \log. v}{n}$$

$$r = \left(\frac{1}{R^n} - \frac{1}{R^p}\right) \times \frac{a}{R - 1}.$$

In this laſt theorem, r denotes the preſent value of an annuity in reverſion, after p years, or not commencing till after the firſt p years, being found by taking the difference between the two values $\frac{R^n - 1}{R - 1} \times \frac{a}{R^n}$ and $\frac{R^p - 1}{R - 1} \times \frac{a}{R^p}$, for n years and p years.

END OF ALGEBRA,

OR OF

PART II. VOL. I.

GEOMETRY:

BEING

PART III. OF VOL. I.

CONTENTS OF PART III.

VOL. I.

GEOMETRY.

	Page
DEFINITIONS - - - - - - -	265
Axioms - - - - - - - - -	271
Remarks and Theorems - - - - - -	271
Of Rations and Proportions—Definitions - -	309
Theorems - - - - - - - -	311
Of Planes and Solids—Definitions - - -	326
Theorems - - - - - - - -	328
Problems - - - - - - - -	343
Application of Algebra to Geometry - - -	361
Theorems - - - - - - - -	362

GEO-

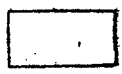
GEOMETRY.

DEFINITIONS.

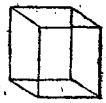
1. A POINT is that which has position, but no magnitude, nor dimensions; neither length, breadth, nor thickness.

2. A Line is length, without breadth or thickness.

3. A Surface or Superficies, is an extension, or a figure, of two dimensions, length and breadth; but without thickness.



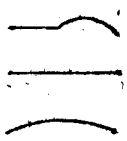
4. A Body or Solid, is a figure of three dimensions, namely, length, breadth, and thickness.



5. Lines are either Right, or Curved, or Mixed of these two.

6. A Right Line, or Straight Line, lies all in the same direction, between its extremities; and is the shortest distance between two points.

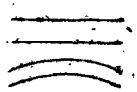
When a Line is mentioned simply, it means a Right Line.



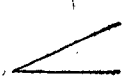
7. A Curve continually changes its direction between its extreme points.

8. Lines are either Parallel, Oblique, Perpendicular, or Tangential.

9. Parallel Lines are always at the same perpendicular distance; and they never meet, though ever so far produced.



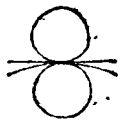
10. Oblique right lines change their distance, and would meet, if produced, on the side of the least distance.



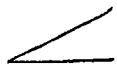
11. One line is Perpendicular to another, when it inclines not more on the one side

than the other; or when the angles on both sides of it are equal.

12. A line or circle is Tangential, or a Tangent to a circle, or other curve, when it touches it, without cutting, when both are produced.

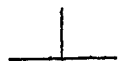


13. An Angle is the inclination, or opening of two lines, having different directions, and meeting in a point.

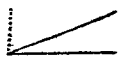


14. Angles are Right or Oblique, Acute or Obtuse.

15. A Right Angle, is that which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.



16. An Oblique Angle, is that which is made by two oblique lines; and is either less or greater than a right angle.



17. An Acute Angle is less than a right angle.

18. An Obtuse Angle is greater than a right angle.

19. Superficies are either Plane or Curved.

20. A Plane Superficies, or a Plane, is that with which a right line may, every way, coincide. Or, if the line touch the plane in two points, it will touch it in every point. But if not, it is curved.

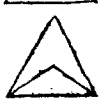
21. Plane figures are bounded either by right lines or curves.

22. Plane Figures that are bounded by right lines, have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

23. A figure of Three sides and angles, is called a Triangle. And it receives particular denominations from the relations of its sides and angles.

24. An Equilateral Triangle, is that whose three sides are all equal.

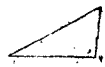
25. An Isosceles Triangle, is that which has two sides equal.



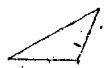
26. A

26. A Scalene Triangle, is that whose three sides are all unequal.

27. A Right-angled Triangle, is that which has one right angle.



28. Other triangles are Oblique-angled, and are either Obtuse or Acute.



29. An Obtuse-angled Triangle, has one obtuse angle.

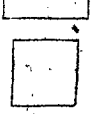
30. An Acute-angled Triangle, has all its three angles acute.

31. A figure of Four sides and angles, is called a Quadrangle, or a Quadrilateral.

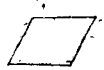
32. A Parallelogram, is a quadilateral which has both its pairs of oppofite fides parallel. And it takes the following particular names, viz. Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram, having all its angles right ones.

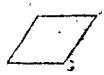
34. A Square is an equilateral rectangle; having all its fides equal, and its angles right ones.



35. A Rhomboid is an oblique-angled parallelogram.



36. A Rhombus is an equilateral rhomboid; having all its fides equal, but its angles oblique.



37. A Trapezium is a quadrilateral which hath not its oppofite fides parallel.

38. A Trapezoid hath only one pair of oppofite fides parallel.

39. A Diagonal is a right line joining any two oppofite angles of a quadrilateral.

40. Plane figures that have More than four fides are, in general, called Polygons: and they receive other particular names according to the number of their fides or angles.

41. A Pentagon is a polygon of five fides; a Hexagon hath fix fides; a Heptagon, feven; an Octagon, eight; a Nonagon, nine; a Decagon, ten; an Undecagon, eleven; and a Dodecagon hath twelve fides.

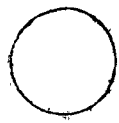
42. A

42. A Regular Polygon hath all its ſides and all its angles equal.—If they are not both equal, the polygon is Irregular.

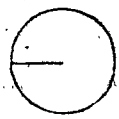
43. An Equilateral Triangle is alſo a Regular Figure of three ſides, and the Square is one of four: the former being alſo called a Trigon, and the latter a Tetragon.

44. A Circle is a plane figure bounded by a curve line, called the Circumference, which is every where equidiſtant from a certain point within, called its Centre.

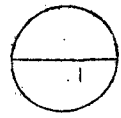
The circumference itſelf is often called a circle, and alſo the Periphery.



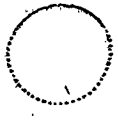
45. The Radius of a circle, is a right line drawn from the centre to the circumference.



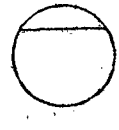
46. The Diameter of a circle, is a right line drawn through the centre, and terminating in the circumference on both ſides.



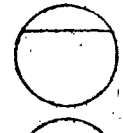
47. An Arc of a circle, is any part of the circumference.



48. A Chord, is a right line joining the extremities of an arc.

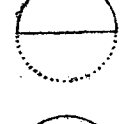


49. A Segment, is any part of a circle, bounded by an arc and its chord.

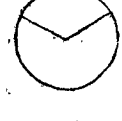


50. A Semicircle, is half the circle, or a ſegment cut off by a diameter.

The half circumference is ſometimes called the Semicircle.

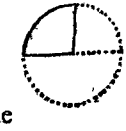


51. A Sector, is any part of a circle which is bounded by an arc, and two radii drawn to its extremities.



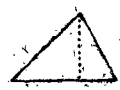
52. A Quadrant, or Quarter of a circle, is a ſector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other.

A quarter of the circumference is ſometimes called a Quadrant.



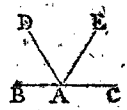
53. The

53. The Height or Altitude of a figure, is a perpendicular let fall from an angle, or its vertex, to the oppoſite ſide, called the baſe.



54. In a right-angled triangle, the ſide oppoſite the right angle, is called the Hypothenuſe; and the other two ſides are called the Legs, or ſometimes the Baſe and Perpendicular.

55. When an angle is denoted by three letters, of which one ſtands at the angular point, and the other two on the two ſides, that which ſtands at the angular point is read in the middle.

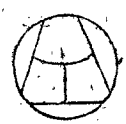


56. The circumference of every circle is ſuppoſed to be divided into 360 equal parts, called Degrees; and each degree into 60 Minutes, each minute into 60 Seconds, and ſo on. Hence a ſemicircle contains 180 degrees, and a quadrant 90 degrees.

57. The Meaſure of a right-lined angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is eſtimated by the number of degrees contained in that arc.

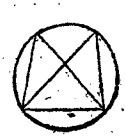
58. Identical figures, are ſuch as have all the ſides and all the angles of the one, reſpectively equal to all the ſides and all the angles of the other, each to each; ſo that if the one figure were applied to, or laid upon the other, all the ſides of the one would exactly fall upon and cover all the ſides of the other; the two becoming as it were but one and the ſame figure.

59. Right lines, or chords, are ſaid to be Equidiſtant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.



60. And the right line on which the Greater Perpendicular falls, is ſaid to be farther from the centre.

61. An Angle In a ſegment, is that which is contained by two lines, drawn from any point in the arc of the ſegment, to the two extremities of that arc.

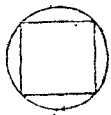


62. An

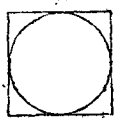
62. An angle On a Segment, or an arc, is that which is contained by two lines, drawn from any point in the oppoſite or ſupplemental part of the circumference, to the extremities of the arc, and containing the arc between them.

63. An angle at the Circumference, is that whoſe angular point is any where in the circumference. And an angle at the Centre, is that whoſe angular point is at the centre.

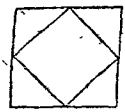
64. A right-lined figure is Inſcribed in a circle, or the circle Circumſcribes it, when all the angular points of the figure are in the circumference of the circle.



65. A right-lined figure Circumſcribes a circle, or the circle is Inſcribed in it, when all the ſides of the figure touch the circumference of the circle.



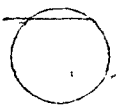
66. One right-lined figure is Inſcribed in another, or the latter Circumſcribes the former, when all the angular points of the former are placed in the ſides of the latter.



67. Similar figures, are thoſe that have all the angles of the one equal to all the angles of the other, each to each, and the ſides about the equal angles proportional.

68. The Perimeter of a figure, is the ſum of all its ſides taken together.

69. A Secant, is a line that cuts a circle, lying partly within, and partly without it.



70. A Propoſition, is ſomething which is either propoſed to be done, or to be demonſtrated, and is either a problem or theorem.

71. A Problem, is ſomething propoſed to be done.

72. A Theorem, is ſomething propoſed to be demonſtrated.

73. A Lemma, is ſomething which is premiſed, or demonſtrated, in order to render what follows more eaſy.

74. A Corollary, is a conſequent truth, gained immediately from ſome preceding truth, or demonſtration.

75. A Scholium, is a remark or obſervation made upon ſomething going before it.

AXIOMS.

AXIOMS.

1. THINGS which are equal to the ſame thing, are equal to each other.

2. When equals are added to equals, the wholes are equal.

3. When equals are taken from equals, the remains are equal.

4. When equals are added to unequals, the wholes are unequal.

5. When equals are taken from unequals, the remains are unequal.

6. Things which are double of the ſame thing, or equal things, are equal to each other.

7. Things which are halves of the ſame thing, are equal.

8. The whole is equal to all its parts taken together.

9. Things which coincide, or fill the ſame ſpace, are identical, or mutually equal in all their parts.

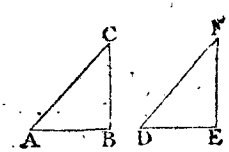
10. All right angles are equal to one another.

11. Angles that have equal meaſures, or arcs, are equal.

THEOREM I.

IF two Triangles have Two Sides and the Included Angle in the one, equal to Two Sides and the Included Angle in the other, the Triangles will be Identical, or equal in all reſpects.

In the two triangles ABC, DEF, if the ſide AC be equal to the ſide DF, and the ſide BC equal to the ſide EF, and the angle C equal to the angle F; then will the two triangles be identical, or equal in all reſpects.



For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in ſuch manner that the point C may

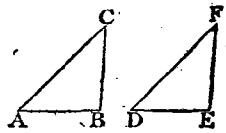
may coincide with the point F, and the ſide CA with the ſide FD, which is equal to it.

Then, ſince the angle F is equal to the angle C (by hyp.), the ſide BC will fall on the ſide EF. Alſo, becauſe AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E; conſequently the ſide AB will coincide with the ſide DE; therefore the two triangles are identical, and have all their other correſponding parts equal (ax. 9), namely, the ſide AB equal to the ſide DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

THEOREM II.

TRIANGLES, which have Two Angles, and the Side which lies between them, equal, are Identical, or have their other ſides and Angle equal.

Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the ſide AB equal to the ſide DE; then theſe two triangles will be identical.



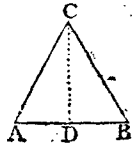
For, conceive the triangle ABC to be placed on the triangle DEF, in ſuch manner that the ſide AB may fall exactly on the equal ſide DE. Then, ſince the angle A is equal to the angle D (by hyp.), the ſide AC muſt fall on the ſide DF: and, in like manner, becauſe the angle B is equal to the angle E, the ſide BC muſt fall on the ſide EF. Thus the three ſides of the triangle ABC will be exactly placed on the three ſides of the triangle DEF; and conſequently the two triangles are identical (ax. 9), having the other two ſides AC, BC equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

THEOREM III.

IN an Iſoſceles Triangle, the Angles at the Baſe are equal. Or, if a Triangle have Two Sides equal, their Oppoſite Angles will alſo be equal.

If the triangle ABC have the ſide AC equal to the ſide BC: then will the angle B be equal to the angle A.

For, conceive the angle C to be biſected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD



Then,

Then, the two triangles ACD, BCD, have two ſides and the contained angle of the one, equal to two ſides and the contained angle of the other, viz. the ſide AC equal to BC, the angle ACD equal to BCD, and the ſide CD common; therefore theſe two triangles are identical, or equal in all reſpects (th. 1.); and conſequently the angle A equal to the angle B. Q. E. D.

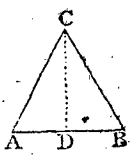
Corol. 1. Hence, the line which biſects the vertical angle of an iſoſceles triangle, biſects the baſe, and is alſo perpendicular to it.

Corol. 2. Hence too it appears, that every equilateral triangle is alſo equiangular, or has all its angles equal.

THEOREM IV.

If a Triangle have Two of its Angles equal to each other, the Sides Oppoſite to them will alſo be equal.

IF the triangle ABC, have the angle A equal to the angle B, it will alſo have the ſide AC equal to the ſide BC.



For, conceive the ſide AB to be biſected in the point D, making AD equal to DB; and join DC, dividing the whole triangle into the two triangles ACD, BCD. Alſo, conceive the triangle ACD to be turned over upon the triangle BCD, ſo that AD may fall on BD.

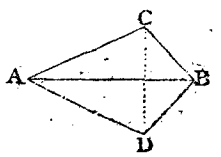
Then, becauſe the line AD is equal to the line DB (by hyp.), the point A coincides with the point B, and the point D with the point D. Alſo, becauſe the angle A is equal to the angle B (by hyp.), the line AC will fall on the line BC, and the extremity C of the ſide AC will coincide with the extremity C of the ſide BC, becauſe DC is common to both; conſequently the ſide AC is equal to BC. Q. E. D.

Corol. Hence every equiangular triangle, is alſo equilateral.

THEOREM V.

TRIANGLES which have their Three Sides mutually equal, are Identical, or have alſo their Three Angles equal, each to each.

Let the two triangles ABC, ABD, have their three ſides reſpectively equal, viz. the ſide AB equal to AB, AC to AD, and BC to BD; then ſhall the two triangles be identical, or have their angles equal, viz.



thoſe angles that are oppoſite to the equal ſides; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

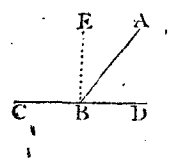
For, conceive the two triangles to be joined together by their longeſt equal ſides; and draw the line CD.

Then, in the triangle ACD, becauſe the ſide AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 2). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, becauſe the ſide BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions, the ſum of the two angles ACD, BCD, is equal to the ſum of the two ADC, BDC (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since then, the two ſides AC, CB, are equal to the two ſides AD, DB, each to each (by hyp.), and their contained angles ACB, ADB, alſo equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, viz. the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

THEOREM VI.

THE Angles which one Right Line makes with another, on the Same Side of it, are together equal to Two Right Angles.



Let the line AB meet the line CD; then will the two angles ABC, ABD, taken together, be equal to two right angles.

For, firſt, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15).

But when the angles are unequal, ſuppoſe BE drawn perpendicular to CD. Then, ſince the two angles EBC, EBD, are right angles (def. 15), and the angle EBD is equal to the two angles EBA, ABD together (ax. 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Conſequently the two angles ABC, ABD, are alſo equal to two right angles. Q. E. D.

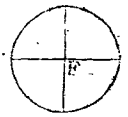
Corol. 1. Hence alſo, converſely, if the two angles ABC, ABD, on both ſides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Corol. 2.

Corol. 2. Hence, all the angles which can be made, at any point B, by any number of lines, on the ſame ſide of the right line CD, are, when taken all together, equal to two right angles.

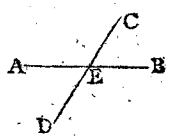
Corol. 3. And, as all the angles that can be made on the other ſide of the line CD are alſo equal to two right angles; therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

Corol. 4. Hence alſo the whole circumference of a circle, being the ſum of the meaſures of all the angles that can be made about the centre F (def. 57), is the meaſure of four right angles. Conſequently a ſemicircle, or 180 degrees, is the meaſure of two right angles; and a quadrant, or 90 degrees, the meaſure of one right angle.



THEOREM VII.

If two Lines Interſect each other, the Oppoſite Angles will be equal.



Let the two lines AB, CD interſect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED equal to the angle CEB.

For, ſince the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles CEB, DEB equal to two right angles.

Therefore the ſum of the two angles AEC, BEC, is equal to the ſum of the two CEB, DEB (ax. 1).

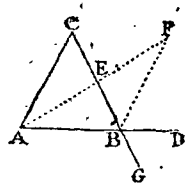
And if the angle CEB, which is common, be taken away from them both, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And in like manner it may be ſhewn, that the angle AED is equal to the oppoſite angle BEC.

THEOREM VIII.

If One Side of a Triangle be produced, the Outward Angle will be Greater than either of the two Inward Oppoſite Angles.

Let ABC be a triangle, having the ſide AB produced to D; then will the outward angle CBD be greater than either of the inward oppoſite angles A or C.



For, conceive the ſide BC to be biſected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

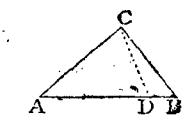
Then, ſince the two triangles AEC, BEF have the ſide AE = the ſide EF, and the ſide BE = the ſide CE (by ſuppoſ.), and the included or oppoſite angles at E alſo equal (th. 7.), therefore thoſe two triangles are equal in all reſpects (th. 1.), and have the angle C = the correſponding angle EBF. But the angle CBD is greater than the angle EBF; conſequently the ſaid outward angle CBD is alſo greater than the angle C.

In like manner, if CB be produced to G, and AB be biſected, it may be ſhewn that the outward angle *ABG*, or its equal *CBD*, is greater than the other angle *A*.

THEOREM IX.

THE Greater Side, of every Triangle, is Oppoſite to the Greater Angle; and the Greater Angle to the Greater Side.

Let ABC be a triangle, having the ſide AB greater than the ſide AC; then will the angle ACB, oppoſite the greater ſide AB, be greater than the angle B, oppoſite the leſs ſide AC.



For, on the greater ſide AB take the part AD equal to the leſs ſide AC, and join CD. Then, ſince BCD is a triangle, the outward angle ADC is greater than the inward oppoſite angle B (th. 8). But the angle ACD is equal to the ſaid outward angle ADC, becauſe AD is equal to AC (th. 3). Conſequently the angle ACD alſo is greater than the angle B. And ſince the angle ACD is only a part of ACB, much more muſt the whole angle ACB be greater than the angle B. Q. E. D.

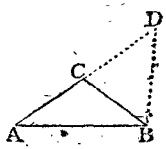
Again, converſely, if the angle C be greater than the angle B, then will the ſide AB, oppoſite the former, be greater than the ſide AC oppoſite the latter.

For, if AB be not greater than AC, it muſt be either equal to it, or leſs than it. But it cannot be equal, for then

then the angle C would be equal to the angle B (th. 3.), which it is not, by the ſuppoſition. Neither can it be leſs, for then the angle C would be leſs than the angle B, by the former part of this; which is alſo contrary to the ſuppoſition. The ſide AB, then, being neither equal to AC, nor leſs than it, muſt neceſſarily be greater. Q. E. D.

THEOREM X.

The Sum of any Two Sides of a Triangle, is Greater than the Third Side.



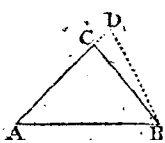
LET ABC be a triangle; then will the ſum of any two of its ſides be greater than the third ſide, as for inſtance, AC + CB greater than AB.

For, produce AC till CD be equal to CB, or AD equal to the ſum of the two AC + CB; and join BD.—Then, becauſe CD is equal to CB (by conſtr.), the angle D is equal to the angle CBD (th. 3). But the angle ABD is greater than the angle CBD, conſequently it muſt alſo be greater than the angle D. And, ſince the greater ſide of any triangle is oppoſite to the greater angle (th. 9), the ſide AD (of the triangle ABD) is greater than the ſide AB. But AD is equal to AC and CD, or AC and CB, taken together (by conſtr.); therefore AC + CB is alſo greater than AB. Q. E. D.

Corol. The ſhorteſt diſtance between two points, is a ſingle right line drawn from the one point to the other.

THEOREM XI.

The Difference of any Two Sides of a Triangle is Leſs than the Third Side.



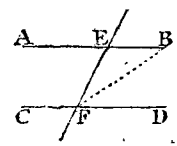
LET ABC be a triangle; then will the difference of any two ſides, as AB — AC, be leſs than the third ſide BC.

For, produce the leſs ſide AC to D, till AD be equal to the greater ſide AB, ſo that CD may be the difference of the two ſides AB — AC; and join BD.—Then, becauſe AD is equal to AB (by conſtr.), the oppoſite angles D and ABD are equal (th. 3). But the angle CBD is leſs than the angle ABD, and conſequently alſo leſs than the equal angle D. And ſince the greater ſide of any triangle

is oppoſite to the greater angle (th. 9), the ſide CD (of the triangle BCD) is leſs than the ſide BC. Q. E. D.

THEOREM XII.

If a Line Interſect two Parallel Lines, it will make the Alternate Angles Equal to each other.



Let the line EF cut the two parallel lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

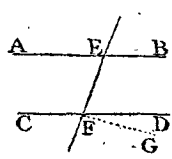
For if they are not equal, one of them muſt be greater than the other; let it be EFD for inſtance which is the greater, if poſſible; and conceive the line FB to be drawn cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, ſince the outward angle AEF, of the triangle BEF, is greater than the inward oppoſite angle EFB (th. 8); and ſince theſe two angles alſo are equal (by the conſtr.), it follows, that thoſe angles are both equal and unequal at the ſame time: which is impoſſible. Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other. Q. E. D.

Corol. Right lines which are perpendicular to one, of two parallel lines, are alſo perpendicular to the other.

THEOREM XIII.

If a Line, Cutting Two other Lines, make the Alternate Angles Equal to each other, thoſe two Lines will be Parallel.



Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE equal to each other; then will AB be parallel to CD.

For if they be not parallel, let ſome other line, as FG, be parallel to AB. Then, becauſe of theſe parallels, the angle AEF is equal to the alternate angle EFG (th. 12). But the angle AEF is equal to the angle EFD (by hyp.) Therefore the angle EFD is equal to the angle EFG (ax. 1), that is, a part is equal to the whole, which is impoſſible. Therefore no line but CD can be parallel to AB. Q. E. D.

Corol. Thoſe lines which are perpendicular to the ſame line, are parallel to each other.

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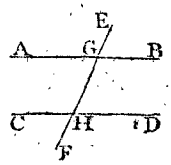
THEOREM XIV.

If a Line Cut two Parallel Lines; the Outward Angle will be Equal to the Inward Oppoſite one, on the Same Side; and the two Inward Angles, on the Same Side, will be equal to Two Right Angles.



Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward oppoſite angle GHD, on the ſame ſide of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.

For, ſince the two lines AB, CD are parallel, the angle AGH is equal to the alternate angle GHD (th. 12). But the angle AGH is equal to the oppoſite angle EGB (th. 7). Therefore the angle EGB is alſo equal to the angle GHD (ax. 1). Q. E. D.

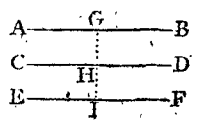
Again, becauſe the two adjacent angles EGB, BGH are together equal to two right angles (th. 6); of which the angle EGB has been ſhewn to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are alſo equal to two right angles.

Corol. 1. And converſely, if one line meeting two other lines, make the angles on the ſame ſide of it equal, thoſe two lines are parallels.

Corol. 2. If a line, cutting two other lines, make the ſum of the two inward angles, on the ſame ſide, leſs than two right angles, thoſe two lines will not be parallel, but will meet each other when produced.

THEOREM XV.

Thoſe Lines which are Parallel to the Same Line, are Parallel to each other.

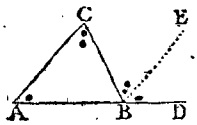


Let the lines AB, CD be each of them parallel to the line EF; then ſhall the lines AB, CD be parallel to each other.

For, let the line GI be perpendicular to EF. Then will this line be alſo perpendicular to both the lines AB, CD (corol. th. 12), and conſequently the two lines AB, CD are parallels (corol. th. 13).

THEOREM XVI.

If one Side of a Triangle be produced, the Outward Angle will be equal to both the Inward Oppoſite Angles taken together.

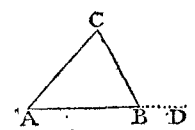


Let the ſide AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the ſum of the two inward oppoſite angles A and C.

For, conceive BE to be drawn parallel to the ſide AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 12). And AD, cutting the ſame two parallels AC, BE, makes the inward and outward angles on the ſame ſide, A and EBD, equal to each other (th. 14). Therefore, by equal additions, the ſum of the two angles A and C, is equal to the ſum of the two CBE and EBD, that is to the whole angle CBD (by ax. 2). Q. E. D.

THEOREM XVII

The Sum of all the Three Angles of any Plane Triangle, is equal to Two Right Angles.



Let ABC be any plane triangle; then the ſum of the three angles A + B + C is equal to two right angles.

For, let the ſide AB be produced to D. Then the outward angle CBD is equal to the ſum of the two inward oppoſite angles A + C (th. 16). To each of theſe equals add the inward angle B, then will the ſum of the three inward angles A + B + C be equal to the ſum of the two adjacent angles ABC + CBD (ax. 2). But the ſum of theſe two laſt adjacent angles is equal to two right angles (th. 6). Therefore alſo the ſum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle, be equal to two angles in another triangle, the third angles will alſo be equal (ax. 3), and the two triangles equiangular.

Corol. 2. If one angle in one triangle, be equal to one angle in another, the ſums of the remaining angles will alſo be equal (ax. 3).

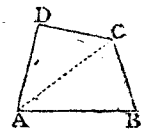
Corol. 3.

Corol. 3. If one angle of a triangle be right, the ſum of the other two will alſo be equal to a right angle, and each of them ſingly will be acute, or leſs than a right angle.

Corol. 4. The two leaſt angles of every triangle are acute, or each leſs than a right angle.

THEOREM XVIII.

In any Quadrangle, the Sum of all the Four Inward Angles, is equal to Four Right Angles.



LET ABCD be a quadrangle; then the ſum of the four inward angles, A + B + C + D is equal to 4 right angles.

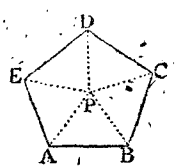
Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC. Then, becauſe the ſum of the three angles of each of theſe triangles, is equal to two right angles (th. 17); it follows that the ſum of all the angles of both triangles, which make up the four angles of the quadrangle, muſt be equal to 4 right angles (ax. 2). Q. E. D.

Corol. 1. Hence, if three of the angles be right ones, the fourth will alſo be a right angle.

Corol. 2. Alſo, if the ſum of two of the four angles be equal to two right angles, the ſum of the remaining two will likewiſe be equal to two right angles.

THEOREM XIX.

IN any Polygon, the Sum of all the Inward Angles, taken together, is equal to Twice as many Right Angles, wanting four, as the Figure has Sides.



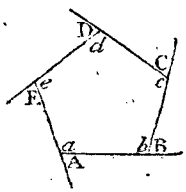
Let ABCDE be any polygon; the ſum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has ſides.

For, from any point P, within it, draw lines PA, PB, PC, &c, to all the angles, dividing the polygon into as many triangles as it has ſides. Now the ſum of the three angles of each of theſe triangles, is equal to two right angles (th. 17); therefore the ſum of the angles of all the triangles is equal to twice as many right angles as the figure has ſides. But the ſum of all the angles about the point P, which are ſo many

many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and muſt be deducted out of the former ſum. Hence it follows that the ſum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has ſides, wanting the ſaid four right angles. Q. E. D.

THEOREM XX.

If every Side of any right-lined Figure be produced out, the Sum of all the Outward Angles thereby made, will be equal to Four Right Angles.

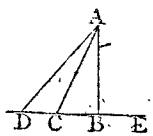


Let A, B, C, &c, be the outward angles of any polygon, made by producing all the ſides; then will the ſum A + B + C + D + E, of all thoſe outward angles, be equal to four right angles.

For, every one of theſe outward angles, together with its adjacent inward angle, make up two right angles, as A + *a* equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many outward, or inward angles, as the figure has ſides. Therefore the ſum of all the inward and outward angles, is equal to twice as many right angles as the figure has ſides. But the ſum of all the inward angles, with four right angles, is equal to twice as many right angles as the figure has ſides (th. 19). Therefore the ſum of all the inward and all the outward angles, is equal to the ſum of all the inward angles and four right angles (by ax. 1). From each of theſe take away all the inward angles, and there remains all the outward angles equal to four right angles (by ax. 3).

THEOREM XXI.

A Perpendicular is the Shorteſt Line that can be drawn from a Given Point to an Indefinite Line. And, of any other Lines drawn from the ſame Point, thoſe that are Neareſt the Perpendicular, are Leſs than thoſe More Remote.



If AB, AC, AD, &c, be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular. Then ſhall the perpendicular AB be leſs than AC, and AC leſs than AD, &c.

For, the angle B being a right one, the angle.

angle C is acute (by cor. 3, th. 17), and therefore leſs than the angle B. But the greater angle of a triangle is ſubtended by the greater ſide (th. 9). Therefore the ſide AC is greater than the ſide AB.

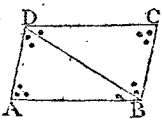
Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuſe (by th. 6). But the angle D is acute (corol. 3, th. 17). Therefore the angle ACD is greater than the angle D. And, ſince the greater ſide is oppoſite to the greater angle, therefore the ſide AD is greater than the ſide AC. Q. E. D.

Corol. A perpendicular is the leaſt diſtance of a given point from a line.

THEOREM XXII.

THE Oppoſite Sides and Angles of any Parallelogram are equal to each other; and the Diagonal divides it into two Equal Parts.

Let ABCD be a parallelogram, of which the diagonal is BD; then will its oppoſite ſides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.



For, ſince the ſides AB and DC are parallel, as alſo the ſides AD and BC (defin. 32), and the line BD meets them; therefore the alternate angles are equal (th. 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have alſo their third angles equal (cor. 1, th. 17), namely, the angle A equal to the angle C, which are two of the oppoſite angles of the parallelogram.

Alſo, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two oppoſite angles of the parallelogram. Q. E. D.

Again, ſince the two triangles are mutually equiangular, and have a ſide in each equal, viz. the common ſide BD; therefore the two triangles are identical (th. 2), or equal in all reſpects, namely, the ſide AB to the oppoſite ſide DC, and AD to the oppoſite ſide BC, and the whole triangle ABD to the whole triangle BCD. Q. E. D.

Corol. 1.

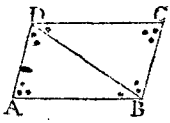
Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will alſo be right angles, and the parallelogram a rectangle.

Corol. 2. Hence alſo, the ſum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

Every Quadrilateral whoſe Oppoſite Sides are Equal, is a Parallelogram.

LET ABCD be a quadrangle, having the oppoſite ſides equal, namely, the ſide AB equal to DC, and AD equal to BC; then ſhall theſe equal ſides be alſo parallel, and the figure a parallelogram.



For, let the diagonal BD be drawn. Then, the triangles ABD, CBD being mutually equilateral (by hyp.), they are alſo mutually equiangular (th. 5); and conſequently the oppoſite ſides are parallel (th. 13); viz. the ſide *AB* parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

THEOREM XXIV.

RIGHT Lines joining the Correſponding Extremes of two Equal and Parallel Lines, are themſelves Equal and Parallel.

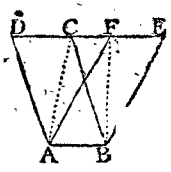
Let AB, DC be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be alſo equal and parallel. [See the fig. above.]

For, draw the diagonal BD. Then, becauſe AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 12). Hence then, the two triangles having two ſides and the contained angles equal, viz. the ſide AB equal to the ſide DC, and the ſide BD common, and the contained angle ABD equal to the contained angle BDC, they will have the remaining ſides and angles alſo reſpectively equal (th. 1); conſequently AD equal to BC, and alſo parallel to it (th. 12). Q. E. D.

THEOREM XXV.

PARALLELOGRAMS, or Triangles, ſtanding on the Same Baſe, and between the Same Parallels, are Equal to each other.

Let



Let ABCD, ABEF be two parallelograms, and ABC, ABF two triangles, ſtanding on the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC to the triangle ABF.

For, ſince the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (th. 14); the two triangles ADF, BCE are therefore equiangular (cor. 1, th. 17); and having the two correſponding ſides AD, BC equal (th. 22), being oppoſite ſides of a parallelogram, theſe two triangles are identical, or equal in all reſpects (th. 2). If each of theſe equal triangles then be taken from the whole ſpace ABED, there will remain the parallelogram ABEF in the one caſe, equal to the parallelogram ABCD in the other (by ax. 3).

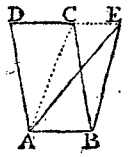
Alſo, the triangles ABC, ABF, on the ſame baſe AB, and between the ſame parallels, are equal, being the halves of the ſaid equal parallelograms (th. 22). Q. E. D.

Corol. 1. Parallelograms, or triangles, having the ſame baſe and altitude, are equal. For the altitude is the ſame as the perpendicular or diſtance between the two parallels, which is every where equal, by the definition of parallels.

Corol. 2. Parallelograms, or triangles, having equal baſes and altitudes, are equal. For, if the one figure be applied with its baſe upon the other, the baſes will coincide or be the ſame; becauſe they are equal; and ſo the two figures, having the ſame baſe and altitude, are equal.

THEOREM XXVI.

If a Parallelogram and a Triangle ſtand on the Same Baſe, and between the Same Parallels, the Parallelogram will be Double the Triangle, or the Triangle Half the Parallelogram.



Let ABCD be a parallelogram, and ABE a triangle, on the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.

For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 22). Then, becauſe the triangles ABC, ABE,

ABE, on the ſame baſe, and between the ſame parallels, are equal (th. 25); and becauſe the triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is alſo equal to half the ſame parallelogram ABCD. Q. E. D.

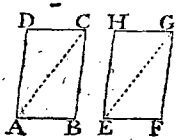
Corol. 1. A triangle is equal to half a parallelogram of the ſame baſe and altitude, becauſe the altitude is the perpendicular diſtance between the parallels, which is every where equal, by the definition of parallels.

Corol. 2. If the baſe of a parallelogram be half that of a triangle, of the ſame altitude, or the baſe of the triangle be double that of the parallelogram, the two figures will be equal to each other.

THEOREM XXVII.

RECTANGLES that are contained under Equal Lines, are Equal to each other.

Let BD, FH be two rectangles, having the ſides AB, BC, equal to the ſides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.



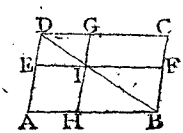
For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Now, the two triangles ABC, EFG, are equal to each other (th. 1), becauſe they have the two ſides AB, BC, and the contained angle B, equal to the two ſides EF, FG, and the contained angle F (by hyp.) But theſe equal triangles are the halves of the reſpective rectangles. And becauſe the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are equal alſo (by ax. 6). Q. E. D.

Corol. The ſquares on equal lines, are alſo equal; for every ſquare is a ſpecies of rectangle.

THEOREM XXVIII.

THE Complements of the Parallelograms, which are about the Diagonal of any Parallelogram, are equal to each other.

Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

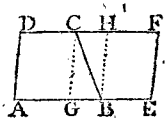


For,

For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 22); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB respectively equal to the parts DGI, IFB, the remaining parts AI, IC must also be equal (by ax. 3). Q. E. D.

THEOREM XXIX.

A TRAPEZOID, or Trapezium having two Sides Parallel, is equal to Half a Parallelogram, whose Base is the Sum of those two Sides, and its Altitude the Perpendicular Distance between them.

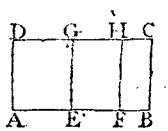


Let ABCD be the trapezoid, having its two sides AB, DC parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitude, are equal (corol. 2, th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CBH; consequently the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q. E. D.

THEOREM XXX.

THE Sum of all the Rectangles contained under one Whole Line, and the several Parts of another Line, any how divided, is Equal to the Rectangle contained under the Two Whole Lines.



Let AD be the one line, and AB the other line, divided into the parts AE, EF, FB; then shall the rectangle contained by AD and AB, be equal to the sum of the rectangles of AD and AE, AD and EF, and AD and FB; thus expressed AD . AB $=$ AD . AE $+$ AD . EF $+$ AD . FB.

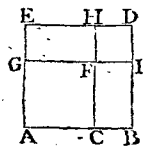
For, make the rectangle AC of the two whole lines AD, AB; and draw EG, FH perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH,

EH, FC. But these rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, because AD is equal to each of the two EG, FH. Therefore the rectangle AD. AB is equal to the sum of all the other rectangles AD. AE, AD . EF, AD . FB.

Corol. If a right line be divided into any two parts; the square on the whole line, is equal to both the rectangles of the whole line and each of the parts.

THEOREM XXXI.

THE Square of the Sum of two Lines, exceeds the Sum of their Squares, by Twice the Rectangle of the said Lines. Or, the Square of a whole Line, is equal to the Squares of its two Parts, together with Twice the Rectangle of those Parts.



Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the squares of AC, CB, together with twice the rectangle of AC, CB. That is, $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

For, let ABDE be the square on the sum or whole line AB, and ACFG the square on the part AC. Produce CF and GF to the other sides at H and I.

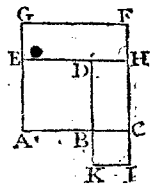
From the lines CH, GI, which are equal, being each equal to the sides of the square AB or BD (th. 22), take the parts CF, GF, which are equal also, being the sides of the square AF, and there remains FH equal to FI, which are also equal to DH, DI, being the opposite sides of a parallelogram. Hence the figure HI is equilateral; and it has all its angles right ones (corol. 1, th. 22); it is therefore a square on the line FI, or the square of its equal CB. Also the figures EF, FB are equal to two rectangles under AC and CB, because GF is equal to AC, and FH or FI equal to CB. But the whole square AD is made up of the four figures, viz. the two squares AF, FD, and the two equal rectangles EF, FB. That is, the square of AB is equal to the squares of AC, CB, together with twice the rectangle of AC, CB.

Corol. Hence, if a line be divided into two equal parts; the square of the whole line, will be equal to four times the square of half the line.

THE-

THEOREM XXXII.

THE Square of the Difference of two Lines, is leſs than the Sum of their Squares, by Twice the Rectangle of the ſaid Lines.



Let AC, BC be any two lines, and AB their difference: then will the ſquare of AB be leſs than the ſquares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2AC.BC$.

For, let ABDE be the ſquare on the difference AB, and ACFG the ſquare on the line AC. Produce ED to H; alſo produce DB and HC, and draw KI, making BI the ſquare of the other line BC.

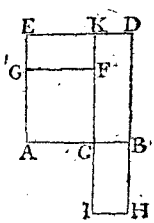
Now it is viſible that the ſquare AD is leſs than the two ſquares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; conſequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again, FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC or equal to AC; and conſequently the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI are two rectangles of the two lines AC, BC; and conſequently the ſquare of AB is leſs than the ſquares of AC, BC, by twice the rectangle AC.BC.

THEOREM XXXIII.

THE Difference of the Squares of any two unequal Lines, is equal to the Rectangle under the Sum and Difference of the ſame Lines.



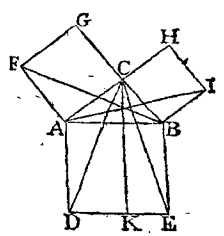
Let AB, AC be any two unequal lines; then will the difference of the ſquares of AB, AC be equal to a rectangle under their ſum and difference. That is, $AB^2 - AC^2 = \overline{AB + AC}.\overline{AB - AC}$.

For, let ABDE be the ſquare of AB, and ACFG the ſquare of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

Then the difference of the two ſquares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB are equal to the two KB, BI, or to the whole KH; and conſequently KH is equal to the difference of the ſquares AD, AF. But KH is a rectangle contained under DH, or the ſum of AB and AC, and KD, or the difference of AB and AC. Therefore the difference of the ſquares of AB, AC, is equal to the rectangle under their ſum and difference. Q. E. D.

THEOREM XXXIV.

In any Right-Angled Triangle, the Square of the Hypothenuſe, is equal to the Sum of the Squares of the other two Sides.



Let ABC be a right-angled triangle, having the right angle C; then will the ſquare of the hypothenuſe AB, be equal to the ſum of the ſquares of the other two ſides AC, CB. Or, $AB^2 = AC^2 + BC^2$.

For, on AB deſcribe the ſquare AE, and on AC, CB the ſquares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, becauſe the line AC meets the two CG, CB, ſo as to make two right angles, theſe two make one ſtraight line GB (corol. 1, th. 6). And becauſe the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a ſquare; to each of theſe equals add the common angle BAC, ſo will the whole angle or ſum FAB, be equal to the whole angle CAD. But the ſide FA is equal to the ſide AC, and the ſide AB to the ſide AD, being ſides of the ſame ſquare; ſo that the two ſides FA, AB, and their included angle FAB, are equal to the two ſides CA, AD and the contained angle CAD, each to each; therefore the whole triangle AFB is equal to the whole triangle ACD (th. 1).

But the ſquare AG is double the triangle AFB, on the ſame baſe FA, and between the ſame parallels FA, GB (th. 26); in like manner, the parallelogram AK is double the triangle ACD, on the ſame baſe AD, and between the ſame parallels AD, CK. And ſince the doubles of equal things,

things, are equal (by ax. 6); therefore the ſquare AG is equal to the parallelogram AK.

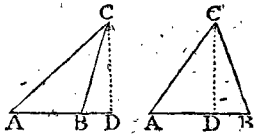
In like manner, the other ſquare BH is proved equal to the other parallelogram BK: And conſequently the two ſquares AG and BH together, equal to the two parallelograms AK and BK together, or to the whole ſquare AE. That is, the ſum of the two ſquares on the two leſs ſides is equal to the ſquare on the greateſt ſide. Q. E. D.

Coral. Hence, the ſquare of either of the two leſs ſides, is equal to the difference of the ſquares of the hypothenuſe and the other ſide (ax. 3); or, equal to the rectangle contained under the ſum and difference of the ſaid hypothenuſe and other ſide (th. 33).

THEOREM XXXV.

In any Triangle, the Difference of the Squares of the two Sides, is Equal to the Difference of the Squares of the two Lines, or Diſtances, included between the Extremes of the Baſe and the Perpendicular.

Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the ſquares of AC, BC, be equal to the difference of the ſquares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.



For, ſince AC^2 is equal to $AD^2 + CD^2$
and BC^2 is equal to $BD^2 + CD^2$ } by th. 34;
Theref. the difference between AC^2 and BC^2
is equal to the difference between $AD^2 + CD^2$
and $BD^2 + CD^2$,
or equal to the difference between AD^2 and BD^2,
by taking away the common ſquare CD^2. Q. E. D.

Corol. The rectangle under the ſum and difference of the two ſides of any triangle, is equal to the rectangle under the ſum and difference of the diſtances between the perpendicular and the two extremes of the baſe, or equal to the rectangle under the baſe and the difference or ſum of the ſegments, according as the perpendicular falls within or without the triangle.

That is, $\overline{AC+BC}.\overline{AC-BC}=\overline{AD+BD}.\overline{AD-BD}$

Or, $\overline{AC+BC}.\overline{AC-BC}=AB.\overline{AD-BD}$ in the 2d figure,

And $\overline{AC+BC}.\overline{AC-BC}=AB.\overline{AD+BD}$ in the 1ſt figure.

THEOREM XXXVI.

In any Obtuſe-angled Triangle, the Square of the Side ſubtending the Obtuſe Angle, is Greater than the Sum of the Squares of the other two Sides, by Twice the Rectangle of the Baſe and the Diſtance of the Perpendicular from the Obtuſe Angle.

Let ABC be a triangle, obtuſe-angled at B, and CD perpendicular to AB; then will the ſquare of AC be greater than the ſquares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2=AB^2+BC^2+2AB.BD$. See the 1ſt fig. above, or below.

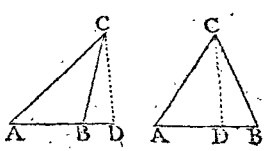
For, ſince the ſquare of the whole line AD is equal to the ſquares of the parts AB, BD, with twice the rectangle of the ſame parts AB, BD (th. 31). If to each of theſe equals there be added the ſquare of CD; then the ſquares of AD, CD, will be equal to the ſquares of AB, BD, CD, with twice the rectangle of AB, BD (by ax. 2).

But the ſquares of AD, CD, are equal to the ſquare of AC; and the ſquares of BD, CD, equal to the ſquare of BC (th. 34); therefore the ſquare of AC is equal to the ſquares of AB, BC, together with twice the rectangle of AB, BD. Q. E. D.

THEOREM XXXVII.

In any Triangle, the ſquare of the Side ſubtending an Acute Angle, is Leſs than the Squares of the Baſe and the other Side, by Twice the Rectangle of the Baſe and the Diſtance of the Perpendicular from the Acute Angle.

Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the ſquare of BC, be leſs than the ſquares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2=AB^2+AC^2-2AB.AD$.



For,

For, in fig. 1, AC^2 is $= BC^2 + AB^2 + 2AB \cdot BD$ (th. 36).
To each of these equals add the square of AB,
then is $AB^2 + AC^2 = BC^2 + 2AB^2 + 2AB \cdot BD$ (ax. 2),
or $= BC^2 + 2AB \cdot AD$ (th. 30).

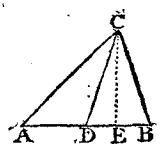
Q. E. D.

Again, in fig. 2, AC^2 is $= AD^2 + DC^2$ (th. 34).
And $AB^2 = AD^2 + DB^2 + 2AD \cdot DB$ (th. 31).
Therefore $AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2 + 2AD\ DB$ (ax. 2),
or $= BC^2 + 2AD^2 + 2AD \cdot DB$ (th. 34),
or $= BC^2 + 2AB \cdot AD$ (th. 30). Q. E. D.

THEOREM XXXVIII.

In any Triangle, the Double of the Square of a Line drawn from the Vertex to the Middle of the Base, together with Double the Square of the Half Base, is Equal to the Sum of the Squares of the other Two Sides.

Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB: then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, BD; or $AC^2 + CB^2 = 2CD^2 + 2DB^2$.

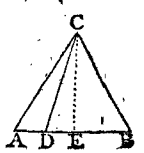


For, let CE be perpendicular to the base AB. Then, since (by th. 36) AC^2 exceeds the sum of the two squares AD^2 and CD^2 (or BD^2 and CD^2) by the double rectangle $2AD \cdot DE$ (or $2BD \cdot DE$); and since (by th. 37) BC^2 is less than the same sum by the said double rectangle; it is manifest that both AC^2 and BC^2 together, must be equal to that sum twice taken; the excess on the one part making up the defect on the other. Q. E. D.

THEOREM XXXIX.

In an Isosceles Triangle, the Square of a Line drawn from the Vertex to any Point in the Base, together with the Rectangle of the Segments of the Base, is Equal to the Square of one of the Equal Sides of the Triangle.

Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC be equal to the square of CD together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \cdot DB$.



For,

For, let CE bisect the vertical angle; then will it also bisect the base AB perpendicularly, making AE = EB (corol. 1, th. 3).

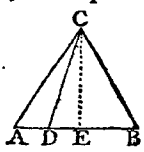
But, in the triangle ACD, obtuse-angled at D, the square AC^2 is $= CD^2 + AD^2 + 2AD \cdot DE$ (th. 36),

or $= CD^2 + AD \cdot \overline{AD + 2DE}$ (th. 30),

or $= CD^2 + AD \cdot \overline{AE + DE}$,

or $= CD^2 + AD \cdot \overline{BE + DE}$,

or $= CD^2 + AD \cdot DB$.

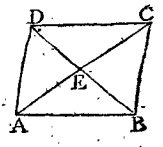


Q. E. D.

THEOREM XL.

In any Parallelogram, the two Diagonals Bisect each other, and the Sum of their Squares is Equal to the Sum of the Squares of all the Four Sides of the Parallelogram.

Let ABCD be a parallelogram, whose diagonals intersect each other in E: then will AE be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,



AE = EC, and BE = ED,

and $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$.

For, the triangles AEB, DEC are equiangular, because they have the opposite angles at E equal (th. 7), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the side AB equal to the side DC (th. 27); therefore these two triangles are identical, and have their corresponding sides equal (th. 2), viz. AE = EC, and BE = ED.

Again, since AC is bisected in E, the sum of the squares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 38).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4DE^2$ (ax. 2).

But, because the square of a whole line is equal to four times the square of half the line (cor. th. 31), that is, $AC^2 = 4AE^2$, and $BD^2 = 4DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ (ax. 1).

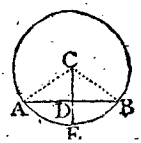
Q. E. D.

THE-

THEOREM XLI.

IF a Right Line, drawn through or from the Centre of a Circle, Biſect a Chord, it will be Perpendicular to it; or if it be Perpendicular to the Chord, it will Biſect both the Chord and the Arc of the Chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord is biſected in the point D, CD will be perpendicular to AB.



For, draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (def. 44), CD common, and AD equal to DB (by hyp.), have all the three ſides of the one, equal to all the three ſides of the other, and ſo have their angles alſo equal (th. 5). Hence then, the angle ADC being equal to the angle BDC, theſe angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord AB be biſected at the point D, or have AD equal to DB, and the arc AEB in the point E, or have AE equal EB.

For, having drawn CA, CB, as before. Then, in the triangle ABC, becauſe the ſide CA is equal to the ſide CB, their oppoſite angles A and B are alſo equal (th. 3). Hence then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore their third angles are alſo equal (corol. 1, th. 17). And having the ſide CD common, they have alſo the ſide AD equal to the ſide DB (th. 2).

Alſo, ſince the angle ACE is equal to the angle BCE, the arc AE which meaſures the former (def. 57), is equal the arc BE which meaſures the latter, ſince equal angles muſt have equal meaſures.

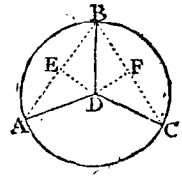
Corol. Hence a line biſecting any chord at right angles, paſſes through the centre of the circle.

THEOREM XLII.

IF More than Two Equal Right Lines can be drawn from any Point within a Circle to the Circumference, that Point will be the Centre.

Let

Let ABC be a circle, and D a point within it; then if any three lines DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.



For, draw the chords AB, BC, which let be biſected in the points E, F, and join DE, DF.

Then, the two triangles DAE, DBE, have the ſide DA equal to the ſide DB by ſuppoſition, the ſide AE equal to the ſide EB by hypotheſis, and the ſide DE common; therefore theſe two triangles are identical, and have the angles at E equal to each other (th. 5); conſequently DE is perpendicular to the chord AB (def. 11), and therefore paſſes through the centre of the circle (corol. th. 41).

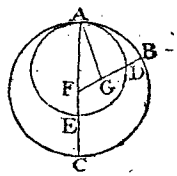
In like manner, it may be ſhewn that DF paſſes through the centre. Conſequently the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii.

Q. E. D.

THEOREM XLIII.

If two Circles touch one another Internally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.

Let the two circles ABC, ADE touch one another internally in the point A; then will the point A and the centres of thoſe circles be all in the ſame right line.



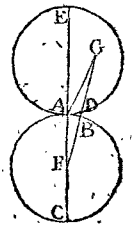
For, let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be ſuppoſed in ſome other point as G; through which draw the line FG cutting the two circles in B and D.

Now, in the triangle AFG, the ſum of the two ſides FG, GA is greater than the third ſide AF (th. 10), or greater than its equal radius FB. From each of theſe take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being ſuppoſed the centre of the inner circle, its two radii GA, GD are equal to one another; conſequently GD will alſo be greater than GB. But ADE being the inner circle, GD is neceſſarily leſs

leſs than GB. So that GD is both greater and leſs than GB; which is abſurd. Conſequently the centre G cannot be out of the line AFC. Q. E. D.

THEOREM XLIV.

If two Circles Touch one another Externally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.



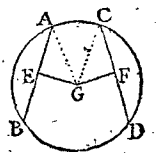
Let the two circles ABC, ADE touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the ſame right line.

For, let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the circle ADE can be out of the line FE, let it, if poſſible, be ſuppoſed in ſome other point G; and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the ſum of the two ſides AF, AG is greater than the third ſide FG (th. 10). But, F and G being the centres of the two circles, the two radii GA, GD are equal, as are alſo the two radii AF, FB. Hence the ſum of GA, AF is equal to the ſum of GD, BF; and therefore this latter ſum alſo GD, BF is greater than GF, which is abſurd. Conſequently the centre G cannot be out of the line FE. Q. E. D.

THEOREM XLV.

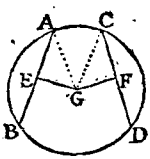
Any two Chords in a Circle, which are Equally Diſtant from the Centre, are Equal to each other; or if they be Equal to each other, they will be Equally Diſtant from the Centre.



Let AB, CD be any two chords at equal diſtances from the centre D; then will theſe two chords AB, CD be equal to each other.

For, draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal diſtances from the centre G. Then, the two right-angled triangles GAE, GCF, having the ſide GA equal the ſide GC, and the ſide GE equal the ſide GF, the difference of the ſquares GA^2, GE^2 will be equal the difference of the

ſquares



ſquares GC^2, GF^2. But AE^2 is equal to the difference of the ſquares GA^2, GE^2, and CF^2 equal the difference of the ſquares GC^2, GF^2 (corol. th. 34); therefore the ſquare AE^2 is equal the ſquare CF^2, and the line AE equal the line CF. But AB is the double of AE, and CD is the double of CF (th. 41); therefore AB is equal CD (by ax. 6). Q. E. D.

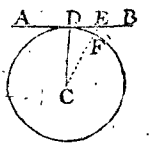
Again, if the chord AB be equal to the chord CD; then will their diſtances from the centre, GE, GF, alſo be equal to each other.

For, ſince AB is equal CD by ſuppoſition, the half AE is equal the half CF. Alſo the radii GA, GC being equal, the difference of the ſquares GA^2, GE^2 will be equal the difference of the ſquares GC^2, GF^2; that is, the ſquare GE^2 equal the ſquare GF^2 (corol. th. 34), and conſequently the diſtance GE equal the diſtance GF. Q. E. D.

Corol. Hence, if two right-angled triangles having equal hypothenuſes, have two other ſides alſo equal; then will the third ſides be equal, and the two triangles identical, or equal in all reſpects.

THEOREM XLVI.

A Right Line Perpendicular to the Extremity of a Radius, is a Tangent to the Circle.



LET the line ADB be perpendicular to the radius CD of a circle; then ſhall AB touch the circle in the point D only.

For, from any other point E in the line AB, draw CFE to the centre, meeting the circle in F.

Then, becauſe the triangle CDE is right-angled at D, the ſquare of the hypothenuſe CE is equal to both the ſquares CD^2 and DE^2 (th. 34). Therefore CE is greater than CD, or than its equal CF. Hence the point E is without a circle; and the ſame for every other point in the line AB. Conſequently the whole line is without the circle, and meets it in the point D only.

Corol. Hence it follows, that a perpendicular is the ſhorteſt line which can be drawn from any point to a given line; ſince the perpendicular CD is ſhorter than any other line which can be drawn from the point C to the line AB, the ſame as theorem 21.

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THEOREM XLVII.

If a Right Line be a Tangent to a Circle; a Radius drawn to the Point of Contact will be Perpendicular to the Tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. [See the laſt figure.]

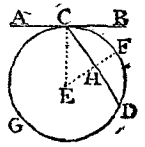
For, the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, muſt paſs out of the circle to arrive at this line. The line CD is therefore the ſhorteſt that can be drawn from the point C to the line AB, and conſequently (th. 21), is perpendicular to that line.

Corol. Hence, converſely, a line drawn perpendicular to a tangent, at the point of contact, paſſes through the centre of the circle.

THEOREM XLVIII.

The Angle formed by a Tangent and Chord is Meaſured by Half the Arc of that Chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C: then is the angle BCD meaſured by half the arc CFD, and the angle ACD meaſured by half the arc CGD.

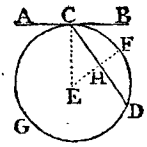


For, draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.

Then, the radius EF, being perpendicular to the chord CD, biſects the arc CFD (th. 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right angle, the ſum of the two remaining angles E and C is equal to a right angle (corol. 3, th. 17), which is equal to the angle BCE, becauſe the radius CE is perpendicular to the tangent: From each of theſe equals take away the common part or angle I, and there remains the angle E equal to the angle BCD. But the angle E is meaſured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD muſt alſo have the ſame meaſure, namely, half the arc CFD of the chord CD.

Again,



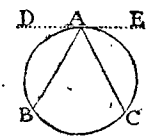
Again, the line GEF, being perpendicular to the chord CD, biſects the arc CGD (th. 41). Therefore CG is half the arc CGD. Now, ſince the line CE, meeting FG, makes the ſum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the ſum of the two angles at C equal to two right angles; if from theſe two equal ſums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of theſe, CEG, being an angle at the centre, is meaſured by the arc CG (def. 57), conſequently the equal angle ACD muſt alſo have the ſame meaſure CG, which is half the arc CGD of the chord CD. Q. E. D.

Coral. 1. The ſum of two right angles is meaſured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are meaſured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Coral. 2. Hence alſo one right angle muſt have for its meaſure a quarter of the circumference, or 90 degrees.

THEOREM XLIX.

An Angle at the Circumference of a Circle, is Meaſured by Half the Arc that ſubtends it.

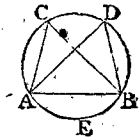


LET BAC be an angle at the circumference; it has for its meaſure, half the arc BC which ſubtends it.

For, ſuppoſe the tangent DE paſſing through the point of contact A. Then, the angle DAC being meaſured by half the arc ABC, and the angle DAB by half the arc AB (th. 48); it follows, by equal ſubtraction, that the difference, or angle BAC, muſt be meaſured by half the arc BC, which it ſtands upon. Q. E. D.

THEOREM L.

All Angles in the Same Segment of a Circle, or Standing on the Same Arc, are Equal to each other.

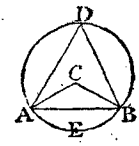


Let ACB, ADB be two angles in the same segment ACDB, or, which is the same thing, standing on the same arc AEB; then will the angle ACB be equal to the angle ADB.

For each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (ax. 11).

THEOREM LI.

An Angle at the Centre of a Circle is Double the Angle at the Circumference, when both of them stand on the Same Arc.

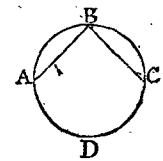


Let ACB be an angle at the centre C, and ADB an angle at the circumference, both standing on the same arc or same chord AB: then will the angle C be double of the angle D, or the angle D equal to half the angle C.

For the angle at the centre C is measured by the whole arc AEB (def. 57), and the angle at the circumference D is measured by half the same arc AEB (th. 49); therefore the angle D is only half the angle C, or the angle C double the angle D.

THEOREM LII.

An Angle in a Semicircle, is a Right Angle.



If ABC, or ADC be a Semicircle; then any angle ABC in that semicircle, is a right angle.

For, the angle B, at the circumference, is measured by half the arc ADC (th. 49), that is, by a quadrant of the circumference. But a quadrant is the measure of a right angle (corol. 4, th. 6; or corol. 2, th. 48). Therefore the angle B is a right angle.

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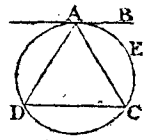
THEOREM LIII.

THE Angle formed by a Tangent to a Circle and a Chord drawn from the Point of Contact, is Equal to the Angle in the Alternate Segment.

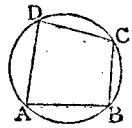


If AB be a tangent, AC a chord, and D any angle in the alternate segment ADC; then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.

For the angle D, at the circumference, is measured by half the arc AEC (th. 49); and the angle BAC, made by the tangent and chord, is also measured by the same half arc AEC (th. 48); therefore these two angles are equal (ax. 11).

THEOREM LIV.

The Sum of any Two Opposite Angles of a Quadrangle Inscribed in a Circle, is Equal to Two Right Angles.

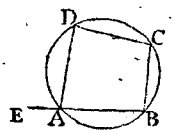


LET ABCD be any quadrilateral inscribed in a circle; then shall the sum of the two opposite angles A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc DCB, which it stands upon, and the angle C by half the arc DAB (th. 49); therefore the sum of the two angles A and C is measured by half the sum of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (corol. 4, th. 6); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner, it is shewn that the sum of the other two opposite angles, D and B, is equal to two right angles. Q. E. D.

THEOREM LV.

IF any Side of a Quadrangle, Inscribed in a Circle, be Produced out, the Outward Angle will be Equal to the Inward Opposite Angle.



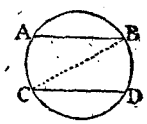
If the side AB, of the quadrilateral ABCD, inscribed in a circle, be produced to E; the outward angle DAE will be equal to the inward opposite angle C.

For,

For, the ſum of the two adjacent angles DAE and DAB is equal to two right angles (th. 6); and the ſum of the two oppoſite angles C and DAB is alſo equal to two right angles (th. 54); therefore the ſum of the two angles DAE and DAB is equal to the ſum of the two C and DAB (ax. 1). From each of theſe equals taking away the common angle DAB, there remains the angle DAE equal the angle C. Q. E. D.

THEOREM LVI.

Two Parallel Chords Intercept Equal Arcs.

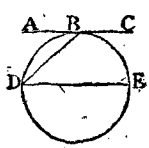


LET the two chords AB, CD be parallel; then will the arcs AC, BD be equal; or AC = BD.

For, draw the line BC. Then, becauſe the lines AB, CD are parallel, the alternate angles B and C are equal (th. 12). But the angle at the circumference B, is meaſured by half the arc AC (th. 49); and the other angle at the circumference C is meaſured by half the arc BD: the halves of the arcs AC, BD, and conſequently the arcs themſelves, are therefore equal. Q. E. D.

THEOREM LVII.

If a Tangent and Chord be Parallel to each other, they Intercept Equal Arcs.



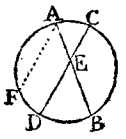
LET the tangent ABC be parallel to the chord DE; then are the arcs BD, BE equal; that is, BD = BE.

For, draw the chord BD. Then, becauſe the lines AB, DE are parallel, the alternate angles D and B are equal (th. 12). But the angle B, formed by a tangent and a chord, is meaſured by half the arc BD (th. 48); and the angle at the circumference D is meaſured by half the arc BE (th. 49); the arcs BD, BE are therefore equal. Q. E. D.

THEOREM LVIII.

THE Angle formed, Within a Circle, by the Interſection of two Chords, is Meaſured by Half the Sum of the Two Arcs Intercepted by thoſe Chords.

Let the two chords AB, CD interſect at the point E; the angle AEC, or DEB, is meaſured by half the ſum of two arcs AC, DB.



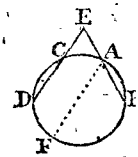
For, draw the chord AF parallel to CD. Then, becauſe the lines AF, CD are parallel, and AB cuts them, the angles on the ſame ſide A and DEB are equal (th. 14). But the angle at the circumference A is meaſured by half the arc BF, or of the ſum of FD and DB (th. 49); therefore the angle E is alſo meaſured by half the ſum of FD and DB.

Again, becauſe the chords AF, CD are parallel, the arcs AC, FD are equal (th. 56); therefore the ſum of the two arcs AC, DB is equal to the ſum of the two FD, DB; and conſequently the angle E, which is meaſured by half the latter ſum, is alſo meaſured by half the former. Q. E. D.

THEOREM LIX.

The Angle formed, Without a Circle, by two Secants, is Meaſured by Half the Difference of the Intercepted Arcs.

LET the angle E be formed by two ſecants AB and CD, this angle is meaſured by half the difference of the two arcs AC, DB, intercepted by the two ſecants.



Draw the chord AF parallel to CD. Then, becauſe the lines AF, CD are parallel, and AB cuts them, the angles on the ſame ſide A and BED are equal (th. 14). But the angle A, at the circumference, is meaſured by half the arc BF (th. 49), or of the difference of DF and DB; therefore the equal angle E is alſo meaſured by half the difference of DF, DB.

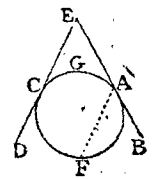
Again, becauſe the chords AF, CD are parallel, the arcs AC, FD are equal (th. 56); therefore the difference of the two arcs AC, DB is equal to the difference of the two DF, DB. Conſequently the angle E, which is meaſured by half the latter difference, is alſo meaſured by half the former.

Q. E. D.

THE-

THEOREM LX.

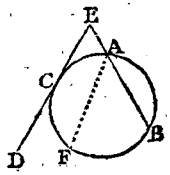
The Angle formed by two Tangents, is Meaſured by Half the Difference of the two Intercepted Arcs.



Let EB, ED be two tangents to a circle at the points A, C; then the angle E is meaſured by half the difference of the two arcs CFA, CGA.

For, draw the chord AF parallel to CD. Then, becauſe the lines AF, CD are parallel, and AB meets them, the angles on the ſame ſide A and E are equal (th. 14). But the angle A, formed by the chord AF and tangent AB, is meaſured by half the arc AF (th. 48); therefore the equal angle E is alſo meaſured by half the ſame arc AF, or half the difference of the arcs CFA and CF.

Again, becauſe the tangent ED and chord AF are parallel, the intercepted arcs CGA, CF, are equal (th. 57); the arc AF therefore, which is the difference of CFA and CF, is alſo the difference of CFA and CGA; conſequently the angle E, which is meaſured by half the former, is alſo meaſured by half the latter.

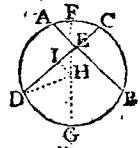


Corol. In like manner it is proved, that the angle E formed by a tangent ECD and a ſecant EAB, is meaſured by half the difference of the two intercepted arcs CA and CFB.

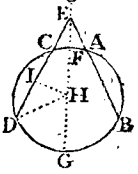
THEOREM LXI.

If two Lines, meeting a Circle each in two Points, Cut one other, either Within it or Without it; the Rectangle of the Parts of the one, will be Equal to the Rectangle of the Parts of the other; the Parts of each being meaſured from the point of meeting to the two interſections with the circumference.

Let the two chords AB, CD meet each other in E; the rectangle of AE, EB, is equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.



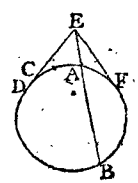
For, through the point E draw the diameter FG; alſo from the centre H, draw the radius DH, and HI perpendicular to CD.



Then, ſince DEH is a triangle, and the perp. HI biſects the chord CD (th. 41), the line CE is equal to the difference of the ſegments DI, EI, the ſum of them being DE. Alſo, becauſe H is the centre of the circle, and the radii DH, FH, GH are all equal, the line EG is equal to the ſum of the ſides DH, HE; and EF is equal to their difference.

But the rectangle of the ſum and difference of the two ſides of a triangle, is equal to the rectangle of the ſum and difference of the ſegments of the baſe (corol. th. 35); therefore the rectangle of FE, EG, is equal to the rectangle of CE, ED. In the ſame manner it is proved, that the rectangle of FE, EG, is equal to the rectangle of AE, EB. Conſequently the rectangle of AE, EB, is alſo equal to the rectangle of CE, ED. Q. E. D.

Corol. 1. When one of the lines in the ſecond caſe, as DE, by revolving about the point E, comes into the poſition of the tangent EC or ED, the two points C and D running into one; then the rectangle CE, ED becomes the ſquare of CE, becauſe CE and DE are then equal. Conſequently the rectangle of the parts of the ſecant, AE . EB, is equal to the ſquare of the tangent CE^2.



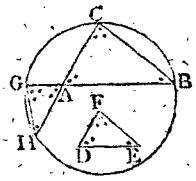
Corol. 2. Hence both the tangents EC, EF, drawn from the ſame point E, are equal; ſince the ſquare of each is equal to the ſame rectangle or quantity.

THEOREM LXII.

In Equiangular Triangles, the Rectangles of the Correſponding or Like Sides, taken alternately, are Equal.

LET

LET ABC, DEF be two equiangular triangles, having the angle A = the angle D, the angle B = the angle E, and the angle C = the angle F; alſo the like ſides AB, DE, and AC, DF, being thoſe oppoſite the equal angles: then will the rectangle of AB, DF, be equal to the rectangle of AC, DE.



In BA produced take AG equal DF; then through the three points B, C, G, conceive a circle BCGH to be deſcribed, meeting CA produced at H, and join GH.

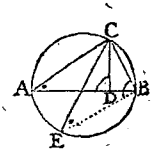
Then the angle G is equal to the angle C on the ſame arc BH, and the angle H equal to the angle B on the ſame arc CG (th. 50); alſo the oppoſite angles at A are equal (th. 7): therefore the triangle AGH is equiangular to the triangle ACB, and conſequently to the triangle DFE alſo. But the two like ſides AG, DF are alſo equal by ſuppoſition; conſequently the two triangles AGH, DFE are identical (th. 2), having the two ſides AG, AH equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 61), and conſequently the rectangle DF . AB equal the rectangle DE . AC. Q. E. D.

THEOREM LXIII.

THE Rectangle of the two Sides of any Triangle, is Equal to the Rectangle of the Perpendicular on the third Side and the Diameter of the Circumſcribing Circle.

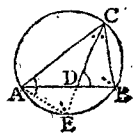
Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC: then the rectangle AC . CB is = the rectangle CD . CE.



For, join BE: then, in the two triangles ACD, ECB, the angles A and E are equal, ſtanding on the ſame arc BC (th. 50); alſo the right angle D is equal the angle B, which is alſo a right angle, being in a ſemicircle (th. 52): therefore theſe two triangles have alſo their third angles equal, and are equiangular. Hence, AC, CE, and CD, CB being like ſides, ſubtending the equal angles, the rectangle AC . CB, of the firſt and laſt of them, is equal to the rectangle CE . CD of the other two (th. 62).

THEOREM LXIV.

THE Square of a Line Biſecting any Angle of a Triangle, and terminating in the Oppoſite Side, together with the Rectangle of the two Segments of that Side, is Equal to the Rectangle of the two Sides including the Biſected Angle.



Let CD biſect the angle C of the triangle ABC; then the ſquare CD^2 + the rectangle AD . DB is = the rectangle AC . CB.

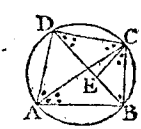
For, let CD be produced to meet the circumſcribing circle at E, and join AE.

Then the two triangles ACE, BCD are equiangular: for the angles at C are equal by ſuppoſition, and the angles B and E are equal, ſtanding on the ſame arc AC (th. 50); conſequently the third angles at A and D are equal (corol. 1, th. 17): alſo AC, CD, and CE, CB are like or correſponding ſides, being oppoſite to equal angles: therefore the rectangle AC . CB is = the rectangle CD . CE (th. 62). But the rectangle CD . CE is = CD^2 + the rectangle CD . DE (th. 30), and therefore alſo the rectangle AC . CB is = CD^2 + CD . DE, or equal CD^2 + AD . DB, ſince CD . DE is = AD . DB (th. 61).

Q. E. D.

THEOREM LXV.

THE Rectangle of the two Diagonals of any Quadrangle Inſcribed in a Circle, is Equal to the Sum of the two Rectangles of the Oppoſite Sides.



Let ABCD be any quadrilateral inſcribed in a circle, and AC, BD its two diagonals: then the rectangle AC . BD is = the rectangle AB . DC + the rectangle AD . BC.

For, let CE be drawn making the angle BCE equal the angle DCA. Then the two triangles ACD, BCE are equiangular; for the angles A and B are equal, ſtanding on the ſame arc DC; and the angles DCA, BCE are equal by ſuppoſition; conſequently the third angles ADC, BEC are equal: alſo, AC, BC, and AD, BE are like or correſponding ſides, being oppoſite the equal angles: therefore the rect. AC . BE is = AD . BC (th. 62).

Again,

Again, the two triangles ABC, DEC are equiangular: for the angles BAC, BDC are equal, ſtanding on the ſame arc BC; and the angle DCE is equal the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are alſo equal: but AC, DC, and AB, DE are the like ſides: therefore the rectangle AC . DE is = the rectangle AB . DC (th. 62).

Hence, by equal additions, the ſum of the rectangles AC . BE + AC . DE is = AD . BC + AB . DC. But the ſum of the rectangles AC . BE + AC . DE is = the rectangle AC . BD (th. 30): therefore the rectangle AC . BD is = the rectangle AD . BC + AB . DC (ax. 1). Q. E. D.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 70. RATIO is the proportion or relation which one magnitude bears to another magnitude of the ſame kind, with reſpect to quantity.

Note. The meaſure, or quantity, of a ratio, is conceived, by conſidering what part or parts the leading quantity, called the Antecedent, is of the other, called the Conſequent; or what part or parts the number expreſſing the quantity of the former, is of the number denoting in like manner the latter. So, the ratio of a quantity expreſſed by the number 2, to a like quantity expreſſed by the number 6, is denoted by 6 divided by 2, or $\frac{6}{2}$ or 3; the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6, is meaſured by $\frac{6}{3}$ or 2; the ratio of 4 to 6 is $\frac{6}{4}$ or $1\frac{1}{2}$; that of 6 to 4 is $\frac{4}{6}$ or $\frac{2}{3}$; &c.

71. Proportion is an equality of ratios. Thus,

72. Three quantities are ſaid to be Proportional, when the ratio of the firſt to the ſecond is equal to the ratio of the ſecond to the third. As of the three quantities A (2), B (4), C (8); where $\frac{4}{2} = \frac{8}{4} = 2$, both the ſame ratio.

73. Four quantities are ſaid to be Proportional, when the ratio of the firſt to the ſecond, is the ſame as the ratio of the third to the fourth. As of the four A (2), B (4), C (5), D (10); where $\frac{4}{2} = \frac{10}{5} = 2$, both the ſame ratio.

Note. To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus, A : B :: C : D; and read thus, as A is to B so is C to D. But when three quantities are proportional, the middle one is repeated, and they are written thus, A : B :: B : C.

74. Of three proportional quantities, the middle one is said to be a Mean Proportional between the other two; and the last, a Third Proportional to the first and second.

75. Of four proportional quantities, the last is said to be a Fourth Proportional to the other three, taken in order.

76. Quantities are said to be Continually Proportional, or in Continued Proportion, when the ratio is the same between every two adjacent terms, viz. when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio.

As in the quantities 1, 2, 4, 8, 16, &c; where the common ratio is equal to 2.

77. In a series or rank of quantities continually proportional, the ratio of the first and third is said to be Duplicate to that of the first and second; and the ratio of the first and fourth, Triplicate to that of the first and second; and so on.

78. Of any number of quantities, A, B, C, D, the ratio of the first A, to the last D, is said to be Compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

79. Inverse ratio is, when the antecedent is made the consequent, and the consequent the antecedent.—Thus, if 1 : 2 :: 3 : 6; then inversely, 2 : 1 :: 6 : 3.

80. Alternate proportion is, when antecedent is compared with antecedent, and consequent with consequent.—As, if 1 : 2 :: 3 : 6; then, by alternation, or permutation, it will be 1 : 3 :: 2 : 6.

81. Compounded ratio is, when the sum of the antecedent and consequent is compared, either with the consequent, or with the antecedent.—Thus, if 1 : 2 :: 3 : 6, then by composition, 1 + 2 : 1 :: 3 + 6 : 3, and 1 + 2 : 2 :: 3 + 6 : 6.

82. Divided ratio is, when the difference of the antecedent and consequent is compared, either with the antecedent or with the consequent.—Thus, if 1 : 2 :: 3 : 6, then, by division, 2 — 1 : 1 :: 6 — 3 : 3, and 2 — 1 : 2 :: 6 — 3 : 6.

THE-

THEOREM LXVI.

Equimultiples of any two Quantities are in the Same Ratio as the Quantities themſelves.

LET A and B be any two quantities, and mA, mB, any equimultiples of them, m being any number whatever; then will mA and mB have the ſame ratio as A and B, or A : B :: mA : mB.

For $\frac{m\text{B}}{m\text{A}} = \frac{\text{B}}{\text{A}}$, the ſame ratio.

Corol. Hence, like parts of quantities have the ſame ratio as the wholes; becauſe the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

THEOREM LXVII.

IF Four Quantities, of the Same Kind, be Proportionals; they will be in Proportion by Alternation or Permutation, or the Antecedents will have the Same Ratio as the Conſequents.

Let A : B :: mA : mB; then will A : mA :: B : mB.

For $\frac{m\text{A}}{\text{A}} = m$, and $\frac{m\text{B}}{\text{B}} = m$, both the ſame ratio.

THEOREM LXVIII.

If Four Quantities be Proportional; they will be in Proportion by Inverſion, or Inverſely.

LET A : B :: mA : mB; then will B : A :: mB : mA.

For $\frac{m\text{A}}{m\text{B}} = \frac{\text{A}}{\text{B}}$, both the ſame ratio.

THEOREM LXIX.

If Four Quantities be Proportional; they will be in Proportion by Compoſition and Diviſion.

LET A : B :: mA : mB;

then will B $\pm$ A : A :: mB $\pm$ mA : mA,

and B $\pm$ A : B :: mB $\pm$ mA : mB.

For, $\frac{m\text{A}}{m\text{B} \pm m\text{A}} = \frac{\text{A}}{\text{B} \pm \text{A}}$; and $\frac{m\text{B}}{m\text{B} \pm m\text{A}} = \frac{\text{B}}{\text{B} \pm \text{A}}$.

Corol.

Corol. It appears from hence, that the Sum of the Greateſt and Leaſt of four proportional quantities, of the ſame kind, exceeds the Sum of the Two Means. For, ſince $A : A + B :: mA : mA + mB$, where A is the leaſt, and $mA + mB$ the greateſt; then $\overline{m + 1} \cdot A + mB$, the ſum of the greateſt and leaſt, exceeds $\overline{m + 1} \cdot A + B$ the ſum of the two means.

THEOREM LXX.

If, of Four Proportional Quantities, there be taken any Equimultiples whatever of the two Antecedents, and any Equimultiples whatever of the two Conſequents; the quantities reſulting will ſtill be proportional.

Let $A : B :: mA : mB$; alſo, let pA and pmA be any equimultiples of the two antecedents, and qB and qmB any equimultiples of the two conſequents; then will $pA : qB :: pmA : qmB$.

For $\frac{qmB}{pmA} = \frac{qB}{pA}$, both the ſame ratio.

THEOREM LXXI.

If there be Four Proportional Quantities, and the two Conſequents be either Augmented or Diminiſhed by Quantities that have the Same Ratio as the reſpective Antecedents; the Reſults and the Antecedents will ſtill be Proportionals.

Let $A : B :: mA : mB$, and nA and nmA any two quantities having the ſame ratio as the two antecedents; then will $A : B \pm nA :: mA : mB \pm nmA$.

For $\frac{mB \pm nmA}{mA} = \frac{B \pm nA}{A}$, both the ſame ratio.

THEOREM LXXII.

If any Number of Quantities be Proportional; either of the Antecedents will be to its Conſequent, as the Sum of all the Antecedents, is to the Sum of all the Conſequents.

Let $A : B :: mA : mB :: nA : nB$, &c; then will $A : B :: A + mA + nA :: B + mB + nB$, &c.

For $\frac{B + mB + nB}{A + mA + nA} = \frac{B}{A}$, the ſame ratio.

THE-

THEOREM LXXIII.

If a Whole Magnitude be to a Whole, as a Quantity taken from the firſt, is to a Quantity taken from the other; the Remainder will be to the Remainder, as the whole to the whole.

Let $A : B :: \frac{m}{n} A : \frac{m}{n} B$;

then will $A : B :: A - \frac{m}{n} A : B - \frac{m}{n} B$.

For $\frac{B - \frac{m}{n} B}{A - \frac{m}{n} A} = \frac{B}{A}$, both the ſame ratio.

THEOREM LXXIV.

If Quantities be Proportional; their Squares, or Cubes, or any Like Powers, or Roots, of them, will alſo be Proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^n A^n : B^n$.

For $\frac{m^n B^n}{m^n A^n} = \frac{B^n}{A^n}$, both the ſame ratio.

THEOREM LXXV.

If there be Two Sets of Proportionals; then the Products or Rectangles of the Correſponding Terms will alſo be Proportional.

Let $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For $\frac{mnBD}{mnAC} = \frac{BD}{AC}$, both the ſame ratio.

THEOREM LXXVI.

If Four Quantities be Proportional; the Rectangle or Product of the two Extremes, will be Equal to the Rectangle or Product of the two Means. And the converſe.

Let $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

THE-

THEOREM LXXVII.

If Three Quantities be Continual Proportionals; the Rectangle or Product of the two Extremes, will be equal to the Square of the Mean. And the converſe.

Let A, mA, m^2A be three proportionals,
or $A : mA :: mA : m^2A$;
then is $A \times m^2A = m^2A^2$, as is evident.

THEOREM LXXVIII.

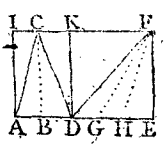
If any Number of Quantities be Continual Proportionals; the Ratio of the Firſt to the Third, will be duplicate or the Square of the Ratio of the Firſt and Second; the Ratio of the Firſt and Fourth will be triplicate or the Cube of that of the Firſt and Second; and ſo on.

Let A, mA, m^2A, m^3A, &c, be proportionals;
then is $\frac{mA}{A} = m$; but $\frac{m^2A}{A} = m^2$; and $\frac{m^3A}{A} = m^3$; &c.

THEOREM LXXIX.

Triangles, and alſo Parallelograms, having the Same Altitude, or that are between the Same Parallels, are to one another in the ſame Ratio as their Baſes.

Let the two triangles ADC, DEF, have the ſame altitude, or be between the ſame parallels AE, CF; then is the ſurface of the triangle ADC to the ſurface of the triangle DEF, as the baſe AD is to the baſe DE. Or, AD : DE :: the triangle ADC : the triangle DEF.



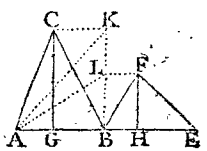
For, let the baſe AD be to the baſe DE, as any one number m (2), to any other number n (3); and divide the reſpective baſes into thoſe parts, AB, BD, DG, GH, HE, all equal to one another; and from the points of diviſion draw the lines BC, FG, FH, to the vertexes C and F. Then will theſe lines divide the triangles ADC, DEF into the ſame number of parts as their baſes, each equal to the triangle ABC, becauſe thoſe triangular parts have equal baſes and altitude (corol. 2, th. 25); namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC is to the triangle DFE, as the number of parts

parts *m* (2) of the former, to the number *n* (3) of the latter, that is, as the baſe AD to the baſe DE (def. 73).

In like manner, the parallelogram ADKI is to the parallelogram DEFK, as the baſe AD is to the baſe DE; each of theſe having the ſame ratio as the number of their parts, *m* to *n*. Q. E. D.

THEOREM LXXX.

Triangles and Parallelograms, having Equal Baſes, are to each other as their Altitudes.



LET ABC, BEF be two triangles, having the equal baſes AB, BE, and whoſe altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF :: CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL = FH; drawing AK and AL.

Then, triangles of equal baſes and heights being equal (corol. 2, th. 25), the triangle ABK is = ABC, and the triangle ABL = BEF. But, conſidering now ABK, ABL as two triangles on the baſes BK, BL, and having the ſame altitude AB, theſe will be as their baſes (th. 79), namely, the triangle ABK : the triangle ABL :: BK : BL.

But the triangle ABK = ABC, and the triangle ABL = BEF, alſo BK = CG, and BL = FH.
Theref. the triangle ABC : triangle BEF :: CG : FH.

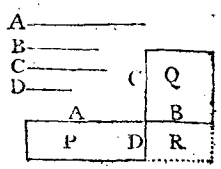
And ſince parallelograms are the doubles of theſe triangles, having the ſame baſes and altitudes, they will likewiſe have to each other the ſame ratio as their altitudes. Q. E. D.

Corol. Since, by this theorem, triangles and parallelograms, when their baſes are equal, are to each other as their altitudes; and by the foregoing one, when their altitudes are equal, they are to each other as their baſes; therefore univerſally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their baſes and altitudes,

THEOREM LXXXI.

If Four Lines be Proportional; the Rectangle of the Extremes will be Equal to the Rectangle of the Means. And, converſely, if the Rectangle of the Extremes, of four Lines, be Equal to the Rectangle of the Means, the Four Lines, taken alternately, will be Proportional.

Let the four lines A, B, C, D be proportionals, or A : B :: C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A.D = B.D.



For, let the four lines be placed, with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the ſame parallels, are to each other as their baſes A and B (th. 79); and the rectangles Q and R, being between the ſame parallels, are to each other as their baſes C and D. But the ratio of A to B is the ſame as the ratio of C to D by hypotheſis; therefore the ratio of P to R, is the ſame as the ratio of Q to R; and conſequently the rectangles P and Q are equal. Q. E. D.

Again, if the rectangle of A and D, be equal to the rectangle of B and C; theſe lines will be proportional, or A : B :: C : D.

For, the rectangles being placed the ſame as before: then, becauſe parallelograms between the ſame parallels, are to one another as their baſes, the rectangle P : R :: A : B, and Q : R :: C : D. But as P and Q are equal, by ſuppoſition, they have the ſame ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B :: C : D. Q. E. D.

Coral. 1. When the two means, namely, the ſecond and third terms, are equal, their rectangle becomes a ſquare of the ſecond term, which ſupplies the place of both the ſecond and third. And hence it follows, that when three lines are proportionals, the rectangle of the two extremes is equal to the

the ſquare of the mean; and, converſely, if the rectangle of the extremes be equal to the ſquare of the mean, the three lines are proportionals.

Corol. 2. Since it appears, by the rules of proportion in Arithmetic and Algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and, by this theorem, the rectangle of the extremes is equal to the rectangle of the two means; it follows, that the area or ſpace of a rectangle is repreſented or expreſſed by the product of its length and breadth multiplied together. And, in general, a rectangle in geometry is ſimilar to the product of the meaſures of its two dimenſions of length and breadth, or baſe and height. Alſo, a ſquare is ſimilar to, or repreſented by, the meaſure of its ſide multiplied by itſelf. So that, what is ſhewn of ſuch products, is to be underſtood of the ſquares and rectangles.

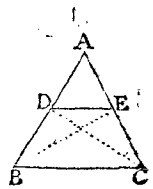
Corol. 3. Since the ſame reaſoning, as in this theorem, holds for any parallelograms whatever, as well as for the rectangles, the ſame property belongs to all kinds of parallelograms, having equal angles, and alſo to triangles which are the halves of parallelograms; namely, that if the ſides about the equal angles of parallelograms, or triangles, be reciprocally proportional, the parallelograms or triangles will be equal; and, converſely, if the parallelograms or triangles be equal, their ſides about the equal angles will be reciprocally proportional.

Corol. 4. Parallelograms, or triangles, having an angle in each equal, are to each other as the rectangles of the ſides which are about theſe equal angles.

THEOREM LXXXII.

If a Line be drawn in a Triangle Parallel to one of its Sides, it will cut the two other Sides Proportionally.

LET DE be parallel to the ſide BC of the triangle ABC; then will AD:DB::AE:EC.



For, draw BE and CD. Then the triangles DBE, DCE are equal to each other, becauſe they have the ſame baſe DE, and are between the ſame parallels DE, BC (th. 25). But the two triangles ADE, BDE, on the baſes AD, DB, have the ſame altitude;

tude; and the two triangles ADE, CDE, on the baſes AE, EC, have alſo the ſame altitude; and becauſe triangles of the ſame altitude are to each other as their baſes, therefore

the triangle ADE : BDE :: AD : DB,
and triangle ADE : CDE :: AE : EC.

But BDE is = CDE; and equals muſt have to equals the ſame ratio; therefore AD : DB :: AE : EC. Q. E. D.

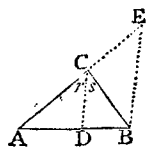
Corol. Hence alſo, the whole lines AB, AC, are proportional to their correſponding proportional ſegments (corol. th. 66),

namely, AB : AC :: AD : AE,
and AB : AC :: BD : CE.

THEOREM LXXXIII.

A RIGHT Line which Biſects any Angle of a Triangle, divides the Side oppoſite to the Biſected Angle, into Two Segments, which are Proportional to the two other Adjacent Sides.

Let the angle ACB, of the triangle ABC, be biſected by the line CD, making the angle *r* equal to the angle *s*: then will the ſegment AD be to the ſegment DB, as the ſide AC is to the ſide CB. Or, AD : DB :: AC : CB.



For, let BE be parallel to CD, meeting AC produced at E. Then, becauſe the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle *s* (th. 12), and therefore alſo equal to the angle *r*, which is equal to *s* by the ſuppoſition. Again, becauſe the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle *r* on the ſame ſide of it (th. 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle *r*, are equal to each other, and conſequently alſo their oppoſite ſides CB, CE alſo equal (th. 3).

But now, in the triangle ABE, the line CD, being drawn parallel to the ſide BE, cuts the two other ſides AB, AE proportionally (th. 82), making AD to DB, as AC to CE or its equal CB. Q. E. D.

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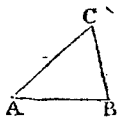
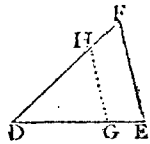
THEOREM LXXXIV.

Equiangular Triangles are Similar, or have their Like Sides Proportional.



LET ABC, DEF be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and consequently the angle C to the angle F: then will AB : AC :: DE : DF.

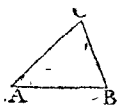


For, make DG = AB, and DH = AC, and join GH. Then the two triangles ABC, DGH, having the two sides AB, AC, equal to the two AG, AH, and the contained angles A and D also equal, are identical, or equal in all respects (th. 1), namely the angles B and C equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore also the angles G and H are equal to the angles E and F (ax. 1), and consequently the line GH is parallel to the side EF (cor. 1, th. 14).

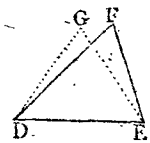
Hence then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH :: DE : DF (cor. th. 82). But DG and DH are equal to AB and BC; therefore — — — AB : AC :: DE : DF. Q. E. D.

THEOREM LXXXV.

Triangles which have their Sides Proportional, are Equiangular.



IN the two triangles ABC, DEF, if AB : DE :: AC : DF :: BC : EF; the two triangles will have their corresponding angles equal.

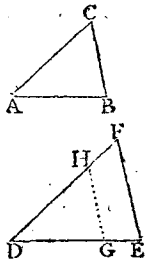


For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible: for if the two triangles ABC, DEG were equiangular, their sides would be proportional (th. 84). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities,

tities, as the two DF, EF, the former DG, EG, would be equal to the latter, DF, EF. Thus then, the two triangles DEF, DEG, having their three ſides equal, would be identical (th. 5); which is abſurd, ſince their angles are unequal.

THEOREM LXXXVI.

TRIANGLES, which have an Angle in the one Equal to an Angle in the other, and the Sides about theſe angles Proportional, are Equiangular.



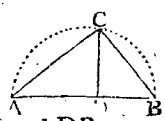
Let ABC, DEF be two triangles, having the angle A = the angle D, and the ſides AB, AC proportional to the ſides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

For, make DG = AB, and DH = AC, and join GH.

Then, the two triangles ABC, DGH, having two ſides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles B and C. But, ſince the ſides DG, DH are proportional to the ſides DE, DF, the line GH is parallel to EF (th. 82); hence the angles E and F are equal to the angles G and H (th. 14), and conſequently to their equals B and C. Q. E. D.

THEOREM LXXXVII.

IN a Right-Angled Triangle, a Perpendicular from the Right Angle, is a Mean Proportional between the Segments of the Hypothenuſe; and each of the Sides, about the Right Angle, is a Mean Proportional between the Adjacent Segment and the Hypothenuſe.



Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuſe AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or, AD : CD :: CD : DB; and AB : AC :: AC : AD; and AB : BC :: BC : BD.

For,

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (cor. 1, th. 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

Hence then, all the three triangles ABC, ADC, BDC, being equiangular, will have their like sides proportional (th. 84);

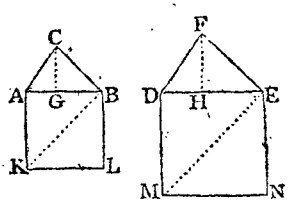
viz. AD : CD :: CD : DB;
and AB : AC :: AC : AD;
and AB : BC :: BC : BD. Q. E. D.

Corol. Because the angle in a semicircle is a right angle (th. 52); it follows that if, from any point C in the periphery of the semicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter: then are AC, BC, CD the mean proportionals as in this theorem, or (by th. 77), $CD^2 = AD \cdot DB$; $AC^2 = AB \cdot AD$; and $BC^2 = AB \cdot BD$.

THEOREM LXXXVIII.

Equiangular or Similar Triangles, are to each other as the Squares of their Like Sides.

LET ABC, DEF be two equiangular triangles, AB and DE being two homologous or like sides: then will the triangle ABC be to the triangle DEF, as the square of AB is to the square of DE, or as AB^2 to DE^2.



For, let AL and DN be the squares on AB and DE; also draw their diagonals BK, EM, and the perpendiculars CG, FH of the two triangles.

Then, since equiangular triangles have their like sides proportional (th. 82), in the two equiangular triangles ABC, DEF, the side AC : DF :: AB : DE; and in the two ACG, DFH, the side AC : DF :: CG : FH; therefore, by equality, CG : FH :: AB : DE, or CG : AB :: FH : DE.

But because triangles on equal bases are to each other as their altitudes, the triangles ABC, ABK, on the same base AB, are to each other, as their altitudes CG, AK, or AB;

and the triangles DEF, DEM, on the ſame baſe DE, are as their altitudes FH, DM or DE;

that is, triangle ABC : triangle ABK :: CG : AB,
and triangle DEF : triangle DEM :: FH : DE.

But it has been ſhewn that CG : AB :: FH : DE;
therefore of equality △ ABC : △ ABK :: △ DEF : △ DEM,
or alternately, as △ ABC : △ DEF :: △ ABK : △ DEM,

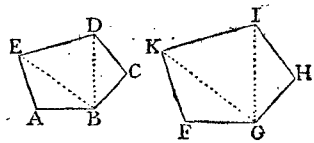
But the ſquares AL, DN, being the double of the triangles ABK, DEM, have the ſame ratio with them;
therefore the △ ABC : △ DEF :: ſquare AL : ſquare DN.

Q. E. D.

THEOREM LXXXIX.

All Similar Figures are in Proportion to each other, as the Squares of their Like Sides.

LET ABCDE, FGHIK be any two ſimilar figures, the like ſides being AB, FG, and BC, GH, and ſo on in the ſame order: then will the figure ABCDE be to the figure FGHIK, as the ſquare of AB to the ſquare of FG, or as AB^2 to FG^2.



For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being ſimilar (by ſuppoſ.), they are equiangular, and have their like ſides proportional (def. 67).

Then, ſince the angle A is = the angle F, and the ſides AB, AE proportional to the ſides FG, FK, the triangles ABE, FGK are equiangular (th. 86). In like manner, the two triangles BCD, GHI, having the angle C = the angle H, and the ſides BC, CD proportional to the ſides GH, HI, are alſo equiangular. Alſo, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; ſo that the two triangles BDE, GIK, having two angles equal, are alſo equiangular. Hence each triangle of the one figure, is equiangular with each correſponding triangle of the other.

But equiangular triangles are ſimilar, and are proportional to the ſquares of their like ſides (th. 88).

Therefore

Therefore the $\triangle$ ABE : $\triangle$ FGK :: AB^2 : FG^2,
and $\triangle$ BCD : $\triangle$ GHI :: BC^2 : GH^2,
and $\triangle$ BDE : $\triangle$ GIK :: DE^2 : IK^2,

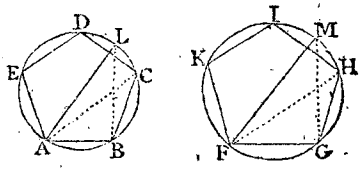
But as the two polygons are similar, their like sides are proportional, and consequently their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and consequently the corresponding triangles also, ABE to FGK, and BCD to GHI, and BDE to GIK, have all the same ratio, viz. that of AB^2 to FG^2; and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz. that of AB^2 to FG^2 (th. 72).

Q. E. D.

THEOREM XC.

SIMILAR Polygons Inscribed in Circles, have their Like Sides, and also their Whole Perimeters, in the Same Ratio as the Diameters of the Circles in which they are inscribed.

Let ABCDE, FGHIK be two similar figures, inscribed in the circles, whose diameters are AL and FM; then will each side AB, BC, &c, of the one figure, be to the like side FG, GH, &c, of the other figure, or the whole perimeter AB + BC, &c, of the one figure, to the whole perimeter FG + GH, &c, of the other figure, as the diameter AL to the diameter FM.



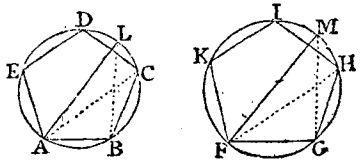
For, draw the two corresponding diagonals AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 67); therefore the two triangles ABC, FGH have the angle B = the angle G, and the sides AB, BC proportional to the two sides FG, GH, and consequently these two triangles are equiangular (th. 86), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (ax. 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and consequently their like sides are proportional

(th. 84); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each ſide BC, CD, &c, has to each ſide GH, HI, &c, the ſame ratio of AL to FM; and conſequently the ſums of them all ſtill in the ſame ratio; viz. AB + BC + CD, &c : FG + GH + HI, &c :: the diam. AL : the diam. FM (th. 72). Q. E. D.

THEOREM XCI.

SIMILAR Polygons Inſcribed in Circles, are in Proportion to each other as the Squares of the Diameters of thoſe Circles.



Let ABCDE, FGHIK be two ſimilar figures, inſcribed in the circles, whoſe diameters are AL and FM; then the ſurface of the polygon ABCDE will be to the ſurface of the polygon FGHIK, as AL^2 to FM^2.

For, the figures, being ſimilar, are to each other as the ſquares of their like ſides, AB^2 to FG^2 (th. 88). But, by the laſt theorem, the ſides AB, FG, are as the diameters AL, FM; and therefore the ſquares of the ſides AB^2 to FG^2, as the ſquares of the diameters AL^2 to FM^2 (th. 74). Conſequently the polygons ABCDE, FGHIK, are alſo to each other as the ſquares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D.

THEOREM XCII.

The Circumferences of Circles are in Proportion to each other as their Diameters.

LET D, d denote the diameters of two circles, and C, c their circumferences;
then will D : d :: C : c, or D : C :: d : c.

For, by theor. 90, ſimilar polygons inſcribed in circles, have their perimeters in the ratio of the diameters of thoſe circles.

Now, as this property belongs to all polygons, whatever the number of the ſides may be; conceive the number of the ſides to be indefinitely great, and the length of each indefinitely ſmall, till they coincide with the circumference of the

the circle, and be equal to it, indefinitely near. Then the perimeter of the polygon of an infinite number of ſides, is the ſame thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the ſame as the perimeters of ſuch polygons, are as the diameters of the circles. Q. E. D.

THEOREM XCIII.

The Areas or Spaces of Circles, are to each other as the Squares of their Diameters, or of their Radii.

LET A, *a* be the areas or ſpaces of two circles, and D, *d* their diameters; then $A : a :: D^2 : d^2$.

For, by theorem 91, ſimilar polygons inſcribed in circles are to each other as the ſquares of the diameters of the circles.

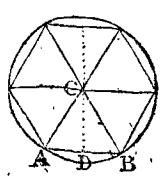
Hence, conceiving the number of the ſides of the polygons to be increaſed more and more, or the length of the ſides to become leſs and leſs, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide and become in effect equal; and then it follows, that the ſpaces of the circles, which are the ſame as of the polygons, will be to each other as the ſquares of the diameters of the circles. Q. E. D.

Corol. The ſpaces of circles are alſo to each other as the ſquares of the circumferences; ſince the circumferences are in the ſame proportion as the diameters, by theorem 92.

THEOREM XCIV.

Every Circle is Equal to the Rectangle of its Radius and a Right Line equal to Half its Circumference.

CONCEIVE a regular polygon to be inſcribed in the circle; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has ſides, one of which is ABC, of which the altitude is the perpendicular CD from the centre to the baſe AB.



Then the triangle ABC, being equal to half a rectangle of equal baſe and altitude (th. 26), is equal to half the rectangle of the baſe AB and altitude CD, or equal

equal to the rectangle of the altitude CD and half the baſe AB; conſequently the whole polygon, or all the triangles added together which compoſe it, is equal to the rectangle of the common altitude CD, and the halves of all the ſides, or the half of the perimeter of the polygon.

Now, conceive the number of ſides of the polygon to be indefinitely increaſed; then will its perimeter coincide with the circumference of the circle, and conſequently the altitude AD will become equal to the radius, and the whole polygon equal to the circle. Conſequently the ſpace of the circle, or of the polygon in that ſtate, is equal to the rectangle of the radius and half the circumference. Q. E. D.

OF PLANES AND SOLIDS.

DEFINITIONS.

DEF. 83. THE Common Section of two Planes, is the line in which they meet, or cut each other.

84. A Right Line is Perpendicular to a Plane, when it is perpendicular to every right line which meets it in that plane.

85. One Plane is Perpendicular to Another, when every right line in the one, which is perpendicular to the line of their common ſection, is perpendicular to the other.

86. The Inclination of one Plane to another, or the angle they form between them, is the angle contained by two right lines, drawn from any point in the common ſection, and at right angles to the ſame, one of theſe lines in each plane.

87. Parallel Planes, are ſuch as being produced ever ſo far both ways will never meet, or which are every where at an equal perpendicular diſtance.

88. A Solid Angle, is that which is made by three or more plane angles, meeting each other in the ſame point.

89. Si-

89. Similar Solids, contained by plane figures, are ſuch as have all their ſolid angles equal, each to each, and are bounded by the ſame number of ſimilar planes.

90. A Priſm, is a ſolid whoſe ends are parallel, equal, and like plane figures, and its ſides, connecting thoſe ends, are parallelograms.

91. A Priſm takes particular names according to the figure of its baſe or ends, whether triangular, ſquare, rectangular, pentagonal, hexagonal, &c.

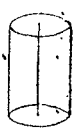
92. An Upright Priſm, is that which has the planes of the ſides perpendicular to the planes of the ends or baſe.

93. A Parallelopiped, or Parallelopipedon, is a priſm bounded by ſix parallelograms, every oppoſite two of which are equal, alike, and parallel.

94. A Rectangular Parallelopipedon, is that whoſe bounding planes are all rectangles, which are perpendicular to each other.

95. A Cube, is a ſquare priſm, being bounded by ſix equal ſquare ſides or faces, and are perpendicular to each other.

96. A Cylinder is a round priſm, having circles for its ends; and is conceived to be formed by the revolution of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.

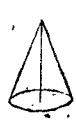


97. The Axis of a Cylinder, is the right line joining the centres of the two parallel circles, about which the figure is deſcribed.

98. A Pyramid, is a ſolid, whoſe baſe is any right-lined plane figure, and its ſides triangles, having all their vertices meeting together in a point above the baſe, called the Vertex of the pyramid.

99. A pyramid, like the priſm, takes particular names from the figure of the baſe.

100. A Cone, is a round pyramid, having a circular baſe; and is conceived to be generated by the revolution of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.



101. The

101. The Axis of a cone, is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

102. Similar Cones and Cylinders, are such as have their altitudes and the diameters of their bases proportional.

103. A Sphere, is a solid bounded by one convex surface, which is every where equally distant from a certain point within, called the Centre. It is conceived to be generated by the revolution of a semicircle about its diameter, which remains fixed.

104. The Axis of a Sphere, is the right line about which the semicircle revolves; and the centre is the same as that of the revolving semicircle.

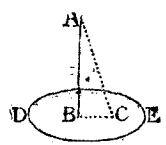
105. The Diameter of a Sphere, is any right line passing through the centre, and terminated both ways by the surface.

106. The Altitude of a Solid, is the perpendicular drawn from the vertex to the opposite side or base.

THEOREM XCV.

A Perpendicular is the Shortest Line which can be drawn from any Point to a Plane.

LET AB be perpendicular to the plane DE; then any other line as AC, drawn from the same point A to the plane, will be longer than the line AB.



In the plane draw the line BC, joining the points B, C.

Then, because the line AB is perpendicular to the plane DE, the angle B is a right angle (def. 84); and therefore the line AB is less than any other line AC (th. 21). Q. E. D.

THEOREM XCVI.

A Perpendicular Measures the Distance of any Point from a Plane.

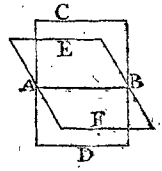
THE distance of one point from another is measured by a right line joining them, because this is the shortest line which can be drawn from one point to another. So, also, the distance from a point to a line, is measured by a perpendicular, because this line is the shortest which can be drawn from

from the point to the line. In like manner, the diſtance from a point to a plane, muſt be meaſured by a perpendicular drawn from that point to the plane, becauſe this is the ſhorteſt line which can be drawn from the point to the plane.

THEOREM XCVII.

The Common Section of Two Planes, is a Right Line.

LET ACBDA, AEBFA be two planes cutting each other, and A, B two points in which the two planes meet; drawing the line AB, this line will be the common interſection of the two planes.



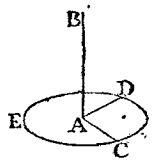
For, becauſe the right line AB touches the two planes in the points A and B, it touches them in all other points (def. 20): this line is therefore common to the two planes. That is, the common interſection of the two planes is a right line.

Q. E. D.

THEOREM XCVIII.

IF a Right Line be Perpendicular to two other Right Lines, at their Common Point of Meeting; it will be Perpendicular to the Plane of thoſe Right Lines.

Let the line AB make right angles with the lines AC, AD; it will be perpendicular to the plane CDE which paſſes through theſe lines.

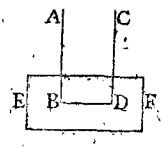


If the line AB were not perpendicular to the plane CDE, another plane might paſs through the point A, to which the line AB would be perpendicular. But this is impoſſible; for, ſince the angles BAC, BAD are right angles, this other plane muſt paſs through the points C, D. Hence, this plane paſſing through the two points A, C of the line AC, and through the two points A, D of the line AD, it will paſs through both theſe two lines, and therefore be the ſame plane with the former. Q. E. D.

THE-

THEOREM XCIX.

If Two Right Lines be Perpendicular to the Same Plane, they will be Parallel to each other.



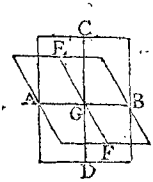
Let the two lines AB, CD be both of them perpendicular to the same plane EBDF; then will AB be parallel to CD.

For, join B, D, by the line BD in the plane. Then, because the lines AB, CD are perpendicular to the plane EF, they are both perpendicular to the line BD (def. 84) in that plane. And because AB and CD are both perpendicular to the same line BD, they are parallel to each other (corol. th. 13). Q. E. D.

Corol. If two lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

THEOREM C.

If Two Planes Cut each other at Right Angles, and a Right Line be drawn in one of the Planes Perpendicular to their Common Intersection, it will be Perpendicular to the other Plane.



Let the two planes ACBD, AEBF cut each other at right angles; and the line CG perpendicular to their common section AB; then will CG be also perpendicular to the other plane AEBF.

For, draw EG, perpendicular to AB. Then, because the two lines GC, GE are perpendicular to the common intersection AB, the angle CGE is the angle of inclination of the two planes (def. 86). But since the two planes cut each other perpendicularly, the angle of inclination CGE is a right angle. And, since the line CG is perpendicular to the two lines GA, GE, in the plane AEBF, it is therefore perpendicular to that plane (th. 98). Q. E. D.

THE-

THEOREM CI.

If one Plane Meet another Plane; it will make Angles with that other Plane, which are together Equal to Two Right Angles.

Let the plane ACB meet the plane AEBF; theſe planes make with each other two angles whoſe ſum is equal to two right angles.

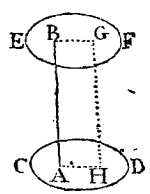
For, through any point G, in the common ſection AB, draw CD, EF perpendicular to AB. Then, the line CG makes with EF two angles together equal to two right angles. But theſe two angles are (by def. 86), the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

Corol. In like manner, it may be demonſtrated, that planes which interſect, have their vertical or oppoſite angles equal; alſo, that parallel planes have their alternate angles equal; and ſo on, as in parallel lines.

THEOREM CII.

If Two Planes be Parallel to each other; a Right Line which is Perpendicular to one of the Planes, will alſo be Perpendicular to the other.

Let the two planes CD, EF be parallel, and let the line AB be perpendicular to the plane CD; then ſhall it alſo be perpendicular to the other plane EF.



For, from any point G, in the plane EF, draw GH perpendicular to the plane CD, and draw AH, BG.

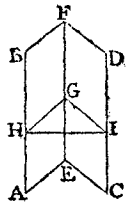
Then, becauſe BA, GH are both perpendicular to the plane CD, the angles A and H are both right angles. And becauſe the planes CD, EF are parallel, the perpendiculars BA, GH are equal (def. 87). Hence it follows that the lines BG, AH are parallel (def. 9). And the line AB being perpendicular to the line AH, is alſo perpendicular to the parallel line BG (cor. th. 12).

In like manner, it is proved, that the line AB is perpendicular to all other lines which can be drawn from the point

B in the plane EF. Therefore the line AB is perpendicular to the plane EF (def. 84). Q. E. D.

THEOREM CIII.

If Two Right Lines be Parallel to a Third Line, though not in the ſame Plane with it; they will be Parallel to each other.



Let the right lines AB, CD be each of them parallel to the third line EF, though not in the ſame plane with it; then will AB be parallel to CD.

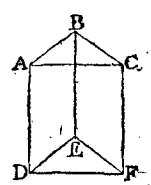
For, from any point G in the line EF, let GH, GI be each perpendicular to EF, in the planes AF, ED of the propoſed parallels.

Then, ſince the line EF is perpendicular to the two lines GH, GI, it is perpendicular to the plane GHI of thoſe lines (th. 98). And becauſe EF is perpendicular to the plane GHI, its parallel AB is alſo perpendicular to that plane (th. 99, cor.) For the ſame reaſon, the line CD is perpendicular to the ſame plane GHI. Hence, becauſe the two lines AB, CD are perpendicular to the ſame plane, theſe two lines are parallel (th. 99).

Q. E. D.

THEOREM CIV.

If Two Lines that meet each other, be Parallel to Two other Lines that meet each other, though not in the ſame Plane with them; the Angles contained by thoſe Lines will be equal.



Let the two lines AB, BC be parallel to the two lines DE, EF; then will the angle ABC be equal to the angle DEF.

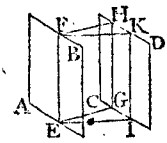
For, make the lines AB, BC, DE, EF all equal to each other; and join AC, DF, AD, BE, CF.

Then, the lines AD, BE, joining the equal and parallel lines AB, DE, are equal and parallel (th. 24). For the ſame reaſon, CF, BE are equal and parallel. Therefore AD, CF are equal and parallel (th. 15); and conſequently alſo AC, DF (th. 24). Hence, the two triangles ABC, DEF, having all their ſides equal, each

each to each, have their angles alſo equal, and conſequently the angle ABC = the angle DEF. Q. E. D.

THEOREM CV.

The Sections made by a Plane cutting two other Parallel Planes, are alſo Parallel to each other.



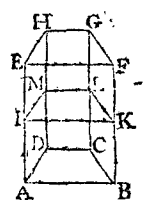
LET the two parallel planes AB, CD be cut by the third plane EFHG, in the lines EF, GH; theſe two ſections EF, GH will be parallel.

Suppoſe EG, FH be drawn parallel to each other in the plane EFHG; alſo let EI, FK be perpendicular to the plane CD; and let IG, KH be joined.

Then EG, FH being parallels, and EI, FK, being both perpendicular to the plane CD, are alſo parallel to each other (th. 99); conſequently the angle HFK is equal the angle GEI (th. 104). But the angle FKH is alſo equal the angle EIG, being both right angles; therefore the two triangles are equiangular (cor. 1, th. 17); and the ſides FK, EI being the equal diſtances between the parallel planes (def. 87), it follows that the ſides FH, EG are alſo equal (th. 2). But theſe two lines are parallel (by ſuppoſ.), as well as equal; and conſequently the two EF, GH, joining thoſe equal parallels, are alſo parallel (th. 24). Q. E. D.

THEOREM CVI.

If a Priſm be cut by a Plane Parallel to its Baſe, the Section will be Equal and Like to the Baſe.



LET AG be any priſm, and IL a plane parallel to the baſe AC; then will the plane IL be equal and like to the baſe AC, or the two planes will have all their ſides and all their angles equal.

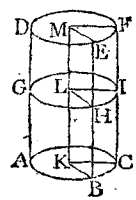
For, the two planes AC, IL being parallel, by hypotheſis; and two parallel planes, cut by a third plane, having parallel ſections (th. 105); therefore IK is parallel to AB, KL to BC, LM to CD, and IM to AD. But AI and BK are parallels (by def. 90); conſequently AK is a parallelogram; and the oppoſite ſides AB, IK are equal (th. 22).

In

In like manner, it is ſhewn that KL is = BC and LM = CD, and IM = AD, or the two planes AC, IL are mutually equilateral. But theſe two planes, having their correſponding ſides parallel, have the angles contained by them alſo equal (th. 104), namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle L, and the angle D = the angle M. So that the two planes AC, IL have all their correſponding ſides and angles equal, or are equal and like. Q. E. D.

THEOREM CVII.

If a Cylinder be cut by a Plane Parallel to its Baſe, the Section will be a Circle, Equal to the Baſe.



Let AF be a cylinder, and GHI any ſection parallel to the baſe ABC; then will GHI be a circle, equal to ABC.

For, let the planes KE, KF paſs through the axis of the cylinder MK, and meet the ſection GHI in the three points H, I, L; and join the points as in the figure.

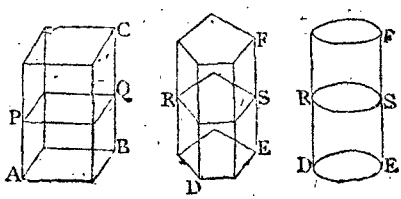
Then, ſince KL, CI are parallel (by def. 96); and the plane KI, meeting the two parallel planes ABC, GHI, makes the two ſections KC, LI parallel (th. 105); the figure KLIC is therefore a parallelogram, and conſequently has the oppoſite ſides LI, KC equal, where KC is a radius of the circular baſe.

In like manner, it is ſhewn that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the ſection GHI, are all equal to radii of the baſe; conſequently GHI is a circle, and equal to ABC. Q. E. D.

THEOREM CVIII.

All Priſms and Cylinders, of Equal Baſes and Altitudes, are Equal to each other.

Let AC, DF be two priſms, and a cylinder, upon equal baſes AB, DE, and having equal altitudes; then will the ſolids AC, DF be equal.



For, let PQ, RS be any

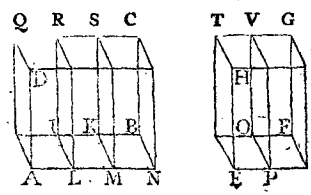
any two ſections parallel to the baſes, and equidiſtant from them. Then, by the laſt two theorems, the ſection PQ is equal to the baſe AB, and the ſection RS equal the baſe DE. But the baſes AB, DE are equal, by the hypotheſis; therefore the ſections PQ, RS are equal alſo. And in like manner, it may be ſhewn, that any other correſponding ſections are equal to one another.

Since then every ſection in the priſm AC, is equal to its correſponding ſection in the priſm, or cylinder RS, the priſms and cylinder themſelves, which are compoſed of thoſe ſections, muſt alſo be equal. Q. E. D.

Corol. Every priſm, or cylinder, is equal to a rectangular parallelopipedon, of an equal baſe and altitude.

THEOREM CIX.

Rectangular Parallelopipedons, of Equal Altitudes, have to each other the ſame Proportion as their Baſes.



Let AC, EG be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will AC be to EG as the baſe AB is to the baſe EF.

For, let the proportion of the baſe AB to the baſe EF, be that of any one number m (3) to any other number n (2). And conceive AB to be divided into m equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM parallel to BN). And let EF be divided, in like manner, into n equal parts, or rectangles, EO, PF: All of theſe parts of both baſes being mutually equal among themſelves. And through the lines of diviſion let the plane ſections LR, MS, PV paſs parallel to AQ, ET.

Then the parallelopipedons AR, LS, MC, EV, PG are all equal, having equal baſes and heights. Therefore the ſolid AC is to the ſolid EG, as the number of parts in AC to the number of equal parts in EG, or as the number of parts in AB to the number of equal parts in EF, that is, as the baſe AB to the baſe EF. Q. E. D.

Corol. From this theorem, and the corollary to the laſt, it appears, that all priſms and cylinders of equal altitudes, are

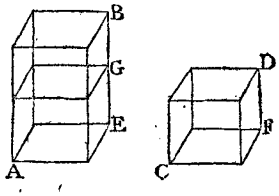
to

to each other as their bases; every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and height.

THEOREM CX.

Rectangular Parallelopipedons, of Equal Bases, are in Proportion to each other as their Altitudes.

LET AB, CD be two rectangular parallelopipedons standing on the equal bases AE, CF; then will AB be to CD as the altitude EB is to the altitude DF.



For, let AG be a rectangular parallelopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

Then AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG be considered as bases, the solids AB, AG, of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG, being parallelograms of equal altitude HE, are to each other as their bases EB, EG; and therefore the two prisms AB, AG are to each other as the lines EB, EG. But AG is equal CD, and EG equal FD; consequently the prisms AB, CD are to each other as their altitudes EB, FD; that is, — — AB : CD :: EB : FD. Q. E. D.

Corol. 1. From this theorem, and the corollary to theorem 108, it appears, that all prisms and cylinders, of equal bases, are to one another as their altitudes.

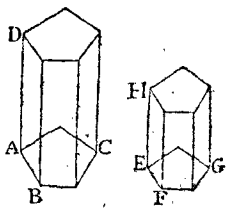
Corol. 2. Because, by corollary 1, prisms and cylinders are as their altitudes, when their bases are equal. And, by the corollary to the last theorem, they are as their bases, when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes. And hence also these products are the proper numeral measures of their quantities or magnitudes.

THEOREM CXI.

SIMILAR Prisms and Cylinders are to each other as the Cubes of their Altitudes, or of any other Like Linear Dimensions.

Let

Let ABCD, EFGH be two ſimilar priſms; then will the priſm CD be to the priſm GH, as AB^3 to EF^3, or as AD^3 to EH^3.



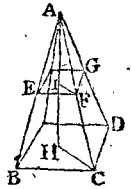
For the ſolids are to each other as the product of their baſes and altitudes (th. 110, cor. 2), that is, as AC . AD to EG . EH. But the baſes, being ſimilar planes, are to each other as the ſquares of their like ſides, that is, AC to EG as AB^2 to EF^2; therefore the ſolid CD is to the ſolid GH as AB^2 . AD to EF^2 . EH. But BD and FH, being ſimilar planes, have their like ſides proportional, that is, AB : EF :: AD : EH, or AB^2 : EF^2 :: AD^2 : EH^2; therefore - - - - - AB^2 . AD : EF^2 . EH :: AB^3 : EF^3, or :: AD^3 : EH^3; and conſequently the ſolid CD : ſolid GH :: AB^3 : EF^3 :: AD^3 : EH^3.

Q. E. D.

THEOREM CXII

In a Pyramid, a Section Parallel to the Baſe is ſimilar to the Baſe; and theſe two planes are to each other as the Squares of their Diſtances from the Vertex.

Let ABCD be a pyramid, and EFG a ſection parallel to the baſe BCD, alſo AIH a line perpendicular to the two planes at H and I: then will BD, EG be two ſimilar planes, and the plane BD will be to the plane EG as AH^2 to AI^2.



For, join CH, FI. Then, becauſe a plane cutting two parallel planes, makes parallel ſections (th. 105), therefore the plane ABC, meeting the two parallel planes BD, EG, makes the ſections BC, EF parallel: In like manner, the plane ACD makes the ſections CD, FG parallel. Again, becauſe two pair of parallel lines make equal angles (th. 104), the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And, in like manner, it is ſhewn, that each angle in the plane EG is equal to each angle in the plane BD, and conſequently thoſe two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles (th. 14), and the angles at A being common, the two triangles ABC, AEF

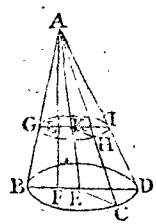
are equiangular, as alſo the two triangles ACD, AFG, and have therefore their like ſides proportional, namely, - - AC : AF :: BC : EF :: CD : FG. And, in like manner, it may be ſhewn, that all the lines in the plane EG are proportional to all the correſponding ones in the baſe BD. Hence theſe two planes, having their angles equal and their ſides proportional, are ſimilar by def. 67.

But, ſimilar planes being to each other as the ſquares of their like ſides, the plane BD:EG::BC^2:EF^2 or ::AC^2:AF^2, by what is ſhewn above. Alſo, the two triangles AHC, AIF, having the angles H and I right ones (th. 98), and the angle A common, are equiangular, and have therefore their like ſides proportional, namely, AC:AF::AH:AI, or AC^2 : AF^2 :: AH^2 : AI^2. Conſequently the two planes BD, EG, which are as the former ſquares AC^2, AF^2, will be alſo as the latter ſquares AH^2, AI^2, that is, - - - BD : EG :: AH^2 : AI^2. Q. E. D.

THEOREM CXIII.

In a Cone, a Section Parallel to the Baſe is a Circle; and this Section is to the Baſe as the Squares of their Diſtances from the Vertex.

Let ABCD be a cone, and GHI a ſection parallel to the baſe BCD: then will GHI be a circle, and BCD, GHI will be to each other as the ſquares of their diſtances from the vertex.



For, draw ALF perpendicular to the two parallel planes; and let the planes ACE, ADE paſs through the axis of the cone AKE, meeting the ſection in the three points H, I, K.

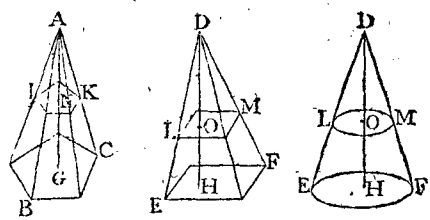
Then, ſince the ſection GHI is parallel to the baſe BCD, and the planes CK, DK meet them, HK is parallel to CE, and IK to DE (th. 105). And becauſe the triangles formed by theſe lines are equiangular, KH:EC::AK:AE::KI:ED. But EC is equal to ED, being radii of the ſame circle; therefore KI is alſo equal to KH. And the ſame may be ſhewn of any other lines drawn from the point K to the circumference of the ſection GHI, which is therefore a circle.

Again, by ſimilar triangles, AL : AF :: AK : AE or ::KI:ED, hence AL^2:AF^2::KI^2:ED^2; but KI^2:ED^2:: circle

circle GHI : circle BCD (th. 93); therefore $AL^2 : AF^2 ::$ circle GHI : circle BCD. Q. E. D.

THEOREM CXIV.

All Pyramids, and Cones, of Equal Baſes and Altitudes, are Equal to one another.



Let ABC, DEF be any pyramids and cone, of equal baſes BC, EF, and equal altitudes AG, DH: then will the pyramids and cone ABC and DEF, be equal.

For, parallel to the baſes, and at equal diſtances AN, DO, from the vertices, ſuppoſe the planes IK, LM to be drawn.

Then, by the two preceding theorems, - - - - - -

$DO^2 : DH^2 :: LM : EF$, and

$AN^2 : AG^2 :: IK : BC$.

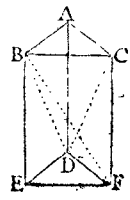
But ſince AN^2, AG^2 are equal to DO^2, DH^2, therefore IK : BC :: LM : EF. But BC is equal to EF, by hypotheſis; therefore IK is alſo equal to LM.

In the ſame manner it is ſhewn, that any other ſections, at equal diſtance from the vertex, are equal to each other.

Since then, every ſection in the cone, is equal to the correſponding ſection in the pyramids, and the heights are equal, the ſolids ABC, DEF, compoſed of thoſe ſections, muſt be equal alſo. Q. E. D.

THEOREM CXV.

Every Pyramid of a Triangular Baſe, is the Third Part of a Priſm of the Same Baſe and Altitude.



Let ABCDEF be a priſm, and BDEF a pyramid, on the ſame triangular baſe DEF: then will the pyramid BDEF be a third part of the priſm ABCDEF.

For, in the planes of the three ſides of the priſm, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD divide the whole priſm into the three pyramids BDEF,

DABC, DBCF, which are proved to be all equal to one another, as follows.

Since the oppoſite ends of the priſm are equal to each other, the pyramid whoſe baſe is ABC and vertex D, is equal to the pyramid whoſe baſe is DEF and vertex B (th. 114), being pyramids of equal baſe and altitude.

But the latter pyramid, whoſe baſe is DEF and vertex B, is the ſame ſolid as the pyramid whoſe baſe is BEF and vertex D, and this is equal to the third pyramid whoſe baſe is BCF and vertex D, being pyramids of the ſame altitude and equal baſes BEF, BCF.

Conſequently all the three pyramids, which compoſe the priſm, are equal to each other, and each pyramid is the third part of the priſm, or the priſm is triple of the pyramid. Q. E. D.

Corol. 1. Every pyramid, whatever its figure may be, is the third part of a priſm of the ſame baſe and altitude; ſince the baſe of the priſm, whatever be its figure, may be divided into triangles, and the whole ſolid into triangular priſms and pyramids.

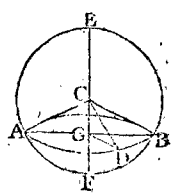
Corol. 2. Any cone is the third part of a cylinder, or of a priſm, of equal baſe and altitude; ſince it has been proved that a cylinder is equal to a priſm, and a cone equal to a pyramid, of equal baſe and altitude.

Scholium. Whatever has been demonſtrated of the proportionality of priſms, or cylinders, holds equally true of pyramids, or cones; the former being always triple the latter; viz. that ſimilar pyramids or cones, are as the cubes of their like linear ſides, or diameters, or altitudes, &c.

THEOREM CXVI.

If a Sphere be cut by a Plane, the Section will be a Circle.

Let the ſphere AEBF be cut by the plane ADB; then will the ſection ADB be a circle.



Draw the chord AB, or diameter of the ſection; perpendicular to which, or to the ſection ADB, draw the axis of the ſphere ECGF, through the centre C, which will biſect the chord AB in the point G (th. 41). Alſo, join CA, CB; and

and draw CD, GD to any point D in the circumference of the ſection ADB.

Then, becauſe CG is perpendicular to the plane ADB, it is perpendicular both to GA and GD (def. 84). So that CGA, CGD are two right-angled triangles, having the perpendicular CG common, and the two hypothenuſes CA, CD equal, being both radii of the ſphere; therefore the third ſides GA, GD are alſo equal (cor. th. 45). In like manner, it is ſhewn, that any other line drawn from the centre G to the circumference of the ſection ADB, is equal to GA or GB; conſequently that ſection is a circle.

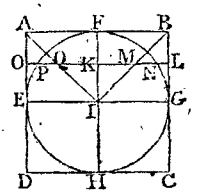
Corol. 1. The centre of every ſection of a ſphere is always in a diameter of the ſphere.

Corol. 2. The ſection through the centre, is a circle having the ſame centre and diameter as the ſphere, and is called a great circle of the ſphere.

THEOREM CXVII.

Every Sphere is Two-Thirds of its Circumſcribing Cylinder.

LET ABCD be a cylinder, circumſcribing the ſphere EFGH; then will the ſphere EFGH be two-thirds of the cylinder ABCD.



For, let the plane AC be a ſection of the ſphere and cylinder through the centre I. Join AI, BI. Alſo, let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC the baſe of the cylinder, the latter line KL meeting BI in M, and the circular ſection of the ſphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the ſquare FG will deſcribe a cylinder AG, the quadrant IFG will deſcribe a hemiſphere EFG, and the triangle IFB will deſcribe a cone IAB. Alſo, in the revolution, the three lines or parts KL, KN, KM, as radii, will deſcribe correſponding circular ſections of thoſe ſolids, namely, KL a ſection of the cylinder, KN a ſection of the ſphere, and KM a ſection of the cone.

Now, FB being equal to FI or IG, and KL parallel to FB, by ſimilar triangles, IK is equal to KM (th. 82). And ſince, in the right-angled triangle IKN, IN^2 is equal $IK^2 + KN^2$ (th. 34); and becauſe KL is equal the radius IG or IN,

IN, and KM = IK, therefore KL^2 is equal $KM^2 + KN^2$, or the ſquare of the longeſt radius, of the ſaid circular ſections, is equal to the ſum of the ſquares of the two others. And becauſe circles are to each other as the ſquares of their diameters, or of their radii, therefore the circle deſcribed by KL, is equal to both the circles deſcribed by KM and KN, or the ſection of the cylinder equal to the correſponding ſections of the ſphere and cone. And as this is always the caſe in every parallel poſition of KL, it follows, that the cylinder EB, which is compoſed of all the former ſections, is equal to the hemiſphere EFG and cone IAB, which are compoſed of all the latter ſections.

But the cone IAB is a third part of the cylinder EB (cor. 2, th. 115); conſequently the hemiſphere EFG is equal to the remaining two-thirds; or the whole ſphere EFGH equal to two-thirds of the whole cylinder ABCD. Q. E. D.

Corol. 1. A cone, hemiſphere, and cylinder, of the ſame baſe and altitude, are to each other as the numbers 1, 2, 3.

Corol. 2. All ſpheres are to each other as the cubes of their diameters; theſe being like parts of their circumſcribing cylinders.

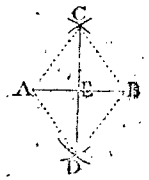
Corol. 3. From the foregoing demonſtration it alſo appears, that the ſpherical zone or fruſtum, EGNP, is equal to the difference between the cylinder EGLO and the cone EMQ, all of the ſame common height IK. And that the ſpherical ſegment PFN, is equal to the difference between the cylinder ABLO and the conic fruſtum AQMB, all of the ſame common altitude FK.

PRO-

PROBLEMS.

PROBLEM I.

To Biſect a Given Line AB; that is, to divide it into two Equal Parts.



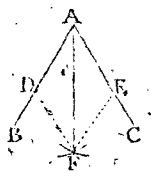
FROM the two centres A and B, with any equal radii, deſcribe arcs of circles, interſecting each other in C and D; and draw the line CD, which will biſect the given line AB in the point E.

For, draw the radii AC, BC, AD, BD. Then, becauſe all theſe four radii are equal, and the ſide CD common, the two triangles ACD, BCD are mutually equilateral: conſequently they are alſo mutually equiangular (th. 5), and have the angle ACE equal to the angle BCE.

Hence, the two triangles ACE, BCE, having the two ſides AC, CE equal to the two ſides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the ſide AE equal to EB. Q. E. D.

PROBLEM II.

To Biſect a Given Angle BAC.



FROM the centre A, with any radius, deſcribe an arc, cutting off the equal lines AD, AE; and from the two centres D, E, with the ſame radius, deſcribe arcs interſecting in F; then draw AF, which will biſect the angle A as required.

For, join DF, EF. Then the two triangles ADF, AEF, having the two ſides AD, DF equal to the two AE, EF (being equal radii), and the ſide AF common, are mutually equilateral; conſequently they are alſo mutually equiangular (th. 5), and have the angle BAF equal the angle CAF.

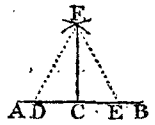
Scholium. In the ſame manner is a given arc of a circle biſected.

PRO-

PROBLEM III.

At a Given Point C, in a Given Line AB, to Erect a Perpendicular.

From the given point C cut off any equal parts CD, CE of the given line; and, from the two centres D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular as required.

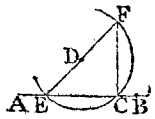


For, draw the two equal radii DF, EF. Then the two triangles CDF, CEF, having the two sides CD, DF, equal to the two CE, EF, and CF common, are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the two adjacent angles at C equal to each other; therefore the line CF is perpendicular to AB (def. 11).

Otherwise.

When the Given Point C is near the End of the Line.

From any point D, assumed above the line, as a centre, through the given point C describe a circle, cutting the given line at E; and through E and the centre D, draw the diameter EDF; then join CF, which will be the perpendicular required.

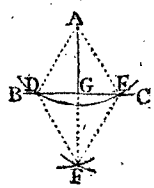


For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 15).

PROBLEM IV.

From a Given Point A, to let fall a Perpendicular on a Given Line BC.

From the given point A as a centre, with any convenient radius, describe an arc, cutting the given line at the two points D and E; and from the two centres D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC as required.



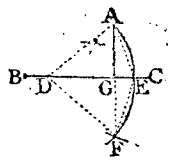
For, draw the equal radii AD, AE, and DF, EF. Then the two triangles ADF, AEF,

AEF, having the two ſides AD, DF, equal to the two AE, EF, and AF common, are mutually equilateral; conſequently they are alſo mutually equiangular (th. 5), and have the angle DAG equal the angle EAG. Hence then, the two triangles ADG, AEG, having the two ſides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1), and have the angles at G equal; conſequently AG is perpendicular to BC (def. 11).

Otherwiſe.

When the Given point is nearly Oppoſite the end of the Line.

FROM any point D, in the given line BC, as a centre, deſcribe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, deſcribe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.

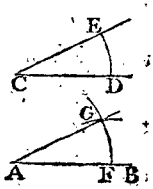


For, draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE will be mutually equilateral; conſequently they are alſo mutually equiangular (th. 5), and have the angles at D equal. Hence, the two triangles DAG, DFG, having the two ſides DA, DG, equal to the two DF, DG, and the included angles at D equal, have alſo the angles at G equal (th. 1); conſequently thoſe angles at G are right angles, and the line AG is perpendicular to DG.

PROBLEM V.

At a Given Point A, in a Given Line AB, to make an Angle Equal to a Given Angle C.

FROM the centres A and C, with any one radius, deſcribe the arcs DE, FG. Then, with centre F, and radius DE, deſcribe an arc cutting FG in G. Through G draw the line AG, and it will form the angle required.

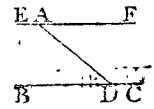


For, conceive the equal lines or radii, DE, FG to be drawn. Then the two triangles

angles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle C.

PROBLEM VI.

Through a Given Point A, to draw a Line Parallel to a Given Line BC.

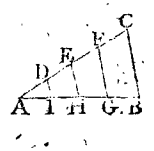


From the given point A draw the line AD to any point in the given line BC. Then draw the line EAF making the angle at A equal to the angle at D (by prob. 5); ſo ſhall EF be parallel to BC as required.

For, the angle D being equal to the alternate angle A, the lines BC, EF, are parallel by th. 13.

PROBLEM VII

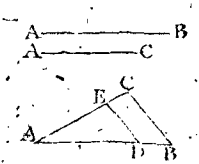
To Divide a Given Line AB into any propoſed Number of Equal Parts.



Draw any other line AC, forming any angle with the given line AB; on which ſet off as many of any equal parts, AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI; then theſe will divide AB in the manner as required.—For thoſe parallel lines divide both the ſides AB, AC proportionally, by th. 82.

PROBLEM VIII.

To find a Third Proportional to Two Given Lines AB, AC.



Place the two given lines AB, AC forming any angle at A; and in AB take alſo AD equal to AC. Join BC, and draw DE parallel to it; ſo will AE be the third proportional ſought.

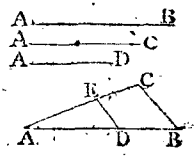
For, becauſe of the parallels BC, DE, the two lines AB, AC are cut proportionally (th. 82); ſo that AB : AC :: AD or AC : AE; therefore AE is the third proportional to AB, AC.

PRO-

PROBLEM IX.

To find a Fourth Proportional to three Given Lines AB, AC, AD.

PLACE two of the given lines AB, AC, making any angle at A; alſo place AD on AB. Join BC; and parallel to it draw DE; ſo ſhall AE be the fourth proportional as required.

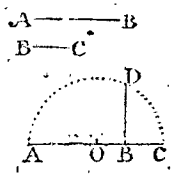


For, becauſe of the parallels BC, DE, the two ſides AB, AC are cut proportionally (th. 82); ſo that AB : AC :: AD : AE.

PROBLEM X.

To find a Mean Proportional between Two Given Lines AB, BC.

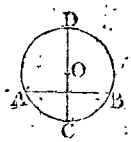
PLACE AB, BC joined in one ſtraight line, which biſect in the point O. Then, with the centre O, and radius OA or OC, deſcribe the ſemicircle ADC; to meet which erect the perpendicular BD, and it will be the mean proportional ſought, between AB and BC (by cor. th. 87).



PROBLEM XI.

To find the Centre of a Given Circle.

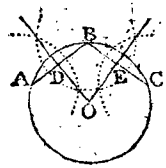
DRAW any chord AB; and biſect it perpendicularly with the line CD, which will be a diameter (th. 41, cor.) Therefore CD biſected in O, will give the centre, as required.



PRO-

PROBLEM XII.

To deſcribe the Circumference of a Circle through Three Given Points A, B, C.

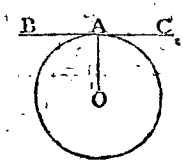


FROM the middle point B draw chords BA, BC, to the two other points, and biſect theſe chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the diſtance of any one of the points, as OA, deſcribe a circle, and it will paſs through the two other points B, C, as required.

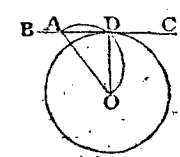
For, the two right-angled triangles OAD, OBD, having the ſides AD, DB equal (by conſtr.), and OD common, with the included right angles at D equal, have their third ſides OA, OB, alſo equal (th. 1). And, in like manner, it is ſhewn that OC is equal to OA. So that all the three OA, OB, OC, being equal, will be radii of the ſame circle.

PROBLEM XIII.

To draw a Tangent to a Given Circle, through a Given Point A.



WHEN the given point A is in the circumference of the circle: Join A and the centre O; perpendicular to which draw BAC, and it will be the tangent, by th. 46.



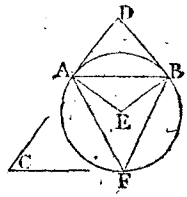
But when the given point A is out of the circle: Draw AO to the centre O; on which as a diameter deſcribe a ſemicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent as required.

For, join DO. Then the angle ADO, in a ſemicircle, is a right angle, and conſequently AD is a tangent (th. 46).

PRO-

PROBLEM XIV.

Upon a Given Line AB to Deſcribe a Segment of a Circle, that may Contain a Given Angle C.

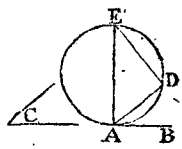


At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE perpendicular to AD, BD; and with the centre E, and radius EA or EB, deſcribe a circle; ſo ſhall AFB be the ſegment required, as any angle F made in it will be equal to the given angle C.

For, the two lines AD, BD, being perpendicular to the radii EA, EB (by conſtr.), are tangents to the circle (th. 46); and the angle A or B, which is equal to the given angle C by conſtruction, is equal to the angle F in the alternate ſegment AFB (th. 53).

PROBLEM XV.

To Cut off a Segment from a Given Circle, that ſhall Contain a given Angle C.

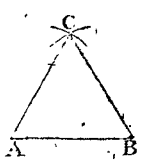


Draw any tangent AB to the given circle; and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the ſegment required, any angle E made in it being equal to the given angle C.

For the angle A, made by the tangent and chord, which is equal the given angle C by conſtruction, is alſo equal to any angle E in the alternate ſegment (th. 53).

PROBLEM XVI.

To make an Equilateral Triangle on a Given Line AB.



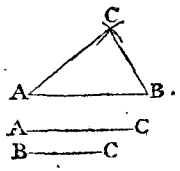
From the centres A and B, with the diſtance AB, deſcribe arcs, interſecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii AC, BC, are, each of them, equal to AB.

PROBLEM XVII.

To make a Triangle with Three Given Lines AB, AC, BC.

WITH the centre A, and diſtance AC, deſcribe an arc. With the centre B, and diſtance BC, deſcribe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.

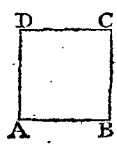


For the radii, or ſides of the triangle, AC, BC, are equal to the given lines AC, BC, by conſtruction.

PROBLEM XVIII.

To make a Square on a Given Line AB.

RAISE AD, BC each perpendicular and equal to AB; and join DC; ſo ſhall ABCD be the ſquare ſought.

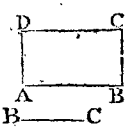


For all the three ſides AB, AD, BC are equal, by the conſtruction, and DC is equal and parallel to AB (by th. 24); ſo that all the four ſides are equal, and the oppoſite ones are parallel. Again, the angle A, or B, of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 22). Hence then, the figure, having all its ſides equal, and all its angles right, is a ſquare (def. 34).

PROBLEM XIX.

To make a Rectangle, or a Parallelogram, of a Given Length and Breadth, AB, BC.

ERECT AD, BC perpendicular to AB, and each equal to BC; then join DC, and it is done.



The demonſtration is the ſame as the laſt problem.

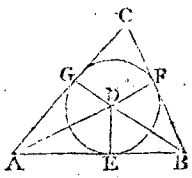
And in the ſame manner is deſcribed any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, inſtead of perpendicular to it.

PRO-

PROBLEM XX.

To Inſcribe a Circle in a Given Triangle ABC.

BISECT the angles at A and B, with the two lines AD, BD. From the interſection D, which will be the centre of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.



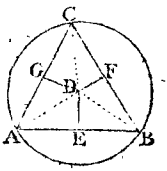
For, ſince the angle DAE is equal the angle DAG, and the angles at E, G right angles (by conſtr.), the two triangles ADE, ADG are equiangular; and, having alſo the ſide AD common, they are identical, and have the ſides DE, DG equal (th. 2). In like manner it is ſhewn, that DF is equal to DE or DG.

Therefore, if with the centre D, and diſtance DE, a circle be deſcribed, it will paſs through all the three points E, F, G, in which points alſo it will touch the three ſides of the triangle (th. 46), becauſe the radii DE, DF, DG are perpendicular to them.

PROBLEM XXI.

To Circumſcribe a Circle about a Given Triangle ABC.

BISECT any two ſides with two of the perpendiculars DE, DF, DG, and D will be the centre.



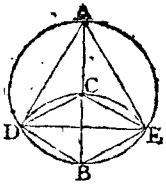
For, join DA, DB, DC. Then the two right-angled triangles DAE, DBE have the two ſides DE, EA equal to the two DE, EB, and the included angles at E equal, thoſe two triangles are therefore identical (th. 1), and have the ſide DA equal DB. In like manner it is ſhewn, that DC is alſo equal to DA or DB. So that all the three DA, DB, DC, being equal, they are radii of a circle paſſing through A, B, and C.

PROBLEM XXII.

To Inſcribe an Equilateral Tringle in a Given Circle.

THROUGH the centre C draw any diameter AB. From the

the point B as a centre, with the radius BC of the given circle, deſcribe an arc DCE. Join AD, AE, and ADE is the equilateral triangle ſought.

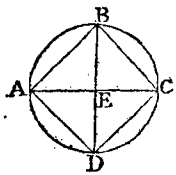


For, join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each ſide equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal the angle CBE, ſtanding on the ſame arc AE; alſo the angle AED is equal the angle CBD, on the ſame arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is alſo equal to the ſame; conſequently that triangle is equilateral.

PROBLEM XXIII.

To Inſcribe a Square in a Given Circle.

Draw two diameters AC, BD, croſſing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and theſe will form the inſcribed ſquare ABCD.

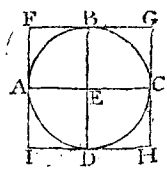


For the four right-angled triangles AEB, BEC, CED, DEA are identical, becauſe they have the ſides EA, EB, EC, ED all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the conſtruction. Therefore all their third ſides AB, BC, CD, DA are equal to one another, and the figure ABCD is equilateral. Alſo, all its four angles A, B, C, D, are right ones, being angles in a ſemicircle. Conſequently the figure is a ſquare.

PROBLEM XXIV.

To Circumſcribe a Square about a Given Circle.

Draw two diameters AC, BD, croſſing at right angles in the centre E. Then through the four extremities of them draw FH, IH parallel to AC, and FI, GH parallel to BD, and they will form the ſquare FGHI.



For

For, the oppoſite ſides of parallelograms being equal, FG and IH are each equal the diameter AC, and FI and GH each equal the diameter BD; ſo that the figure is equilateral. Again, becauſe the oppoſite angles of parallelograms are equal, all the four angles F, G, H, I are right angles, being equal to the oppoſite angles at E. So that the figure FGH, having its ſides equal, and its angles right ones, is a ſquare, and its ſides touch the circle at the four points A, B, C, D, being perpendicular to the radii drawn to thoſe points.

PROBLEM XXV.

To Inſcribe a Circle in a Given Square.

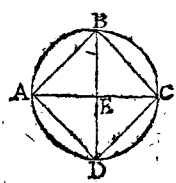
BISECT the two ſides FG, FI in the points A and B (laſt fig.) Then through theſe two points draw AC parallel to FG or IH, and BD parallel to FI or GH. Then the point of interſection E will be the centre, and the four lines EA, EB, EC, ED radii of the inſcribed circle.

For, becauſe the four parallelograms EF, EG, EH, EI have their oppoſite ſides and angles equal; therefore all the four lines EA, EB, EC, ED are equal, being each equal to half a ſide of the ſquare. So that a circle deſcribed from the centre E, with the diſtance EA, will paſs through all the points A, B, C, D, and will be inſcribed in the ſquare; or will touch its four ſides in thoſe points, becauſe the angles there are right ones.

PROBLEM XXVI.

To Circumſcribe a Circle about a Given Square.

DRAW the diagonals AC, BD, and their interſection E will be the centre.

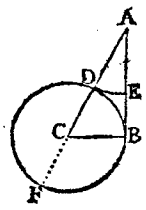


For the diagonals of a ſquare biſect each other (th. 40), making EA, EB, EC, ED all equal, and conſequently theſe are radii of a circle paſſing through the four points A, B, C, D.

PROBLEM XXVII.

To Cut a Given Line in Extreme and Mean Proportion.

LET AB be the given line to be divided in extreme and mean ratio, that is, ſo as that the whole line may be to the greater part, as the greater part is to the leſs part.



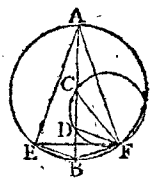
Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and diſtance CB, deſcribe the circle BD; then with centre A and diſtance AD, deſcribe the arc DE; ſo ſhall AB be divided in E in extreme and mean ratio, or ſo that AB : AE :: AE : EB.

For, produce AC to the circumference at F. Then, ADF being a ſecant, and AB a tangent, becauſe B is a right angle; therefore the rectangle AF . AD is equal AB^2 (cor. 1, th. 61); conſequently the means and extremes of theſe are proportional (th. 77), or AB : AF or AD + DF :: AD : AB. But AE is equal AD by conſtruction, and AB = 2BC = DF; therefore, AB : AE + AB :: AE : AB; and by diviſion, AB : AE :: AE : EB.

PROBLEM XXVIII.

To Inſcribe an Iſoſceles Triangle in a Given Circle, that ſhall have each of the Angles at the Baſe Double the Angle at the Vertex.

DRAW any diameter AB of the given circle; and divide the radius CB, in the point D, in extreme and mean ratio, by the laſt problem. From the point B apply the chords BE, BF each equal to CD; then join AE, AF, EF, and AEF will be the triangle required.



For, through the three points C, D, F deſcribe the circle CDF; and draw the lines CF, DF.

Then, becauſe CB : CD :: CD : DB (by conſtr.), the rectangle BC . BD = CD^2 (th. 77) = BF^2 by conſtruction; conſequently BF is a tangent to the circle CDF (corol. 1, th. 61). Hence the angle BFD, made by the tangent BF and chord FD, is equal to the angle BCF in the alternate ſegment;

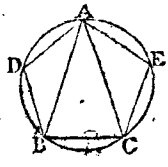
ſegment; to each of theſe equal angles add the angle DFC, then the whole angle GFB, or its equal CBF (th. 3), is equal to the ſum of the two DFC, DCF, which again is equal to the external angle BDF of the triangle CDF (th. 16). Since then the two angles FBD, FDB are equal, their oppoſite ſides BF, FD are alſo equal (th. 4). But BF, CD are equal (by conſtr.); therefore CD, DF are equal, being each equal to BF, and conſequently their oppoſite angles DCF, DFC are alſo equal (th. 3); and the ſum of the two together, or the equal external angle BDF, is double of one of them BCF. But BCF is equal to EAF (th. 51), and BDF equal to DBF, or to AEF on the ſame arc AF (th. 50); conſequently this angle AEF is double of EAF.

Again, the two angles AEB, AFB, being angles in a ſemicircle are right angles, and the two ſides BE, BA, of the right-angled triangle AEB, are equal to the two ſides BF, BA, of the right-angled triangle AFB; therefore the third ſides AE, AF are equal (cor. th. 45). Hence the triangle AEF is iſoſceles, and has each angle at the baſe double the angle at the vertex.

PROBLEM XXIX.

To Inſcribe a Regular Pentagon in a Given Circle.

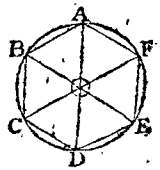
INSCRIBE the iſoſceles triangle ABC having each of the angles ABC, ACB double the angle BAC (prob. 28). Then biſect the two arcs ADB, AEC, in the points D, E; and draw the chords AD, DB, AE, EC, ſo ſhall ADBCE be the inſcribed equilateral pentagon required.



For, becauſe equal angles ſtand on equal arcs, and double angles on double arcs, alſo the angles ABC, ACB being each double the angle BAC, therefore the arcs ADB, AEC, ſubtending the two former angles, are each double the arc BC ſubtending the latter. And ſince the two former arcs are biſected in D and E, it follows that all the five arcs AD, DB, BC, CE, EA are equal to each other, and conſequently the chords alſo which ſubtend them, or the five ſides of the pentagon, are all equal.

PROBLEM XXX.

To Inſcribe a Regular Hexagon in a Given Circle.



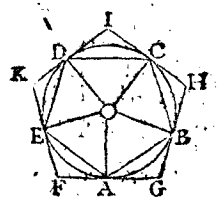
APPLY the radius of the given circle AO, as a chord AB, and it will be a ſide of the regular hexagon ABCDEF.

For, draw the radii AO, BO, CO, DO, EO, FO. Then the triangle ABO being equilateral (by conſtr.), its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one-third of two right angles (th. 17), or one-ſixth of four right angles. But the whole circumference is the meaſure of four right angles (cor. 4, th. 6). Therefore the arc AB is one-ſixth of the circumference of the circle, and conſequently its chord AB one ſide of an equilateral hexagon inſcribed in the circle.

Corol. The ſide of a regular hexagon is equal to the radius of the circumſcribing circle, or to the chord of one-ſixth part of the circumference.

PROBLEM XXXI.

To deſcribe a Regular Pentagon or Hexagon about a Given Circle.



IN the given circle inſcribe a regular polygon of the ſame name or number of ſides, as ABCDE, by one of the foregoing problems. Then to all the angular points of it draw tangents (by prob. 13), and theſe will form the circumſcribing polygon required.

For, all the chords, or ſides of the inſcribing figure, AB, BC, &c, being equal, and all the radii OA, OB, &c, being equal, all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right-angles; therefore OEF + OAF = two right angles, and OAG + OBG = two right angles; conſequently, alſo, AOE + AFE = two right angles, and AOB + AGB = two right angles (cor. 2, th. 18). Hence, then, the angles AOE + AFE being = AOB + AGB, of which AOB is = AOE;

AOE; conſequently the remaining angles F and G are alſo equal. In the ſame manner it is ſhewn, that all the angles F, G, H, I, K are equal.

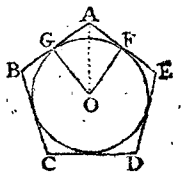
Again, the tangents from the ſame point FE, FA are equal, as alſo the tangents AG, GB (th 61. cor. 2); alſo, the angles F and G of the iſoſceles triangles AFE, AGB, are equal, as well as their oppoſite ſides AE, AB; conſequently thoſe two triangles are identical (th. 1). and have their other ſides EF, FA, AG, GB all equal, and FG equal to the double of any one of them In like manner it is ſhewn that all the other ſides GH, HI, IK, KF are equal to FG, or double of the tangents GB, BH, &c.

Hence, then, the circumſcribed figure is both equilateral and equiangular, which was to be ſhewn.

Corol. The inſcribed circle touches the middles of the ſides of the polygon.

PROBLEM XXXII.

To Inſcribe a Circle in a Regular Polygon.

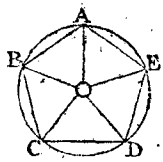


BISECT any two ſides of the polygon by the perpendiculars GO, FO and their interſection O will be the centre of the inſcribed circle, and OG or OF will be the radius.

For the perpendiculars to the tangents AF, AG, paſs through the centre (cor. th. 47); and the inſcribed circle touches the middle points F, G, by the laſt corollary. Alſo, the two ſides AG, AO, of the right-angled triangle AOG, being equal to the two ſides AF, AO, of the right-angled triangle AOF, the third ſides OF, OG will alſo be equal (cor th. 45). Therefore the circle deſcribed with the centre O and radius OG, will paſs through F, and will touch the ſides in the points G and F. And the ſame for all the other ſides of the figure.

PROBLEM XXXIII.

To deſcribe a Circle about a Regular Polygon.

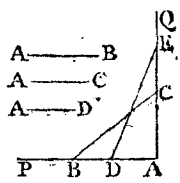


BISECT any two of the angles, C and D, with the lines CO, DO; then their interſection O will be the centre of the circumſcribing circle; and OC, or OD, the radius.

For, draw OB, OA, OE, &c, to the angular points of the given polygon. Then the triangle OCD is iſoſceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore their oppoſite ſides CO, DO are equal (th. 4). But the two triangles OCD, OCB, having the two ſides OC, CD equal to the two OC, CB, and the included angles OCD, OCB alſo equal, will be identical (th. 1), and have their third ſides BO, OD equal. In like manner it is ſhewn, that all the lines OA, OB, OC, OD, OE are equal. Conſequently a circle deſcribed with the centre O and radius OA, will paſs through all the other angular points, B, C, D, &c, and will circumſcribe the polygon.

PROBLEM XXXIV.

To make a Square Equal to the Sum of two Given Squares.



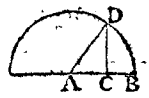
LET AB and AC be the ſides of the two given ſquares. Draw two indefinite lines AP, AQ at right angles to each other; in which place the ſides AB, AC of the given ſquares; join BC; then a ſquare deſcribed on BC will be equal to the ſum of the two ſquares deſcribed on AB and AC (th. 34).

Scholium. In the ſame manner, a ſquare may be made equal to the ſum of three or more given ſquares. For, if AB, AC, AD be taken as the ſides of the given ſquares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the ſquare on DE will be equal to the ſum of the three ſquares on AB, AC, AD. And ſo on for more ſquares.

PRO-

PROBLEM XXXV.

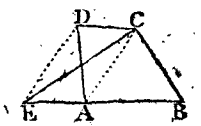
To make a Square Equal to the Difference of two Given Squares.



LET AB and AC, taken in the ſame ſtraight line, be equal to the ſides of the two given ſquares.—From the centre A, with the diſtance AB, deſcribe a circle; and make CD perpendicular to AB, meeting the circumference in D: ſo ſhall a ſquare deſcribed on CD be equal to $AD^2 - AC^2$ or $AB^2 - AC^2$, as required (cor. th. 34).

PROBLEM XXXVI.

To make a Triangle Equal to a Given Quadrilateral ABCD.

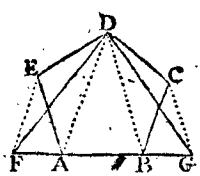


DRAW the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE: then will the triangle CEB be equal to the given quadrilateral ABCD.

For, the two triangles ACE, ACD, being on the ſame baſe AC, and between the ſame parallels AC, DE, are equal (th. 25); therefore, if ABC be added to each, BCE will be equal to ABCD (ax. 2).

PROBLEM XXXVII.

To make a Triangle Equal to a Given Pentagon ABCDE.



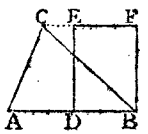
DRAW DA and DB, and alſo EF, CG parallel to them, meeting AB produced at F and G; then draw DF and DG; ſo ſhall the triangle DFG be equal to the given pentagon ABCDE.

For the triangle DFA = DEA, and the triangle DGB = DCB (th. 25); therefore, by adding DAB to the equals, the ſums are equal (ax. 2), that is, DAB + DAF + DBG = DAB + DAE + DBC, or the triangle DFG = the pentagon ABCDE.

PRO-

PROBLEM XXXVIII.

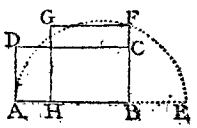
To make a Rectangle Equal to a Given Triangle ABC.



BISECT the base AB in D; then raise DE and BF perpendicular to AB, and meeting CF, parallel to AB, at E and F; so shall DF be the rectangle equal to the given triangle ABC (by cor. 2, th. 26).

PROBLEM XXXIX.

To make a Square Equal to a Given Rectangle ABCD.



PRODUCE one side AB, till BE be equal to the other side BC. On AE as a diameter describe a circle, meeting BC produced at F; then will BF be the side of the square BFGH, equal to the given rectangle BD, as required, as appears by Cor. Th. 87, and Th. 77.

APPLI-

APPLICATION OF ALGEBRA TO GEOMETRY.

WHEN it is propoſed to reſolve a geometrical problem algebraically, or by algebra, it is proper, in the firſt place, to draw a figure that ſhall repreſent the ſeveral parts or conditions of the problem, and to ſuppoſe that figure to be the true one. Then, having conſidered attentively the nature of the problem, the figure is next to be prepared for a ſolution, if neceſſary, by producing or drawing ſuch lines in it, as appear moſt conducive to that end. This done, the uſual ſymbols or letters, for known and unknown quantities, are employed to denote the given parts of the figure, or as many of them as neceſſary, as alſo ſuch unknown line or lines as may be eaſieſt found, whether required or not. Then proceed to the operation, by obſerving the relations that the ſeveral parts of the figure have to each other; from which, and the proper theorems in the foregoing elements of geometry, make out as many equations, independent of each other, as there are unknown quantities, employed in them: the reſolution of which equations, in the ſame manner as in arithmetical problems, will determine the unknown quantities, and reſolve the problem propoſed.

As no general rule can be given for drawing the lines, and ſelecting the fitteſt quantities to ſubſtitute for, ſo as always to bring out the moſt ſimple concluſions, becauſe different problems require different methods of ſolution; the beſt way to gain experience in this matter, is to try the ſolution of the ſame problem in different ways, and then apply that which ſucceeds beſt, to other caſes of the ſame kind, when they afterwards occur. The following particular directions, however, may be of ſome uſe in this matter.

1*ſt*, In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or ſo as to form ſimilar triangles. And if an angle be given, it will be proper to let the perpendicular be oppoſite to that angle, and to fall from one end of a given line, if poſſible.

2*nd*,

2*nd*, In ſelecting the quantities proper to ſubſtitute for, thoſe are to be choſen, whether required or not, which lie neareſt the known or given parts of the figure, and by means of which the next adjacent parts may be expreſſed by addition and ſubtraction only, without the intervention of ſurds.

3*d*, When two lines or quantities are alike related to other parts of the figure or problem, the beſt way is, not to make uſe of either of them ſeparately, but to ſubſtitute for their ſum, or difference, or rectangle, or the ſum of their alternate quotients, or for ſome line or lines, in the figure, to which they have both the ſame relation.

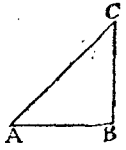
4*th*, When the area, or the perimeter, of a figure, is given, or ſuch parts of it as have only a remote relation to the parts required; it is ſometimes of uſe to aſſume another figure ſimilar to the propoſed one, having one ſide equal to unity, or ſome other known quantity. For, from hence the other parts of the figure may be found, by the known proportions of the like ſides, or parts, and ſo an equation be obtained.

For examples of this matter take the following problems.

PROBLEM I.

In a Right-angled Triangle, having given the Baſe (3), *and the Sum of the Hypothenuſe and Perpendicular* (9); *to find both theſe two Sides.*

LET ABC repreſent the propoſed triangle, right-angled at B. Put the baſe AB $= 3 = b$, and the ſum AC + BC of the hopothenuſe and perpendicular $= 9 = s$; alſo, let x denote the perpendicular BC; hence it follows that the hypothenuſe AC will be expreſſed by $s - x$.



But $AC^2 = AB^2 + BC^2$ (by theor. 34. Geom.); that is, $(s-x)^2 = b^2 + x^2$, or $s^2 - 2sx + x^2 = b^2 + x^2$, or $s^2 - b^2 = 2sx$; hence $x = \frac{s^2 - b^2}{2s} = 4 = BC$, and $s - x = \frac{s^2 + b^2}{2s} = 5 = AC$, the perpendicular and hypothenuſe as required.

PRO-

PROBLEM II.

In a Right-angled Triangle, having given the Hypothenuſe (5); *and the Sum of the Baſe and Perpendicular* (7); *to find both theſe two Sides.*

LET ABC repreſent the propoſed triangle, right-angled at B. Put the given hypothenuſe AC $= 5 = a$, and the ſum AB + BC of the baſe and perpendicular $= 7 = s$; alſo, let x denote the baſe AB; hence it follows that the perpendicular BC will be expreſſed by $s - x$.

But $AC^2 = AB^2 + BC^2$ by the nature of right-angled triangles; that is, $a^2 = x^2 + (s - x)^2 = x^2 + s^2 - 2sx + x^2$, or $2x^2 - 2sx = a^2 - s^2$, or $x^2 - sx = \frac{1}{2}a^2 - \frac{1}{2}s^2$; hence, by completing the ſquare, &c, is found $x = \frac{1}{2}s - \frac{1}{2}\sqrt{2a^2 - s^2} = 3 = AB$; and $s - x = \frac{1}{2}s + \frac{1}{2}\sqrt{2a^2 - s^2} = 4 = BC$; the baſe and perpendicular required.

Otherwiſe.

LET z denote the difference of the baſe and perpendicular, the ſum being denoted by s, as above. Then, becauſe that, in any two quantities, half their difference added to half the ſum gives the greater quantity, and the ſame ſubtracted gives the leſs quantity (by Ex. 2, Alg. pag. 227), it follows that $\frac{1}{2}s + \frac{1}{2}z$ and $\frac{1}{2}s - \frac{1}{2}z$ will denote the two legs, or the perpendicular and baſe. Hence then $a^2 = (\frac{1}{2}s + \frac{1}{2}z)^2 + (\frac{1}{2}s - \frac{1}{2}z)^2 = \frac{1}{2}s^2 + \frac{1}{2}z^2$; or $z^2 = 2a^2 - s^2$, and $z = \sqrt{2a^2 - s^2}$. Then the half of this being added and ſubtracted with $\frac{1}{2}s$ the half ſum, gives - - - - - - - -

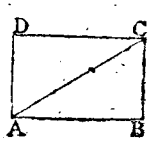
$\frac{1}{2}s + \frac{1}{2}z = \frac{1}{2}s + \frac{1}{2}\sqrt{2a^2 - s^2}$, and

$\frac{1}{2}s - \frac{1}{2}z = \frac{1}{2}s - \frac{1}{2}\sqrt{2a^2 - s^2}$, the ſame as before.

PROBLEM III.

In a Rectangle, having given the Diagonal (10), *and the Perimeter, or Sum of all the Four Sides* (28); *to find each of the Sides ſeverally.*

LET ABCD be the propoſed rectangle; and put the diagonal AC $= 10 = d$, and half the perimeter AB + BC or AD + DC $= 14 = a$; alſo put one ſide AB $= x$; then the other ſide BC will be denoted by $a - x$. Hence, by right-angled triangles,



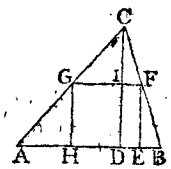
AC^2

$AC^2 = AB^2 + BC^2$, that is, $d^2 = x^2 + (a - x)^2 = 2x^2 - 2ax + a^2$: conſeq. $x^2 - ax = \frac{1}{2}d^2 - \frac{1}{4}a^2$, and by completing the ſquare, &c, $x = \frac{1}{2}a + \frac{1}{2}\sqrt{2d^2 - a^2} = 8 = AB$, and $a - x = \frac{1}{2}a - \frac{1}{2}\sqrt{2d^2 - a^2} = 6 = BC$, the two ſides; and the ſame for the other two AD, DC.

Or, by ſubſtituting for the ſum and difference of the two ſides, AB, BC, this problem may be reſolved, like the latter method of the foregoing one, by a ſimple quadratic, and conſequently without completing the ſquare.

PROBLEM IV.

Having given the Baſe and Perpendicular of any Triangle; to find the Side of a Square Inſcribed in the ſame.

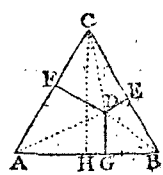


Let ABC repreſent the given triangle, and EFGH its inſcribed ſquare. Put the baſe $AB = b$, the perpendicular $CD = a$, and the ſide of the ſquare GF or $GH = DI = x$; then will $CI = CD - DI = a - x$.

Then, becauſe the like lines in the ſimilar triangles ABC, GFC, are proportional, (by theor. 84, Geom.), AB : CD :: GF : CI, that is, $b : a :: x : a - x$. Hence $ab - bx = ax$, or $ab = ax + bx$, and conſequently $x = \frac{ab}{a+b} = GF$ or GH the ſide of the inſcribed ſquare; which therefore is of the ſame magnitude, whatever the ſpecies or the angles of the triangle may be.

PROBLEM V.

In an Equilateral Triangle, having given the lengths of the three Perpendiculars, drawn from a certain Point within, on the three Sides; to determine the Sides.



Let ABC repreſent the equilateral triangle, and DE, DF, DG the given perpendiculars from the point D. Draw the lines DA, DB, DC to the three angular points; and let fall the perpendicular CH on the baſe AB. Put the three given perpendiculars $DE = a$, $DF = b$, $DG = c$, and put $x = AH$ or BH, half the ſide of the

the equilateral triangle. Then is AC or BC $= 2x$, and by right-angled triangles the perpendicular CH $= \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$.

Now, since the area or space of a rectangle is expressed by the product of the base and height (cor. 2, th. 81, Geom.), and that a triangle is equal to half a rectangle of equal base and height (cor. 1, th. 26), it follows that,

the whole triangle ABC is $= \frac{1}{2}AB \times CH = x \times x\sqrt{3} = x^2\sqrt{3}$,
the triangle ABD $= \frac{1}{2}AB \times DG = x \times c = cx$,
the triangle BCD $= \frac{1}{2}BC \times DE = x \times a = ax$,
the triangle ACD $= \frac{1}{2}AC \times DF = x \times b = bx$.

But the three last triangles make up, or are equal to, the whole former or great triangle;

that is, $x^2\sqrt{3} = ax + bx + cx$; hence then

$x = \frac{a+b+c}{\sqrt{3}}$, half the side of the triangle sought.

Also, since the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH, is just equal to the sum of all the three smaller perpendiculars DE + DF + DG taken together, wherever the point D is situated.

PROBLEM VI.

In a Right-angled Triangle, having given the Base (3), and the Difference between the Hypothenuse and Perpendicular (1); to find both these two Sides.

PROBLEM VII.

In a Right-angled Triangle, having given the Hypothenuse (5), and the Difference between the Base and Perpendicular (1); to determine both these two Sides.

PROBLEM VIII.

Having given the Area, or Measure of the Space, of a Rectangle, inscribed in a given Triangle; to determine the Sides of the Rectangle.

PROBLEM IX.

In a Triangle, having given the Ratio of the two Sides, together with both the Segments of the Baſe, made by a Perpendicular from the Vertical Angle; to determine the Sides of the Triangle.

PROBLEM X.

In a Triangle, having given the Baſe, the Sum of the other two Sides, and the Length of a Line drawn from the Vertical Angle to the Middle of the Baſe; to find the ſides of the Triangle.

PROBLEM XI

In a Triangle, having given the two Sides, about the Vertical Angle, and the Line biſecting that Angle, and terminating in the Baſe; to find the Baſe.

PROBLEM XII.

To determine the Radii of three Equal Circles, deſcribed in a given Circle, to touch each other and alſo the Circumference of the given Circle.

PROBLEM XIII.

In a right-angled Triangle, having given the Perimeter or Sum of all the Sides, and the Perpendicular let fall from the Right Angle on the Hypothenuſe; to determine the Triangle, that is, its Sides.

PROBLEM XIV.

To determine a Right-angled Triangle; having given, the Perimeter, and the Radius of its Inſcribed Circle.

PROBLEM XV.

To determine a Right-angled Triangle; having given, the Lengths of two Lines drawn from the acute Angles, to the Middle of the oppoſite Sides.

PRO-

PROBLEM XVI.

To determine a Right-angled Triangle; having given, the Hypothenuſe, and the Difference of two Lines drawn from the two acute Angles to the Centre of the Inſcribed Circle.

PROBLEM XVII.

To determine a Triangle; having given, the Baſe, the Perpendicular, and the Ratio of the two Sides.

PROBLEM XVIII.

To determine a Right-angled Triangle; having given, the Hypothenuſe, and the Side of the Inſcribed Square.

PROBLEM XIX.

To determine a Triangle; having given, the Baſe, the Perpendicular, and the Difference of the two other Sides.

PROBLEM XX.

To determine a Triangle; having given, the Baſe, the Perpendicular, and the Rectangle or Product of the two Sides.

PROBLEM XXI.

To determine a Triangle; having given, the Lengths of three Lines drawn from the three Angles, to the Middle of the oppoſite Sides.

PROBLEM XXII.

In a Triangle, having given, all the three Sides; to find the Radius of the Inſcribed Circle.

PROBLEM XXIII.

To determine a Right-angled Triangle; having given, the Side of the Inſcribed Square, and the Radius of the Inſcribed Circle.

PRO-

PROBLEM XXIV.

To determine a Triangle, and the Radius of the Inſcribed Circle; having given, the Lengths of three Lines drawn from the three Angles, to the Centre of that Circle.

PROBLEM XXV.

To determine a Right-angled Triangle; having given, the Hypothenuſe, and the Radius of the Inſcribed Circle.

PROBLEM XXVI.

To determine a Triangle; having given, the Baſe, the Line biſecting the Vertical Angle, and the Diameter of the Circumſcribing Circle.

END OF VOL. I.

COURSE

OF

MATHEMATICS,

IN TWO VOLUMES:

COMPOSED, AND MORE ESPECIALLY DESIGNED,

FOR THE USE OF THE GENTLEMEN CADETS

IN THE ROYAL MILITARY ACADEMY AT WOOLWICH.

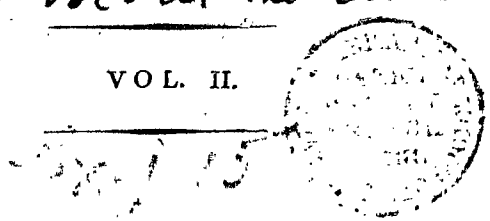
BY

CHARLES HUTTON, LL.D., F.R.S.,

AND PROFESSOR OF MATHEMATICS IN THE SAID ACADEMY.

1st Vol. at the end

VOL. II.



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CONTENTS OF VOL. II.

	Page
PLANE Trigonometry	1
Heights and Diſtances	17
Menſuration of Planes or Areas	25
Menſuration of Solids	41
Land Surveying	50
Artificers Works	84
Timber Meaſuring	93
Conic Sections	96
Of the Ellipſe	100
Of the Hyperbola	108
Of the Parabola	118
Laws of Motion, Forces, &c.	132
Compoſition and Reſolution of Forces	140
Colliſion of Bodies	144
Laws of Gravity, Projectiles, &c.	151
Practical Gunnery	163
Inclined Planes, Pendulums, &c.	166
Mechanical Powers	177
Centre of Gravity	191
Centres of Percuſſion, Oſcillation, and Gyration	201
Of Hydroſtatics	211
Of Hydraulics	222
Of Pneumatics	226
Of the Syphon	236
Of the Common Pump	237
Of the Air Pump	238
Diving Bell, and Condenſer	240

Of

Page

Of the Barometer - - - - - - - 241
Of the Thermometer - - - - - - - 243
Meaſurement of Altitudes by the Barometer and Thermometer 244
Reſiſtance of Fluids - - - - - - - 245
Practical Exerciſes in Menſuration - - - 248
——————— *on Specific Gravity* - - - 256
Weights and Dimenſions of Balls and Shells - - 259
Of the Piling of Balls and Shells - - - 263
Of Diſtances by the Velocity of Sound - - - 265
Practical Exerciſes in Mechanics, Statics, Hydroſtatics, Sound, Motion, Gravity, Projectiles, and other Branches of Natural Philoſophy - - - - - - 266
The Doctrine of Fluxions - - - - - - 276
The Direct Method of Fluxions - - - - 279
The Inverſe Method of Fluxions - - - - 287
Of Maxima and Minima - - - - - - 301
The Method of Tangents - - - - - - 306
Rectification of Curves - - - - - - 308
Quadrature of Curves - - - - - - 310
Surfaces of Bodies - - - - - - - 312
Cubature of Solids - - - - - - - 313
To make Logarithms - - - - - - - 314
Inflexion of Curves - - - - - - - 316
Radius of Curvature - - - - - - - 318
Involutes and Evolutes - - - - - - 320
Centres of Gravity - - - - - - - 323
Practical Queſtions in Fluxions - - - - 326
Practical Exerciſes on Forces - - - - 328
The Motion of Bodies in Fluids - - - - 350

PLANE

PLANE TRIGONOMETRY.

DEFINITIONS.

1. PLANE TRIGONOMETRY treats of the relations and calculations of the ſides and angles of plane triangles.

2. The circumference of every circle (as before obſerved in Geom. Def. 56) is ſuppoſed to be divided into 360 equal parts, called Degrees; alſo each degree into 60 Minutes, each minute into 60 Seconds, and ſo on.

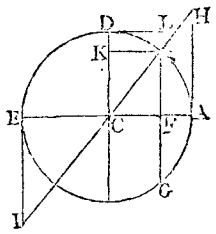
Hence a ſemicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Meaſure of any angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is eſtimated by the number of degrees contained in that arc.

Hence, a right angle, being meaſured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the ſum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the ſum of two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the ſum of the other two angles.

4. Degrees are marked at the top of the figure with a ſmall °, minutes with ′, ſeconds with ″, and ſo on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 ſeconds.

5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.



6. The Supplement of an arc, is what it wants of a ſemicircle, or 180°. Thus, if ADE be a ſemicircle, then BDE is the ſupplement of the arc AB; and, reciprocally, AB is the ſupplement of the arc BDE. So that, if AB be an arc of 50°, then its ſupplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter paſſing through the other extremity. Thus, BF is the ſine of the arc AB, or of the arc BDE.

Corol. Hence the ſine (BF) is half the chord (BG) of the double arc (BAG).

8. The Verſed Sine of an arc, is the part of the diameter intercepted between the arc and its ſine. So, AF is the verſed ſine of the arc AB, and EF the verſed ſine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity: which laſt line is called the Secant of the ſame arc. Thus, AH is the tangent, and CH the ſecant, of the arc AB. Alſo, EI is the tangent, and CI the ſecant, of the ſupplemental arc BDE. And this latter tangent and ſecant are equal to the former, but are accounted negative, as being drawn in an oppoſite or contrary direction to the former.

10. The Coſine, Cotangent, and Coſecant, of an arc, are the ſine, tangent, and ſecant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD being the complements of each other, the ſine, tangent or ſecant of the one of theſe, is the coſine, cotangent or coſecant of the other. So, BF, the ſine of AB, is the coſine of BD; and BK, the ſine of BD, is the coſine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB: alſo, CH, the ſecant of AB, is the coſecant of BD; and CL, the ſecant of BD, is the coſecant of AB.

Corol.

Corol. Hence ſeveral remarkable properties eaſily follow from theſe definitions; as,

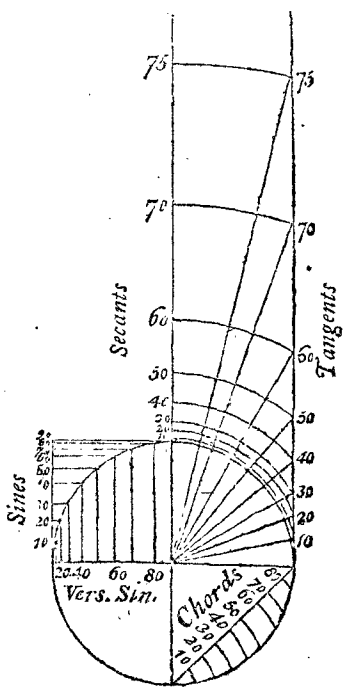
1*ſt*, That an arc and its ſupplement have the ſame ſine, tangent, and ſecant; but the two latter, the tangent and ſecant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2*d*, When the arc is 0, or nothing, the ſine and tangent are nothing, but the ſecant is then the radius CA.—But when the arc is a quadrant AD, then the ſine is the greateſt it can be, being the radius CD of the circle; and both the tangent and ſecant are infinite.

3*d*, Of any arc AB, the verſed ſine AF, and coſine BK, or CF, together make up the radius CA of the circle.—The radius CA, tangent AH, and ſecant CH, form a right-angled triangle CAH. So alſo do the radius, ſine, and coſine, form another right-angled triangle CBF or CBK. As alſo the radius, cotangent and coſecant, another right-angled triangle CDL. And all theſe right-angled triangles are ſimilar to each other.

11. The ſine, tangent, or ſecant of an angle, is the ſine, tangent, or ſecant of the arc by which the angle is meaſured, or of the degrees, &c. in the ſame arc or angle.

12. The method of conſtructing the ſcales of chords, ſines, tangents, and ſecants, uſually engraven on inſtruments, for practice, is exhibited in the annexed figure.



13. A Trigonometrical Canon, is a table exhibiting the length of the ſine, tangent, and ſecant, to every degree and minute of the quadrant, with reſpect to the radius, which is expreſſed by unity or 1, and conceived to be divided into 10000000 or more decimal parts. And farther, the logarithms of theſe ſines, tangents, and ſecants are alſo ranged in the tables; which are moſt commonly uſed, as they perform the calculations by only addition and ſubtraction, inſtead of the multiplication

and diviſion by the natural ſines, &c, according to the nature of logarithms.

Upon this table depends the numeral ſolution of the ſeveral caſes in trigonometry. It will therefore be proper to begin with the mode of conſtructing it, which may be done in the following manner:

PROBLEM I.

To find the Sine and Coſine of a Given Arc.

THIS problem is reſolved after various ways. One of theſe is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known ſeries for the ſine and coſine, hereafter demonſtrated. Thus, the ſemicircumference of the circle, whoſe radius is 1, being 3·14159265358979 3 &c, the proportion will therefore be,

as the number of degrees or minutes in the ſemicircle,
is to the degrees or minutes in the propoſed arc,
ſo is 3·14159265 &c, to the length of the ſaid arc.

This length of the arc being denoted by the letter a; alſo its ſine and coſine by s and c; then will theſe two be expreſſed by the two following ſeries, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$
$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$
$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c.$$
$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

Example 1. If it be required to find the ſine and coſine of one minute. Then, the number of minutes in 180° being 10800, it will be firſt, as 10800 : 1 :: 3·14159265 &c : ·0002908882 08665 = the length of an arc of one minute. Therefore, in this caſe,

a = ·0002908882
and $\frac{1}{6}a^3$ = 0000000000 4 &c,
the diff. is s = ·0002908882 the ſine of 1 minute.
Alſo, from 1·
take $\frac{1}{2}a^2$ = 0·0000000846159 &c,
leaves c = ·9999999153 the coſine of 1 minute.

Exam-

Example 2. For the fine and cofine of 5 degrees.
Here, as $180° : 5° :: 3{\cdot}14159265$ &c : $\cdot08726646 = a$ the length of 5 degrees. Hence $a = \cdot08726646$

$$-\tfrac{1}{6}a^3 = -\cdot00011076$$
$$+\tfrac{1}{120}a^5 = \cdot00000004$$

thefe collected give $s = \cdot08715574$ the fine of 5°.

And, for the cofine, $1 = 1\cdot$

$$-\tfrac{1}{2}a^2 = -\cdot00380771$$
$$+\tfrac{1}{24}a^4 = \cdot00000241$$

thefe collected give $c = \cdot99619470$ the cofine of 5°.

After the fame manner, the fine and cofine of any other arc may be computed. But the greater the arc is, the flower the feries will converge, in which cafe a greater number of terms muft be taken to bring out the concluſion to the fame degree of exactneſs.

Or, having found the fine, the cofine will be found from it, by the property of the right-angled triangle CBF, viz. the cofine $CF = \sqrt{CB^2 - BF^2}$, or $c = \sqrt{1 - s^2}$.

There are alſo other methods of conſtructing the canon of fines and cofines, which, for brevity's fake, are here omitted.

PROBLEM II.

To compute the Tangents and Secants.

THE fines and cofines being known, or found by the foregoing problem; the tangents and fecants will be eaſily found, from the principle of fimilar triangles, in the following manner:

In the firſt figure, where, of the arc AB, BF is the fine, CF or BK the cofine, AH the tangent, CH the fecant, DL the cotangent, and CL the cofecant, the radius being CA or CB or CD; the three fimilar triangles CFB, CAH, CDL give the following proportions:

1*ſt*, CF : FB :: CA : AH; whence the tangent is known, being a fourth proportional to the cofine, fine, and radius.

2*d*, CF : CB :: CA : CH; whence the fecant is known, being a third proportional to the cofine and radius.

3*d*, BF : FC :: CD : DL; whence the cotangent is known, being a fourth proportional to the ſine, coſine, and radius.

4*th*, BF : BC :: CD : CL; whence the coſecant is known, being a third proportional to the ſine and radius.

Having given an idea of the calculation of ſines, tangents and ſecants, we may now proceed to reſolve the ſeveral caſes of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and obſervations, as below.

Note 1. There are uſually three methods of reſolving triangles, or the caſes of trigonometry; namely, Geometrical Conſtruction, Arithmetical Computation, and Inſtrumental Operation.

In the Firſt Method, The triangle is conſtructed, by making the parts of the given magnitudes, namely the ſides from a ſcale of equal parts, and the angles from a ſcale of chords, or by ſome other inſtrument. Then, meaſuring the unknown parts, by the ſame ſcales or inſtruments, the ſolution will be obtained near the truth.

In the Second Method, Having ſtated the terms of the proportion according to the proper rule or theorem, reſolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the ſecond and third together, and dividing the product by the firſt, in working with the natural numbers: or, in working with the logarithms, add the logs. of the ſecond and third terms together, and from the ſum take the log. of the firſt term; then the natural number anſwering to the remainder is the fourth term ſought.

In the Third Method, Or Inſtrumentally, as ſuppoſe by the log lines on one ſide of the common two-foot ſcales; Extend the Compaſſes from the firſt term, to the ſecond or third, which happens to be of the ſame kind with it; then that extent will reach from the other term to the fourth term, as required, taking both extents towards the ſame end of the ſcale.

Note 2. In every triangle, or caſe in trigonometry, there muſt be given three parts, to find the other three. And, of the three parts that are given, one of them at leaſt muſt be a ſide; becauſe the ſame angles are common to an infinite number of triangles.

Note 3.

Note 3. All the cafes in trigonometry, may be compriſed in three varieties only; viz.

1*ſt*, When a ſide and its oppoſite angle are given.
2*d*, When two ſides and the contained angle are given.
3*d*, When the three ſides are given.

For there cannot poſſibly be more than theſe three varieties of caſes; for each of which it will therefore be proper to give a ſeparate theorem, as follows:

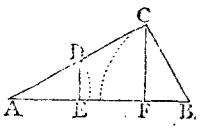
THEOREM I.

When a Side and its Oppoſite Angle, are two of the Given Parts.

THEN the ſides of the triangle have the ſame proportion to each other, as the ſines of their oppoſite angles have. That is, As any one ſide,

Is to the ſine of its oppoſite angle;
So is any other ſide,
To the ſine of its oppoſite angle.

Demonſtr. For, let ABC be the propoſed triangle, having AB the greateſt ſide, and BC the leaſt. Take AD = BC, conſidering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the ſines of the angles A and B, to the radius AD or BC.



But the triangles ADE, ACF are equiangular, and therefore AC : CF :: AD or BC : DE; that is, AC is to the ſine of its oppoſite angle B, as BC to the ſine of its oppoſite angle A.

Note 1. In practice, to find an angle, begin the proportion with a ſide oppoſite a given angle. And to find a ſide, begin with an angle oppoſite a given ſide.

Note 2. An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuſe, unleſs it be a right angle, or unleſs its magnitude be ſuch as to prevent the ambiguity; becauſe the ſine anſwers to two angles, which are ſupplements to each other: and accordingly the geometrical conſtruction forms two triangles with the ſame parts that are given, as in the example below; and when there is no reſtriction or limitation included in the queſtion, either of them may be taken. The degrees in the table, anſwering to the ſine, is the acute angle; but if the angle be obtuſe,

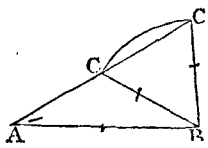
ſubtract thoſe degrees from 180°, and the remainder will be the obtuſe angle. When a given angle is obtuſe, or a right one, there can be no ambiguity; for then neither of the other angles can be obtuſe, and the geometrical conſtruction will form only one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards
BC 232 yards
∠A 37° 20′

Required the other parts.



1. *Geometrically.*

Draw an indefinite line, upon which ſet off AB = 345, from ſome convenient ſcale of equal parts.—Make the angle A = 37°$\frac{1}{3}$.—With a radius of 232, taken from the ſame ſcale of equal parts, and centre B, croſs AC in the two points C, C.—Laſtly, join BC, BC, and the figure is conſtructed, which gives two triangles, and ſhewing that the caſe is ambiguous.

Then, the ſides AC meaſured by the ſcale of equal parts, and the angles B and C meaſured by the line of chords, or other inſtrument, will be found to be nearly as below; viz.

AC	∠B	∠C
174	27°	115°$\frac{1}{2}$
or 374$\frac{1}{2}$	or 78$\frac{1}{4}$	or 64$\frac{1}{2}$

2. *Arithmetically.*

Firſt, to find the angles at C.

As ſide	BC	232	-	-		log.	2·3654880
To ſin. op.	∠A	37°	20′	-	-		9·7827958
So ſide	AB	345	-	-	-		2·5378191
To ſin. op.	∠C	115°	36′	or 64°	24′	-	9·9551269
add	∠A	37	20	37	20		
the ſum		152	56	or 101	44		
taken from		180	00	180	00		
leaves	∠B	27	04	or 78	16		

Then,

Then, to find the side AC.

As sine	∠A	37° 20′	- -	log. 9·7827958
To op. side	BC	232	- -	2·3654880
So sin. ∠B		27° 04′	- -	9·6580371
		78 16	- -	9·9908291
To op. side	AC	174·07	- -	2·2407293
	or	374·56	- -	2·5735213

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 232 to 345 upon the line of numbers; then that extent will reach, on the sines, from $37°\frac{1}{3}$ to $64°\frac{1}{2}$, the angle C.

In the second proportion.—Extend the compasses from $37°\frac{1}{3}$ to 27° or $78°\frac{1}{4}$, on the sines; then that extent will reach, on the line of numbers, from 232 to 174 or 374·5, the two values of the side AC.

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles, ∠A 57° 12′, ∠B 24 45 } Anf. { ∠C 98° 3′, AC 154·33, BC 309·86 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 feet, BC 112 feet, ∠A 57° 27′ }

Anf. { ∠B 64° 34′ 21″ or 115 25 39, ∠C 57 58 39 or 7 7 21, AB 112·65 feet or 16·47 feet. }

Required the other parts.

THEOREM II.

When two Sides and their Contained Angle are given.

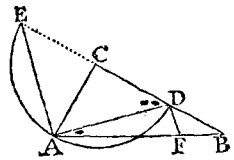
THEN it will be,

As the sum of those two sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their op. angles,
To the tang. of half the diff. of the same angles.

Hence,

Hence, becauſe it has been ſhewn (Alg. p. 227), that the half ſum of any two quantities increaſed by their half difference, gives the greater, and diminiſhed by it gives the leſs, if the half difference of the angles, ſo found, be added to their half ſum, it will give the greater angle, and ſubtracting it will leave the leſs angle.

Then all the angles being now known, the unknown ſide will be found by the former theorem.



Demonſtr. Let ABC be the propoſed triangle, having the two given ſides AC, BC, including the given angle C. With the centre C, and radius CA, the leſs of theſe two ſides, deſcribe a ſemicircle, meeting the other ſide BC produced in D and E. Join AE, AD, and draw DF parallel to AE.

Then, BE is the ſum, and BD the diff. of the two given ſides CB, CA. Alſo, the ſum of the two angles CAB, CBA, is equal to the ſum of the two CAD, CDA, theſe ſums being each the ſupplement of the vertical angle C to two right angles: but the two latter CAD, CDA are equal to each other, being oppoſite to the two equal ſides CA, CD: hence either of them, as CDA, is equal to half the ſum of the two unknown angles CAB, CBA. Again, the exterior angle CDA is equal to the two interior angles B and DAB; therefore the angle DAB is equal to the diff. between CDA and B, or between CAD and B; conſequently the ſame angle DAB is equal to half the difference of the unknown angles B and CAB; of which it has been ſhewn that CDA is the half ſum.

Now the angle DAE, in a ſemicircle, is a right angle, or AE is perp. to AD; and DF, parallel to AE, is alſo perp. to AD: conſeq. AE is the tangent of CDA the half ſum, and DF the tangent of DAB the half difference of the angles, to the ſame radius AD, by the definition of a tangent. But, the tangents AE, DF being parallel, it will be as BE : BD :: AE : DF; that is, as the ſum of the ſides is to the difference of the ſides, ſo is the tangent of half the ſum of the oppoſite angles, to the tangent of half their difference.

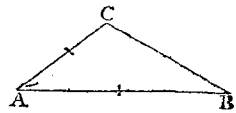
Note. The ſum of the unknown angles is found, by taking the given angle from 180°.

EXAM-

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards
AC 174·07 yards
∠A 37° 20′



Required the other parts.

1. *Geometrically.*

Draw AB = 345 from a ſcale of equal parts. Make the angle A = 37° 20′. Set off AC = 174 by the ſcale of equal parts. Join BC, and it is done.

Then the other parts being meaſured, they are found to be nearly as follow; viz. the ſide BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

As ſum of ſides AB, AC, - -	519·07	log. 2·7152259
To diff. of ſides AB, AC, - -	170·93	- 2·2328183
So tang. half ſum ∠s C and B -	71° 20′	- 10·4712979
To tang. half diff. ∠s C and B -	44 16	- 9·9888903

their ſum gives ∠ C	115 36
their diff. gives ∠ B	27 4

Then, by the former theorem,

As ſin. ∠ C 115° 36′ or 64° 24′ -	log. 9·9551259
To its op. ſide AB 345 - - -	2·5378191
So ſin. ∠ A 37° 20′ - - - -	9·7827958
To its op. ſide BC 232 - - -	2·3654890

3. *Inſtrumentally.*

In the firſt proportion.—Extend the compaſſes from 519 to 171, on the line of numbers; and that extent will reach, on the tangents, from 71°⅓ (the contrary way, becauſe the tangents are ſet back again from 45°) a little beyond 45, which being ſet ſo far back from 45, falls upon 44°¼, the fourth term.

In the ſecond proportion.—Extend from 64°⅓ to 37°⅓, on the ſines; and that extent will reach, on the numbers, from 345 to 232, the fourth term ſought.

EXAM-

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles, AC 154·33, ∠A 57° 12′ } Anſ. { BC 309·86, ∠B 24° 45, ∠C 98 3 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 yards, BC 112 yards, ∠C 57° 58′ 39″ } Anſ. { AB 112·65, ∠A 57° 27 0′, ∠B 64 34 21 }

Required the other parts.

THEOREM III.

When the Three Sides of the Triangle are given.

THEN, having let fall a perpendicular from the greateſt angle upon the oppoſite ſide, or baſe, dividing it into two ſegments, and the whole triangle into two right-angled triangles; it will be,

As the baſe, or ſum of the ſegments,
Is to the ſum of the other two ſides;
So is the difference of thoſe ſides,
To the diff. of the ſegments of the baſe.

Then half the difference of the ſegments being added to the half ſum, or the half baſe, gives the greater ſegment; and the ſame ſubtracted gives the leſs ſegment.

Hence, in each of the two right-angled triangles, there will be known two ſides, and the angle oppoſite to one of them; conſequently the other angles will be found by the firſt problem.

Demonſtr. By cor. to theor. 35, Geom. the rectangle under the ſum and difference of the two ſides, is equal to the rectangle under the ſum and difference of the two ſegments. Therefore, by forming the ſides of theſe rectangles into a proportion, it will appear that the ſums and differences are proportional as in this theorem, by theor. 76, Geometry.

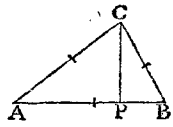
EXAM-

EXAMPLE I.

In the plane triangle ABC,

Given the sides { AB 345 yards, AC 232, BC 174·07

To find the angles



1. *Geometrically.*

Draw the base AB = 345 by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follow, viz.

∠ A 27°, ∠ B 37°$\frac{1}{3}$, and ∠ C 115°$\frac{1}{2}$.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,
As the base AB : AC + BC :: AC − BC : AP − BP,
that is, as 345 : 406·07 :: 57·93 : 68·18 = AP − BP,

its half is -	34·09
the half base is	172·50
the sum of these is	206·59 = AP
and their diff.	138·41 = BP

Then, in the triangle APC, right-angled at P,

As the side AC - -	232	- log.	2·365488
To sin. op. ∠ P - -	90°	- -	10·0000000
So is side AP - -	206·59	- -	2·3151093
To sin. op. ∠ ACP -	62° 56′	-	9·9496213
which taken from -	90 00		
leaves the ∠ A	27 04		

Again,

Again, in the triangle BPC, right-angled at P,

As the ſide BC	- -	174·07	-	log. 2·2407239
To ſin. op. ∠ P	- -	90°	- -	10·0000000
So is ſide BP	- -	138·41	- -	2·1411675
To ſin. op. ∠ BCP	-	52° 40′	- -	9·9004436
which taken from	-	90 00		
leaves the ∠ B		37 20		

Alſo, the -	∠ ACP	62° 56′
added to -	∠ BCP	52 40
gives the whole	∠ ACB	115 36

So that all the three angles are as follow, viz.

the ∠ A 27° 4′; the ∠ B 37° 20′; the ∠ C 115° 36′.

3. *Inſtrumentally.*

In the firſt proportion.—Extend the compaſſes from 345 to 406, on the line of numbers; then that extent will reach, on the ſame line, from 58 to 68·2 nearly, which is the difference of the ſegments of the baſe.

In the ſecond proportion.—Extend from 232 to 206½, on the line of numbers; then that extent will reach, on the lines, from 90° to 63°.

In the third proportion.—Extend from 174 to 138½; then that extent will reach from 90° to 52°⅔ on the lines.

EXAMPLE II.

In the plane triangle ABC,

Given the ſides	AB 365 poles AC 154·33 BC 309·86	Anſ.	∠ A 57° 12′ ∠ B 24 45 ∠ C 98 3

To find the angles.

In the plane triangle ABC,

Given the ſides	AB 120 AC 112·6 BC 112	Anſ.	∠ A 57° 27′ 00″ ∠ B 57 58 39 ∠ C 64 34 21

To find the angles.

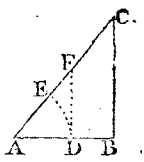
The three foregoing theorems include all the caſes of plane triangles, both right-angled and oblique; beſides which, there

there are other theorems ſuited to ſome particular forms of triangles, which are ſometimes more expeditious in their uſe than the general ones; one of which, as the caſe for which it ſerves ſo frequently occurs, may be here taken, as follows:

THEOREM IV.

When, in a Right-angled Triangle, there are given one Leg and the Angles; to find the other Leg or the Hypothenuſe; it will be,

As radius, i. e. ſin. of 90° or tang. of 45°
Is to the given leg,
So is tang. of its adjacent angle
To the other leg;
And ſo is ſecant of the ſame angle
To the hypothenuſe.



Demonſtr. AB being the given leg, in the right-angled triangle ABC; with the centre A, and any aſſumed radius AD, deſcribe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Now it is evident, from the definitions, that DF is the tangent, and AF the ſecant, of the arc DE, or of the angle A which is meaſured by that arc, to the radius AD. Then, becauſe of the parallels BC, DF, it will be, - - as AD : AB :: DF : BC :: AF : AC, which is the ſame as the theorem is in words.

EXAMPLE I.

In the right-angled triangle ABC,

Given { the leg AB 162 / ∟ A 53° 7′ 48″ } To find AC and BC.

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = 53° 7′ 48″; then raiſe the perpendicular BC, meeting AC in C. So ſhall AC meaſure 270, and BC 216.

2. *Arith-*

2. *Arithmetically.*

As radius -	tang. 45° - -	log. 10·0000000
To leg AB -	- 162 - - -	2·2095150
So tang. ∟ A	- 53° 7′ 48″ - -	10·1249371
To leg BC	- 216 - - -	2·3344521
So fecant ∟ A	- 53° 7′ 48″ - -	10·2218477
To hyp. AC	- 270 - - -	2·4313627

3. *Inſtrumentally.*

Extend the compaſſes from 45° to 53°$\frac{1}{8}$, on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

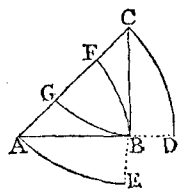
EXAMPLE II.

In the right angled triangle ABC,

Given { the leg AB 180 / the ∟A 62° 40′ } Anſ. { AC 392·0469 / BC 348·2464 }

To find the other two ſides.

Note. There is ſometimes given another method for right-angled triangles, which is this:



ABC being ſuch a triangle, make one leg AB radius, that is, with centre A, and diſtance AB, deſcribe an arc BF. Then it is evident that the other leg BC repreſents the tangent, and the hypothenuſe AC the ſecant, of the arc BF, or of the angle A.

In like manner, if the leg BC be made radius; then the other leg AB will repreſent the tangent, and the hypothenuſe AC the ſecant, of the arc BG or angle C.

But if the hypothenuſe be made radius; then each leg will repreſent the ſine of its oppoſite angle; namely, the leg AB the ſine of the arc AE or angle C, and the leg BC the ſine of the arc CD or angle A.

And then the general rule for all theſe caſes, is this, namely, that the ſides of the triangle bear to each other, the ſame proportion as the parts which they repreſent.

And this is called, Making every ſide radius.

OF

Of HEIGHTS and DISTANCES, &c.

BY the menſuration and protraction of lines and angles, are determined the lengths, heights, depths, and diſtances of bodies or objects.

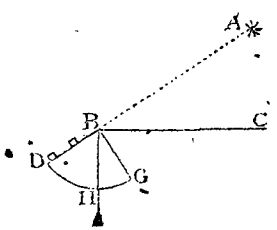
Acceſſible lines are meaſured by applying to them ſome certain meaſure a number of times, as an inch, or foot, or yard. But inacceſſible lines muſt be meaſured by taking angles, or by ſuch like method, drawn from the principles of geometry.

When inſtruments are uſed for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of ſimilar triangles, without regard to the meaſure of the angles.

Angles of elevation, or of depreſſion, are uſually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furniſhed with a plummet ſuſpended from the centre, and two ſides fixed on one of the radii, or elſe with teleſcopic ſights.

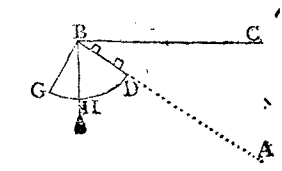
To take an Angle of Altitude and Depreſſion with the Quadrant.

Let A be any object, as the ſun, moon, or a ſtar, or the top of a tower, or hill, or other eminence: and let it be required to find the meaſure of the angle ABC, which a line drawn from the object makes with the horizontal line BC.



Fix the centre of the quadrant in the angular point, and move it round there as a centre till, with one eye at D, the other being ſhut, you perceive the object A through the ſights: then will the arc GH of the quadrant, cut off by the plumb line BH, be the meaſure of the angle ABC as required.

The angle ABC of depreſſion of any object A, is taken in the ſame manner; except that here the eye is applied to the centre, and the meaſure of the angle is the arc GH, on the other ſide of the plumb line.



The following examples are to be conſtructed and calculated by the foregoing methods, treated of in Trigonometry.

EXAMPLE I.

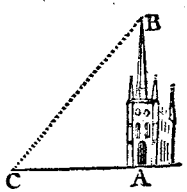
Having meaſured a diſtance of 200 feet, in a direct horizontal line, from the bottom of a ſteeple, the angle of elevation of its top, taken at that diſtance, was found to be 47° 30′: from hence it is required to find the height of the ſteeple.

Conſtruction.

Draw an indefinite line, upon which ſet off AC = 200 equal parts, for the meaſured diſtance. Erect the indefinite perpendicular AB; and draw CB ſo as to make the angle C = 47° 30′, the angle of elevation; and it is done. Then AB, meaſured on the ſcale of equal parts, is nearly 218¼.

Calculation.

As radius - -	10·0000000
To AC 200 - -	2·3010300
So tang. ∠C 47° 30′	10·0379475
To AB 218·26 required	2·3389775



EXAMPLE II.

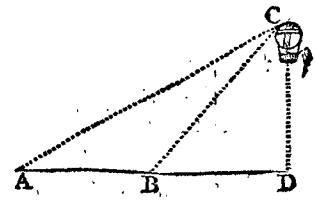
What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two obſervers, at the ſame time, both on the ſame ſide of it, and in the ſame vertical plane; their diſtance as under being half a mile or 880 yards. And what was its diſtance from the ſaid two obſervers?

Conſtruction.

Draw an indefinite ground line, upon which ſet off the given diſtance AB = 880; then A and B are the places of the obſervers. Make the angle A = 35°, and the angle B = 64°; and the interſection of the lines at C will be the place of the balloon: from whence the perpendicular CD, being let fall, will be its perpendicular height. Then, by meaſurement, are found the diſtances and height nearly as follows, viz. AC 1631, BC 1041, DC 936.

Cal-

Calculation.



First, from ∠ B	64°	
take ∠ A	35	
leaves ∠ ACB	29	

Then, in the triangle ABC,

As sin. ∠ ACB	29°	9·6855712
To op. side ∠B	880	2·9444827
So sin ∠ A	35°	9·7585913
To op. side BC	1041·125	3·0175028

As sin. ∠ ACB	29°	9·6855712
To op. side AB	880	2·9444827
So sin. ∠ B	116° or 64°	9·9536602
To op. side AC	1631·442	3·2125717

And, in the triangle BCD,

As sin. ∠ D	90°	10·0000000
To op. side BC	1041·125	3·0175028
So sin. ∠ B	64°	9·9536602
To op. side CD	935·757	2·9711630

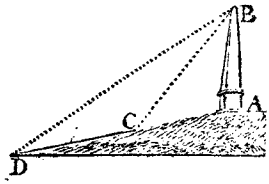
EXAMPLE III.

Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obelisk, 41°; but, after measuring on in the same direction 60 feet further, the like angle was only 23° 45′. What then was the height of the obelisk?

Construction.

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from whence set off the distance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B where the two lines meet will be the top of the obelisk. Therefore AB, joined, will be its height.

Calculation.



From the ∠C	41° 00′	
take the ∠D	23 45	
leaves the ∠DBC	17 15	

Then, in the triangle DBC,

As sin. ∠DBC	17° 15′	9·4720856
To op. side DC	69	1·7781513
So sin. ∠D	23 45	9·6050320
To op. side CB	81·488	1·9110977

And, in the triangle ABC,

As sum of sides CB, CA	121·488	2·0845833
To diff. of sides CB, CA	41·488	1·6179225
So tang. half sum ∠s A, B	69° 30′	10·4272623
To tang. half diff. ∠s A, B	42 24½	9·9606516

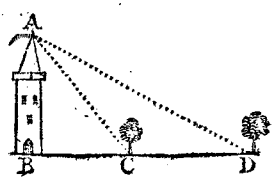
the diff. of these is ∠CBA 27 5½

Lastly, as sin. ∠CBA	27° 5′½	9·6582842
To op. side CA	40	1·6020600
So sin. ∠C	41° 0′	9·8169429
To op. side AB	57·623	1·7607187

EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower, 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be 33° and 64°½. What then may be the distance between the two objects?

Construction.



Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD making the two angles BAC, BAD equal to the given

given angles 33° and 64°½. So shall C and D be the places of the two objects.

Calculation.

First, In the right-angled triangle ABC,

As radius	- - - - -		10·0000000
To AB	- 120	- - -	2·0791812
So tang. ∠BAC	33°	- - -	9·8125174
To BC	- 77·929	- - -	1·8916986

And, in the right-angled triangle ABD,

As radius	- - - - -		10·0000000
To AB	- - 120	- -	2·0791812
So tang. ∠BAD	- 64°½	- -	10·3215039
To BD	- - 251·585	- -	2·4006851
from which take BC	77·929		
leaves the dist. CD	173·656 as required.		

EXAMPLE V.

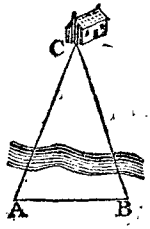
Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then at each end of this line of distance took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What then were the distances from each end to the house?

Construction.

Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So shall the point C be the place of the house required.

Calculation.

To the given ∠A	68°	2′
add the given ∠B	73	15
then their sum	141	17
being taken from	180	0
leaves the third ∠C	38	43



Hence,	As ſin. ∠ C	38° 43′	-	9·7962062
	To op. ſide AB	200	-	2·3010300
	So ſin. ∠ A	68° 2′	-	9·967 679
	To op. ſide BC	296·54	-	2·4720917
And,	As ſin. ∠ C	38° 43′	-	9·7962062
	To op. ſide AB	200	-	2·3010300
	So ſin. ∠ B	73° 15′	-	9·9811711
	To op. ſide AC	306·19	-	2·4859949

EXAM. VI. From the edge of a ditch, of 36 feet wide, ſurrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my ſtation to the top of it?

Anſ. { height of wall 69·64,
ladder 78·4 feet.

EXAM. VII. Required the length of a ſhoar, which, being to ſtrut 11 feet from the upright of a building, will ſupport a jamb 23 feet 10 inches from the ground?

Anſ. 26 feet 3 inches.

EXAM. VIII. A ladder, 40 feet long, can be ſo planted, that it ſhall reach a window 33 feet from the ground, on one ſide of the ſtreet; and by turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high, on the other ſide: required the breadth of the ſtreet? Anſ. 56·649 feet.

EXAM. IX. A Maypole, whoſe top was broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the foot of the pole: what was the height of the whole maypole, ſuppoſing the broken piece to meaſure 39 feet in length?

Anſ. 75 feet.

EXAM. X. At 170 feet diſtance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower. Anſ. 221 feet.

EXAM. XI. From the top of a tower, by the ſea-ſide, of 143 feet height, it was obſerved that the angle of depreſſion of a ſhip's bottom, then at anchor, meaſured 35°; what then was the ſhip's diſtance from the bottom of the wall?

Anſ. 204·22 feet.

EXAM. XII. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and

and 200 yards farther off, on a level with the bottom of it, the angle was 31° ? Anſ. 286·28 yards.

EXAM. XIII. Wanting to know the height of an inacceſſible tower; at the leaſt diſtance from it, on the ſame horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my diſtance from it at the firſt ſtation?

Anſ. { height 307·53
diſtance 182·15

EXAM. XIV. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inacceſſible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the tower equal 51°; then meaſuring in a line directly from it to the diſtance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′. What then is the height of the tower?

Anſ. 93·33148 feet.

EXAM. XV. From a window near the bottom of a houſe, which ſeemed to be on a level with the bottom of a ſteeple, I took the angle of elevation of the top of the ſteeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: what then is the height and diſtance of the ſteeple?

Anſ. { height 210·44
diſtance 250·79

EXAM. XVI. Wanting to know the height of, and my diſtance from, an object on the other ſide of a river, which ſeemed to be on a level with the place where I ſtood, cloſe by the ſide of the river; and not having room to meaſure backwards, on the ſame plane, becauſe of the immediate riſe of the bank, I placed a mark where I ſtood, and meaſured in a direction from the object, up the aſcending ground to the diſtance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depreſſion were found to be, viz. of the mark left at the river's ſide 42°, of the bottom of the object 27°, and of its top 19°. Required then the height of the object, and the diſtance of the mark from its bottom?

Anſ. { height 57·26
diſtance 151·50

EXAM. XVII. If the height of the mountain called the Pike of Teneriff be 4 miles, and the angle taken at the top of it, as formed between a plumb line and a line conceived

to touch the earth in the horizon, or farthest visible point, be 87° 25′ 55″; it is required from hence to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly round?

Anſ. { dist. 178·458 miles, diam. 7957·818

EXAM. XVIII. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angles which the other ship and the fort subtends, which angles were 83° 45′ and 85° 15′. What then was the distance between each ship and the fort?

Anſ. { 2292·26 yards, 2298·05

EXAM. XIX. Being on the side of a river, and wanting to know the distance to a house which was seen at a distance on the other side; I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house, were 68° 2′ and 73° 15′. What then was the distance between each station and the house?

Anſ. { 593·08 yards, 612·38

EXAM. XX. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close on the bank on the other side of the river, to be 53° and 79° 12′. What then was the perpendicular breadth of the river?

Anſ. 529·48 yards.

EXAM. XXI. Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also, the horizontal angle subtended between these two lines was 55° 40′. What then was the distance required? Anſ. 741·2 yards.

EXAM. XXII. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required

to

to determine the place of that headland, and the ſhip's diſtance from it at the laſt obſervation. Anſ. 26·0728 miles.

EXAM. XXIII. Wanting to know the diſtance between a houſe and a mill, which were ſeen at a diſtance on the other ſide of a river, I meaſured a baſe line along the ſide where I was of 600 yards, and at each end of it took the angles ſubtended by the other end and the houſe and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the diſtance between the houſe and mill?

Anſ. 959·5866 yards.

EXAM. XXIV. Wanting to know my diſtance from an inacceſſible object O, on the other ſide of a river; and having no inſtrument for taking angles, but only a chain or cord for meaſuring diſtances; from each of two ſtations, A and B, which were taken at 500 yards aſunder, I meaſured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; alſo the diagonal AD meaſured 550 yards, and the diagonal BC 560. What then was the diſtance of the object O from each ſtation A and B?

Anſ. { AO 536·25
{ BO 500·09

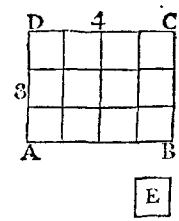
MENSURATION OF PLANES.

THE Area of any plane figure, is the meaſure of the ſpace contained within its extremes or bounds; without any regard to thickneſs.

This area, or the content of the plane figure, is eſtimated by the number of little ſquares that may be contained in it; the ſide of thoſe little meaſuring ſquares being an inch, a foot, a yard, or any other fixed quantity. And hence, the area or content is ſaid to be ſo many ſquare inches, or ſquare feet, or ſquare yards, &c.

Thus,

Thus, if the figure to be meaſured be the rectangle ABCD, and the little ſquare E, whoſe ſide is one inch, be the meaſuring unit propoſed: then, as often as the ſaid little ſquare is contained in the rectangle, ſo many ſquare inches the rectangle is ſaid to contain, which in the preſent caſe is 12.



PROBLEM I.

To find the Area of any Parallelogram, whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area *.

EXAMPLES.

Ex. 1. To find the area of a parallelogram, whoſe length is 12·25, and height 8·5.

```
  12·25 length
    8·5 breadth
  ------
   6125
  9800
  ------
 104·125 area
  ------
```

* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The ſame is otherwiſe proved thus: Let the foregoing rectangle be the figure propoſed; and let the length and breadth be divided into equal parts, each equal to the lineal meaſuring unit, being here 4 for the length, and 3 for the breadth; and let the oppoſite points of diviſion be connected by right lines.—Then, it is evident that theſe lines divide the rectangle into a number of little ſquares, each equal to the ſquare meaſuring unit E; and farther, that the number of theſe little ſquares, or the area of the figure, is equal to the number of lineal meaſuring units in the length, repeated as often as there are lineal meaſuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geom. theor. 25, cor. 2), that a rectangle is equal to any oblique parallelogram, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

Ex. 2.

Ex. 2. To find the area of a ſquare, whoſe ſide is 35·25 chains. Anſ. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whoſe length is $12\frac{1}{2}$ feet, and breadth 9 inches. Anſ. $9\frac{3}{8}$ feet.

Ex. 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular height 5·45. Anſ. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of ſquare yards of painting in a rhomboid, whoſe length is 37 feet, and breadth 5 feet 3 inches. Anſ. $21\frac{7}{12}$ ſquare yards.

PROBLEM II.

To find the Area of a Triangle.

RULE I. MULTIPLY the baſe by the perpendicular height, and half the product will be the area *. Or, multiply the one of theſe dimenſions by half the other.

EXAMPLES.

Ex. 1. To find the area of a triangle, whoſe baſe is 625, and perpendicular height 520 links?

Here $625 \times 260 = 162500$ ſquare links,
or equal 1 acre, 2 roods, 20 perches, the anſwer.

Ex. 2. How many ſquare yards contains the triangle, whoſe baſe is 40, and perpendicular 30 feet?
Anſ. $66\frac{2}{3}$ ſquare yards.

Ex. 3. To find the number of ſquare yards in a triangle, whoſe baſe is 49 feet, and height $25\frac{1}{4}$ feet?
Anſ. $68\frac{5}{72}$, or 68·7361.

Ex. 4. To find the area of a triangle, whoſe baſe is 18 feet 4 inches, and height 11 feet 10 inches?
Anſ. 108 feet, $5\frac{2}{3}$ inches.

---

* The truth of this rule is evident, becauſe any triangle is the half of a parallelogram of equal baſe and altitude, by Geom. theor. 26.

RULE II.

RULE II. When two ſides and their contained angle are given: Multiply the two given ſides together, and take half their product: Then ſay, as radius is to the ſine of the given angle, ſo is that half product, to the area of the triangle.

Or, multiply that half product by the ſine of the ſaid angle *.

Ex. 1. What is the area of a triangle, whoſe two ſides are 30 and 40, and their contained angle 28° 57′ 18″?

Here $\frac{1}{2} \times 40 \times 30 = 600$,
therefore 1 : ·4841226 nat. ſin. 28° 57′ 18″
600

290·47356 the anſwer.

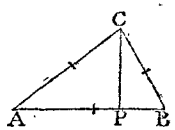
Ex. 2. How many ſquare yards contains the triangle, of which one angle is 45°, and its containing ſides 25 and $21\frac{1}{4}$ feet? Anſ. 20·86947.

RULE III. When the three ſides are given: Add all the three ſides together, and take half that ſum. Next, ſubtract each ſide ſeverally from the ſaid half ſum, obtaining three remainders. Laſtly, multiply the ſaid half ſum and thoſe three remainders all together, and extract the ſquare root of the laſt product, for the area of the triangle †.

Ex. 1.

---

* For, let AB, AC be the two given ſides, including the given angle A. Now $\frac{1}{2}$AB × CP is the area, by the firſt rule, CP being perpendicular. But, by trigon. as ſin. ∠P, or radius, is to ſin. ∠A :: AC : CP = ſin. ∠A × AC, taking radius = 1. Therefore the area $\frac{1}{2}$AB × CP is = $\frac{1}{2}$AB × AC × ſin. ∠A, to radius 1; or as radius : ſin. ∠A :: $\frac{1}{2}$AB × AC : the area.



† For, let *b* denote the baſe AB of the triangle ABC (ſee the laſt fig.), alſo *a* the ſide AC, and *c* the ſide BC. Then, by th. 3, Trigon. as $b : a + c :: a - c : \frac{aa - cc}{b}$ = AP − PC the diff. of the ſegments;

therefore

Ex. 1, To find the area of the triangle whoſe three ſides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
—	25 1ſt rem.	15 2d rem.	5 3d rem.
2)90	—	—	—
45 half ſum			
—			

Then $45 \times 25 \times 15 \times 5 = 84375$.
The root of which is 290·4737, the area.

Ex. 2. How many ſquare yards of plaſtering are in a triangle, whoſe ſides are 30, 40, 50? Anſ. $66\frac{2}{3}$.

Ex. 3. How many acres, &c. contains the triangle, whoſe ſides are 2569, 4900, 5025 links?
Anſ. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the Area of a Trapezoid.

ADD together the two parallel ſides; then multiply their ſum by the perpendicular breadth or diſtance between them; and half the product will be the area; by Geom. theor. 29.

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theref. $\frac{1}{2}b + \frac{aa - cc}{2b} = \frac{bb + aa - cc}{2b}$ = the ſegment AP;
hence $\sqrt{AC^2 - AP^2}$ = the perp. CP, that is,

$$\sqrt{(aa - (\frac{bb + aa - cc}{2b})^2)} = \text{ - - - - - - - - - - - -}$$

$$\sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4bb}} = \text{CP.}$$

But $\frac{1}{2}$AB × CP is the area, that is,

$$\frac{1}{2}b \times \text{CP} = \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{16}}$$

$$= \sqrt{(\frac{aa - bb - cc + 2bc}{4} \times \frac{-aa + bb + cc + 2bc}{4})}$$

$$= \sqrt{(\frac{a + b + c}{2} \times \frac{-a + b + c}{2} \times \frac{a - b + c}{2} \times \frac{a + b - c}{2})}$$

$= \sqrt{(s \times \overline{s - a} \times \overline{s - b} \times \overline{s - c})}$, which is the rule, where s denotes half the ſum of the three ſides.

Ex. 1.

Ex. 1. In a trapezoid, the parallel ſides are 750 and 1225, and the perpendicular diſtance between them 1540 links: to find the area.

1225
750

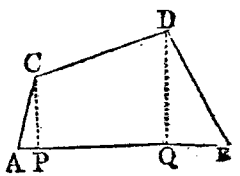
1975 × 770 = 152075 ſquare links = 15 acr. 33 perc.

Ex. 2. How many ſquare feet are contained in the plank, whoſe length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the leſs end 11 inches?

Anſ. $13\frac{13}{24}$ feet.

Ex. 3. In meaſuring along one ſide AB of a quadrangular field, that ſide and the two perpendiculars let fall on it from the two oppoſite corners, meaſured as below: required the content.

AP = 110 links.
AQ = 745
AB = 1110
CP = 352
DQ = 595



Anſ. 4 acres, 3 roods, 17·92 perches.

PROBLEM IV.

To find the Area of any Trapezium.

DIVIDE the trapezium into two triangles by a diagonal; then find the areas of theſe triangles, and add them together.

Note. If two perpendiculars be let fall on the diagonal, from the other two oppoſite angles, the ſum of theſe perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.

Ex. 1. To find the area of the trapezium, whoſe diagonal is 42, and the two perpendiculars on it 16 and 18.

Here 16 + 18 = 34, its half is 17.
Then 42 × 17 = 714 the area.

Ex. 2. How many ſquare yards of paving are in the trapezium, whoſe diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Anſ. $222\frac{1}{12}$ yards.

Ex. 3.

Ex. 3. In the quadrangular field ABCD, on account of obſtructions there could only be taken the following meaſures, viz. the two ſides BC 265 and AD 220 yards, the diagonal AC 378, and the two diſtances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the area in acres, when 4840 ſquare yards make an acre? Anſ. 17 acres, 2 roods, 21 perches.

PROBLEM V.

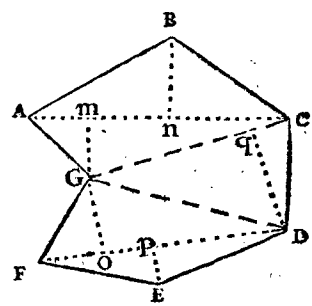
To find the Area of an Irregular Polygon.

DRAW diagonals dividing the propoſed polygon into trapeziums and triangles. Then find the areas of all theſe ſeparately, and add them together for the content of the whole polygon.

EXAM. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars: namely,

AC 55
FD 52
GC 44
Gm 13
Bn 18
Go 12
Ep 8
Dq 23

Anſ. 18785.



PROBLEM VI.

To find the Area of a Regular Polygon.

RULE 1. MULTIPLY the perimeter of the polygon, or ſum of its ſides, by the perpendicular drawn from its centre on one of its ſides, and take half the product for the area *.

* This is only in effect reſolving the polygon into as many equal triangles as it has ſides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

Ex. 1.

Ex. 1. To find the area of the regular pentagon, each ſide being 25 feet, and the perpendicular from the centre on each ſide is 17·2047737.

Here $25 \times 5 = 125$ is the perimeter.
And $17{\cdot}2047737 \times 125 = 2150{\cdot}59675$.
Its half 1075·298375 is the area ſought.

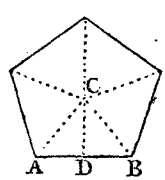
RULE II. Square the ſide of the polygon; then multiply that ſquare by the area or multiplier ſet againſt its name in the following table, and the product will be the area *.

No. of Sides.	Names.	Areas, or Multipliers.
3	Trigon, or triangle	0·43301 27
4	Tetragon, or ſquare	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

EXAM.

* This rule is founded on the property, that like polygons, being ſimilar figures, are to one another as the ſquares of their like ſides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the reſpective polygons to the ſide 1. Whence the rule is manifeſt.

Note. The areas in the table, to each ſide 1, may be computed in the following manner: From the centre C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has ſides; and let ABC be one of thoſe triangles, the perpendicular of which is CD. Divide 360 degrees by the number of ſides in the polygon, the quotient gives the angle at the centre ACB. The half



of

EXAM. Taking here the ſame example as before, namely a pentagon, whoſe ſide is 25 feet.

Then 25^2 being $= 625$,
And the tabular area 1·7204774;
Theref. $1{\cdot}7204774 \times 625 = 1075{\cdot}298\,75$, as before.

Ex. 2. To find the area of the trigon, or equilateral triangle, whoſe ſide is 20. Anſ. 173·20508.

Ex. 3. To find the area of the hexagon, whoſe ſide is 20. Anſ. 1039·23048.

Ex. 4. To find the area of an octagon, whoſe ſide is 20. Anſ. 1931·37084.

Ex. 5. To find the area of a decagon, whoſe ſide is 20. Anſ. 3077·68352.

PROBLEM VII.

To find the Diameter and Circumference of any Circle, the one from the other.

THIS may be done nearly by either of the two following proportions, viz.

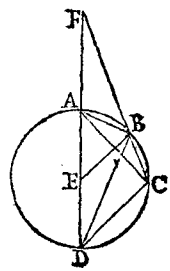
As 7 is to 22, ſo is the diameter to the circumference.
Or, As 1 is to 3·1416, ſo is the diameter to the circumference *.

Ex. 1.

---

of this gives the angle ACD; and this taken from 90°, leaves the angle CAD. Then, as radius is to AD, ſo is tang. angle CAD to the perpendicular CD. This multiplied by AD, gives the area of the triangle ABC; which being multiplied by the number of the triangles, or of the ſides of the polygon, gives its whole area, as in the table.

* For, let ABCD be any circle, whoſe centre is E, and let AB, BC be any two equal arcs. Draw the ſeveral chords as in the figure, and join BE; alſo draw the diameter DA, which produce to F, till BF be equal to the chord BD.



Then the two iſoſceles triangles DEB, DBF are equiangular, becauſe they have the angle at D common; conſequently DE : DB :: DB : DF. But the two triangles AFB, DCB are identical, or equal in all reſpects, becauſe they have the angle F = the angle BDC, being each equal the angle ADB, theſe being ſubtended by the equal

Ex. 1. To find the circumference of the circle whoſe diameter is 20.

By the firſt rule, as $7 : 22 :: 20 : 62\frac{6}{7}$, the anſwer.

Ex. 2.

---

arcs AB, BC; alſo the exterior angle FAB of the quadrangle ABCD, is equal the oppoſite interior angle at C; and the two triangles have alſo the ſide BF = the ſide BD; therefore the ſide AF is alſo equal the ſide DC. Hence the proportion above, viz. DE : DB :: DB : DF = DA + AF, becomes DE : DB :: DB : 2DE + DC. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE.DC$.

Now, if the radius DE be taken = 1, this expreſſion becomes $DB^2 = 2 + DC$, and hence $DB = \sqrt{2 + DC}$. That is, If the meaſure of the ſupplemental chord of any arc be increaſed by the number 2, the ſquare root of the ſum will be the ſupplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be ſucceſſively biſected by the above theorem: thus, the chord AC, of $\frac{1}{6}$ of the circumference, is the ſide of the inſcribed regular hexagon, and is therefore equal the radius AE or 1; hence, in the right-angle triangle ACD, it will be $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1.732050807\overline{6}$, the ſupplemental chord of $\frac{1}{6}$ of the periphery.

Then, by the foregoing theorem, by always biſecting the arcs, and adding 2 to the laſt ſquare root, there will be found the ſupplemental chords of the 12th, the 24th, the 48th, the 96th, &c. parts of the periphery; thus,

$\sqrt{3.732050807\overline{6}}$	$= 1.9318516525$	for the ſupplemental chord of	$\frac{1}{12}$
$\sqrt{3.9318516525}$	$= 1.9828897227$		$\frac{1}{24}$
$\sqrt{3.9828897227}$	$= 1.9957178465$		$\frac{1}{48}$
$\sqrt{3.9957178465}$	$= 1.9989291743$		$\frac{1}{96}$
$\sqrt{3.9989291743}$	$= 1.9997322757$		$\frac{1}{192}$
$\sqrt{3.9997322757}$	$= 1.9999330678$		$\frac{1}{384}$
$\sqrt{3.9999330678}$	$= 1.9999832669$		$\frac{1}{768}$
$\sqrt{3.9999832669}$	$= \ldots$		$\frac{1}{1536}$ of the periphery.

Since then it is found that 3.9999832669 is the ſquare of the ſupplemental chord of the 1536th part of the periphery, let this number be taken from 4, the ſquare of the diameter, and the remainder 0.0000167331 will be the ſquare of the chord of the ſaid 1536th part of the periphery, and conſequently the root $\sqrt{0.0000167331} = 0.0040906112$ is the length of that chord; this number then being multiplied by 1536, gives 6.2831788 for the perimeter of a regular polygon of 1536 ſides inſcribed in the circle; which, as the

Ex. 2. If the circumference of the earth be 25000 miles, what is its diameter?

By the 2d rule, as 3·1416 : 1 :: 25000 : 7957$\frac{3}{4}$ nearly the diameter.

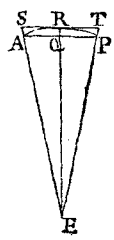
PROBLEM VIII.

To find the Length of any Arc of a Circle.

MULTIPLY the degrees in the given arc by the radius of the circle, and the product again by the decimal ·01745, for the length of the arc *.

Ex. 1.

ſides of the polygon nearly coincide with the circumference of the circle, muſt alſo expreſs the length of the circumference itſelf, very nearly.



But now, to ſhew how near this determination is to the truth, let AQP = 0·0040906112 repreſent one ſide of ſuch a regular polygon of 1536 ſides, and SRT a ſide of another ſimilar polygon deſcribed about the circle; and from the centre E let the perpendicular EQR be drawn, biſecting AP and ST in Q and R. Then, ſince AQ is = $\frac{1}{2}$AP = 0.0020453056, and EA = 1, therefore $EQ^2 = EA^2 - AQ^2$ = ·9999958167, and conſequently its root gives EQ = ·9999979084; then, becauſe of the parallels AP, ST, it is EQ : ER :: AP : ST :: as the whole inſcribed perimeter : to the circumſcribed one, that is, as ·9999979084 : 1 :: 6·2831788 : 6·2831920 the perimeter of the circumſcribed polygon. But the circumference of the circle being greater than the perimeter of the inner polygon, and leſs than that of the outer, it muſt conſequently be greater than 6·2831788, but leſs than 6·2831920, and muſt therefore be nearly equal $\frac{1}{2}$ their ſum, or 6·2831854, which in fact is true to the laſt figure, which ſhould be a 3 inſtead of the 4.

Hence, the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·14·5927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416 is very near. Alſo, the other ratio in the rule 7 to 22 or 1 to 3$\frac{1}{7}$ = 3·1428 &c. is another near approximation.

* It having been found, in the demonſtration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which conſiſts of 360 degrees;

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Anſ. 4·7115.

Ex. 2. To find the length of an arc of 12° 10′, or $12°\frac{1}{6}$, the radius being 10 feet. Anſ. 2·1231.

PROBLEM IX.

To find the Area of a Circle.

* RULE I. MULTIPLY half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

RULE II. Square the diameter, and multiply that ſquare by the decimal ·7854, for the area.

Ex. 1. To find the area of a circle whoſe diameter is 10, and its circumference 31·416.

By Rule 1.		By Rule 2.
31·416		·7854
10		$100 = 10^2$
4)314·16		
78·54	the area	78·54

---

therefore as 360° : 6·28318 54 :: 1° : ·01745 &c. the length of the arc of 1 degree. Hence, the number ·01745 multiplied by any number of degrees, will give the length of the arc of thoſe degrees. And, becauſe the circumferences, and arcs, are as the diameters, or radii of the circles, therefore as the radius 1 is to any other radius r, ſo is the length of the arc above mentioned to $r \times$ ·01745 $\times$ degrees in the arc, which is the length of that arc, as in the rule.

* This firſt rule is proved in the Geom. theor. 94.

And the 2d rule is deduced from the firſt in this manner: It appears, by the demonſtr. of prob. 7, that when the diameter of a circle is 1, its circumference is 3·1415927, or nearly 3·1416; then, by the firſt rule, $1 \times 3{\cdot}1416 \div 4 =$ ·7854, which is therefore the area of the circle whoſe diameter is 1. But the areas of different circles are to each other as the ſquare of their diameters, by Geom. theor. 93; therefore as $1^2 : d^2 ::$ ·7854 : ·7854d^2, the area of the circle whoſe diameter is d, as in the 2d rule.

Ex. 2.

Ex. 2. To find the area of a circle, whoſe diameter is 7, and circumference 22. Anſ. $38\frac{1}{2}$.

Ex. 3. How many ſquare yards are in a circle, whoſe diameter is $3\frac{1}{2}$ feet? Anſ. 1·069.

PROBLEM X.

To find the Area of a Circular Ring, or Space included between two Concentric Circles.

TAKE the difference between the areas of the two circles, as found by the laſt problem.—Or, which is the ſame thing, ſubtract the ſquare of the leſs diameter from the ſquare of the greater, and multiply their difference by ·785. Or laſtly, multiply the ſum of the diameters by the difference of the ſame, and that product by ·7854; which is ſtill the ſame thing, becauſe the product of the ſum and difference of any two quantities, is equal to the difference of their ſquares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the ſum, and $10 - 6 = 4$ the diff.
Therefore $·7854 \times 16 \times 4 = ·7854 \times 64 = 50·2656$, the area.

Ex. 2. What is the area of the ring, the diameters of whoſe bounding circles are 10 and 20? Anſ. 235·62.

PROBLEM XI.

To find the Area of the Sector of a Circle.

RULE I. MULTIPLY the radius, or half the diameter, by half the arc of the ſector, for the area. Or, multiply the whole diameter by the whole arc of the ſector, and take $\frac{1}{4}$ of the product. The reaſon of which is the ſame as for the firſt rule to problem 9.

RULE II. As 360 is to the degrees in the arc of the ſector, ſo is the area of the whole circle, to the area of the ſector.

This is evident, becauſe the ſector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular ſector, whoſe arc contains 18 degrees; the diameter being 3 feet?

1. By the 1ſt Rule.

Firſt, $3{\cdot}1416 \times 3 = 9{\cdot}4248$, the circumference.
And $360 : 18 :: 9{\cdot}4248 : {\cdot}47124$, the length of the arc.
Then ${\cdot}47124 \times 3 \div 4 = {\cdot}11781 \times 3 = {\cdot}35343$, the area.

2. By the 2d Rule.

Firſt, ${\cdot}7854 \times 3^2 = 7{\cdot}0686$, the area of the whole circle.
Then, as $360 : 18 :: 7{\cdot}0686 : {\cdot}35343$, the arc of the ſector.

Ex. 2. To find the area of a ſector, whoſe radius is 10, and arc 20. Anſ. 100.

Ex. 3. Required the area of a ſector, whoſe radius is 25, and its arc containing 147° 29′. Anſ. 804·4917.

PROBLEM XII.

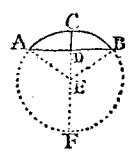
To find the Area of a Segment of a Circle.

Rule 1. Find the area of the ſector having the ſame arc with the ſegment, by the laſt problem.

Find alſo the area of the triangle, formed by the chord of the ſegment and the two radii of the ſector.

Then take the ſum of theſe two for the anſwer, when the ſegment is greater than a ſemicircle: or take their difference for the anſwer, when it is leſs than a ſemicircle.—As is evident by inſpection.

Ex. 1. To find the area of the ſegment ACBDA, its chord AB being 12, and the radius AE or CE 10.



Firſt, As AE : AD :: ſin. ∠D90° : ſin. 36° $52\frac{1}{5}' = 36{\cdot}87$ degrees, the degrees in the ∠AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now ${\cdot}7854 \times 400 = 314{\cdot}16$, the area of the whole circle,

Therefore $360° : 73{\cdot}74 :: 314{\cdot}16 : 64{\cdot}3504$, area of the ſector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 = DE$.

Theref. $AD \times DE = 6 \times 8 = 48$, the area of the triangle AEB.

Hence, ſector ACBA — triangle AEB = 16·3504, area of ſeg. ACBDA.

Rule 11.

RULE 11. Divide the height of the ſegment by the diameter, and find the quotient in the column of heights in the following tablet:—Take out the correſponding area in the next column on the right hand; and multiply it by the ſquare of the circle's diameter, for the area of the ſegment *.

Note. When the quotient is not found exactly in the table, proportion may be made between the next leſs and greater area, in the ſame manner as is done for logarithms, or any other table.

Table of the Areas of Circular Segments.

Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.
·01	·00133	·11	·04701	·21	·11990	·31	·20738	·41	·30319
·02	·00375	·12	·05339	·22	·12811	·32	·21667	·42	·31304
·03	·00687	·13	·06000	·23	·13646	·33	·22603	·43	·32293
·04	·01054	·14	·06683	·24	·14494	·34	·23547	·44	·33284
·05	·01468	·15	·07387	·25	·15354	·35	·24498	·45	·34278
·06	·01924	·16	·08111	·26	·16226	·36	·25455	·46	·35274
·07	·02417	·17	·08853	·27	·17109	·37	·26418	·47	·36272
·08	·02944	·18	·09613	·28	·18002	·38	·27386	·48	·37270
·09	·03502	·19	·10390	·29	·18905	·39	·28359	·49	·38270
·10	·04088	·20	·11182	·30	·19817	·40	·29337	·50	·39270

Ex. 2. Taking the ſame example as before, in which are given the chord AB 12, and the radius 10, or diameter 20.

And having found, as above, DE = 8; then CE − DE = CD = 10 − 8 = 2. Hence, by the rule, CD ÷ CF = 2 ÷ 20 = ·1 the tabular height. This being found in the firſt column of the table, the correſponding tabular area is ·04088. Then ·04088 × 20^2 = ·04088 × 400 = 16·352, the area, nearly the ſame as before.

* The truth of this rule depends on the principle of ſimilar plane figures, which are to one another as the ſquare of their like linear dimenſions. The ſegments in the table are thoſe of a circle whoſe diameter is 1; and the firſt column contains the correſponding heights or verſed ſines divided by the diameter. Thus then, the area of the ſimilar ſegment, taken from the table, and multiplied by the ſquare of the diameter, gives the area of the ſegment to this diameter.

Ex. 3. What is the area of the ſegment, whoſe height is 18, and diameter of the circle 50? Anſ. 636·375.

Ex. 4. Required the area of the ſegment whoſe chord is 16, the diameter being 20? Anſ. 44·7292.

PROBLEM XIII.

To meaſure long Irregular Figures.

* TAKE or meaſure the breadth in ſeveral places, at equal diſtances. Then add all theſe breadths together, and divide the ſum by the number of them, for the mean breadth; which multiply by the length, for the area.

Note 1. Take half the ſum of the extreme breadths for one of the ſaid breadths.

Note 2. If the perpendiculars or breadths be not at equal diſtances, compute all the parts ſeparately, as ſo many trapezoids, and add them all together for the whole area.

Or elſe, add all the perpendicular breadths together, and divide their ſum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

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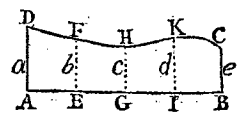
* This rule is made out as follows:—Let ABCD be the irregular piece; having the ſeveral breadths AD, EF, GH, IK, BC, at the equal diſtances AE, EG, GI, IB. Let the ſeveral breadths in order be denoted by the correſponding letters *a*, *b*, *c*, *d*, *e*, and the whole length AB by *l*; then compute the areas of the parts into which the figure is divided by the perpendiculars, as ſo many trapezoids, by prob. 3, and add them all together. Thus, the ſum of the parts is,



$$\frac{a+b}{2} \times \mathrm{AE} + \frac{b+c}{2} \times \mathrm{EG} + \frac{c+d}{2} \times \mathrm{GI} + \frac{d+e}{2} \times \mathrm{IB}$$

$$= \frac{a+b}{2} \times \tfrac{1}{4}l + \frac{b+c}{2} \times \tfrac{1}{4}l + \frac{c+d}{2} \times \tfrac{1}{4}l + \frac{d+e}{2} \times \tfrac{1}{4}l$$

$$= (\tfrac{1}{2}a + b + c + d + \tfrac{1}{2}e) \times \tfrac{1}{4}l = (m + b + c + d)\,\tfrac{1}{4}l,$$

which is the whole area, agreeing with the rule; *m* being the arithmetic mean between the extremes, and 4 the number of the parts. And the ſame for any other number of parts.

Ex. 1.

Ex. 1. The breadths of an irregular figure, at five equidiſtant places, being 8·2, 7·4, 9·2, 10·2, 8·6; and the whole length 39; required the area?

Firſt, $(8{\cdot}2 + 8{\cdot}6) \div 2 = 8{\cdot}4$, the mean of the two extremes.
Then $8{\cdot}4 + 7{\cdot}4 + 9{\cdot}2 + 10{\cdot}2 = 35{\cdot}2$, ſum of breadths.
And $35{\cdot}2 \div 4 = 8{\cdot}8$, the mean breadth.
Hence $8{\cdot}8 \times 39 = 343{\cdot}2$, the anſwer.

Ex. 2. The length of an irregular figure being 84, and the breadths at ſix equidiſtant places 17·4, 20·6, 14·2, 10·5, 20·1, 24·4; what is the area? Anſ. 1550·64.

MENSURATION of SOLIDS.

BY the Menſuration of Solids are determined the ſpaces included by contiguous ſurfaces; and the ſum of the meaſures of theſe including ſurfaces, is the whole ſurface or ſuperficies of the body.

The meaſure of a ſolid, is called its ſolidity, capacity, or content.

Solids are meaſured by cubes, whoſe ſides are inches, or feet, or yards, &c. And hence the ſolidity of a body is ſaid to be ſo many cubic inches, feet, yards, &c, as will fill its capacity or ſpace, or another of equal magnitude.

The leaſt ſolid meaſure is the cubic inch, other cubes being taken from it according to the proportion in the following table.

Table of Cubic or Solid Meaſures.

1728	cubic inches make	1 cubic foot
27	cubic feet -	1 cubic yard
$166\frac{2}{3}$	cubic yards -	1 cubic pole.
64000	cubic poles -	1 cubic furlong
512	cubic furlongs -	1 cubic mile.

PRO-

PROBLEM I.

To find the Superficies of a Prism.

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required *.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length o each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and diameter of its base is 2 feet. Ans. 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at 2d. a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches. Ans. 2l. 3s. $10\frac{1}{2}$d.

PROBLEM II.

To find the Surface of a Pyramid or Cone.

MULTIPLY the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

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* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

Ex. 1.

Ex. 1. What is the upright ſurface of a triangular pyramid, the ſlant height being 20 feet, and each ſide of the baſe 3 feet? Anſ. 90 feet.

Ex. 2. Required the convex ſurface of a cone, or circular pyramid, the ſlant height being 50 feet, and the diameter of its baſe $8\frac{1}{2}$ feet. Anſ. 667·59.

PROBLEM III.

To find the Surface of the Fruſtum of a Pyramid or Cone; being the lower part, when the top is cut off by a plane parallel to the baſe.

ADD together the perimeters of the two ends, and multiply their ſum by the ſlant height, taking half the product for the anſwer.—As is evident, becauſe the ſides of the ſolid are trapezoids, having the oppoſite ſides parallel.

Ex. 1. How many ſquare feet are in the ſurface of the fruſtum of a ſquare pyramid, whoſe ſlant height is 10 feet; alſo, each ſide of the baſe or greater end being 3 feet 4 inches, and each ſide of the leſs end 2 feet 2 inches? Anſ. 110 feet.

Ex. 2. To find the convex ſurface of the fruſtum of a cone, the ſlant height of the fruſtum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4. Anſ. 90 feet.

PROBLEM IV.

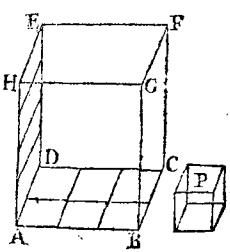
To find the Solid Content of any Priſm or Cylinder.

* FIND the area of the baſe, or end, whatever the figure of it may be; and multiply it by the length of the priſm or cylinder, for the ſolid content.

Ex. 1.

---

* This rule appears from the Geom. theor. 110, cor. 2, and is more particularly ſhewn as follows: Let the rectangular parallelopipedon be the ſolid to be meaſured, and the cube P the ſolid meaſuring unit, its ſide being 1 inch, or 1 foot, &c.; alſo, let the length and breadth of the baſe ABCD, and the height AH, be divided into ſpaces equal to the length of the baſe of the cube P, namely here 3 in the length and 2 in the breadth, making 3 times 2 or 6 ſquares in



Ex. 1. To find the ſolid content of a cube, whoſe ſide is 24 inches. Anſ. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickneſs 2 feet 6 inches? Anſ. $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the ciſtern contain, whoſe dimenſions are the ſame as in the laſt example, when 282 cubic inches are contained in one gallon? Anſ. $129\frac{17}{47}$.

Ex. 4. Required the ſolidity of a triangular priſm, whoſe length is 10 feet, and the three ſides of its triangular end or baſe, are 3, 4, 5 feet. Anſ. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whoſe length is 20 feet, and circumference 5 feet 6 inches. Anſ. 48·1459.

PROBLEM V.

To find the Content of any Pyramid or Cone.

* FIND the area of the baſe, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content.

Ex. 1. Required the ſolidity of the ſquare pyramid, each ſide of its baſe being 30, and its ſlant height 25. Anſ. 6000.

Ex. 2. To find the content of a triangular pyramid, whoſe perpendicular height is 30, and each ſide of the baſe 3. Anſ. 38·97117.

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in the baſe AC, each equal to the baſe of the cube P. Hence it is manifeſt that the parallelopipedon will contain the cube P, as many times as the baſe AC contains the baſe of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelopipedon is found, by multiplying the area of the baſe by the altitude of the parallelopipedon. And, becauſe all priſms and cylinders are equal to parallelopipedons of equal baſes and altitudes, by Geom. theor. 108, it follows that the rule is general for all ſuch ſolids, whatever the figure of the baſe may be.

* This rule follows from that of the priſm, becauſe any pyramid is $\frac{1}{3}$ of a priſm of equal baſe and altitude; by Geom. theor. 115, cor. 1 and 2.

Ex. 3.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three ſides of its baſe 5, 6, 7. Anſ. 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each ſide of its baſe 2 feet? Anſ. 27·5276.

Ex. 5. What is the content of the hexagonal pyramid, whoſe height is 6·4, and each ſide of its baſe 6 inches? Anſ. 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its baſe 9 feet. Anſ. 22·56093.

PROBLEM VI.

To find the Solidity of the Fruſtum of a Cone or Pyramid.

* ADD into one ſum, the areas of the two ends, and the mean proportional between them; or the ſquare root of their product; and $\frac{1}{3}$ of that ſum will be a mean area; which being multiplied by the perpendicular height or length of the fruſtum, will give its content.

Ex. 1. To find the number of ſolid feet in a piece of timber, whoſe baſes are ſquares, each ſide of the greater end

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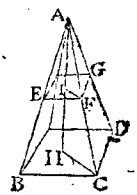
* Let ABCD be any pyramid, of which BCDGFE is a fruſtum. And put a^2 for the area of the baſe BCD, b^2 the area of the top EFG, h the height IH of the fruſtum, and c the height AI of the top part above it. Then $c + h =$ AH the height of the whole pyramid.



Hence, by the laſt prob. $\frac{1}{3}a^2(c+h)$ is the content of the whole pyramid ABCD, and $\frac{1}{3}b^2c$ the content of the top part AEFG; therefore the difference $\frac{1}{3}a^2(c+h) - \frac{1}{3}b^2c$ is the content of the fruſtum BCDGFE. But, by Geom. theor. 112, $a^2 : b^2 :: (c+h)^2 : c^2$, or $a : b :: c+h : c$, and $a-b : b :: h : c$; hence $c = \frac{bh}{a-b}$, and $c+h = \frac{ah}{a-b}$; then theſe values of c and $c+h$ being ſubſtituted for them in the expreſſion for the content of the fruſtum, gives that content $= \frac{1}{3}a^2 \times \frac{ah}{a-b} - \frac{1}{3}b^2 \times \frac{bh}{a-b} = \frac{1}{3}h \times \frac{a^3-b^3}{a-b} = \frac{1}{3}h \times (a^2+ab+b^2)$; which is the rule above given.

being

being 15 inches, and each ſide of the leſs end 6 inches; alſo, the length or perpendicular altitude 24 feet? Anſ. 19½.

Ex. 2. Required the content of a pentagonal fruſtum, whoſe height is 5 feet, each ſide of the baſe 18 inches, and each ſide of the top or leſs end 6 inches. Anſ. 9·31925 feet.

Ex. 3. To find the content of a conic fruſtum, the altitude being 18, the greateſt diameter 8, and the leaſt diameter 4. Anſ. 527·7888.

Ex. 4. What is the ſolidity of the fruſtum of a cone, the altitude being 25, alſo the circumference at the greater end being 20, and at the leſs end 10? Anſ. 464·205.

Ex. 5. If a caſk, which is two equal conic fruſtums joined together at the baſes, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Anſ. 79·0613.

PROBLEM VII.

To find the Surface of a Sphere, or any Segment.

* RULE I. MULTIPLY the circumference of the ſphere by its diameter, and the product will be the whole ſurface of it.

RULE II.

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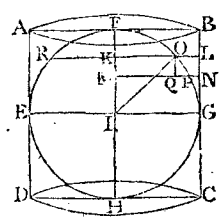
* Theſe rules come from the following theorems for the ſurface of a ſphere, viz. That the ſaid ſurface is equal to the curve ſurface of its circumſcribing cylinder; or that it is equal to 4 great circles of the ſame ſphere, or of the ſame diameter; which are thus proved.

Let ABCD be a cylinder, circumſcribing the ſphere EFGH, the former generated by the revolution of the rectangle FBCH about the axis or diameter FH, and the latter by the revolution of the ſemicircle FGH about the ſame diameter FH. Draw two lines KL, MN perpendicular to the axis, intercepting the parts LN, OP of the cylinder and ſphere; then ſhall the ring or cylindric ſurface generated by the revolution of LN, be equal to the ring or ſpherical ſurface generated by the arc OP. For, firſt, ſuppoſe the parallels KL and MN to be indefinitely near together; drawing IO, and alſo OQ parallel to LN. Then, the two triangles IKO, OQP being equiangular, it



is,

RULE II. Multiply the ſquare of the diameter by 3·1416, and the product will be the ſurface.

Note. For the ſurface of a ſegment or fruſtum, multiply the whole circumference by the height of the part required.

Ex. 1. Required the convex ſuperficies of a ſphere, whoſe diameter is 7, and circumference 22. Anſ. 154.

Ex. 2. Required the ſuperficies of a globe, whoſe diameter is 24 inches. Anſ. 1809·5616.

Ex. 3. Required the area of the whole ſurface of the earth, its diameter being 7957$\frac{3}{4}$ miles, and its circumference 25000 miles. Anſ. 19894375o ſq. miles.

Ex. 4. The axis of a ſphere being 42 inches, what is the convex ſuperficies of the ſegment whoſe height is 9 inches? Anſ. 1187·5248 inches.

Ex. 5. Required the convex ſurface of a ſpherical zone, whoſe breadth or height is 2 feet, and cut from a ſphere of 12$\frac{1}{2}$ feet diameter. Anſ. 78·54 feet.

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is, as OP : OQ or LN :: 10 or KL : KO :: circumference deſcribed by KL : circumf. deſcribed by KO; therefore the rectangle OP × circumf. of KO is equal to the rectangle LN × circumf. of KL; that is, the ring deſcribed by OP on the ſphere, is equal to the ring deſcribed by LN on the cylinder.

And as this is every where the caſe, therefore the ſums of any correſponding number of theſe are alſo equal; that is, the whole ſurface of the ſphere, generated by the whole ſemicircle FGH, is equal to the whole curve ſurface of the cylinder, generated by the height BC; as well as the ſurface of any ſegment generated by FO, equal to the ſurface of the correſponding ſegment generated by BL.

Corol. 1. Hence the ſurface of the ſphere is equal to 4 of its great circles, or equal to the circumference EFGH, or of DC, multiplied by the height BC, or the diameter FH.

Corol. 2. Hence alſo, the ſurface of any ſegment, or fruſtum, or zone, is equal to the ſame circumference of the ſphere, multiplied by the height of the ſaid ſegment or fruſtum.

PROBLEM VIII.

To find the Solidity of a Sphere or Globe.

* RULE I. MULTIPLY the ſurface by the diameter, and take $\frac{1}{6}$ of the product for the content.

RULE II. Multiply the cube of the diameter by the decimal ·5236, for the content.

EX. 1. To find the content of a ſphere whoſe axis is 12.
Anſ. 904 7808.

EX. 2. To find the ſolid content of the globe of the earth, ſuppoſing its circumference to be 25000 miles.
Anſ. 263858149120 miles.

PROBLEM IX.

To find the Solid Content of a Spherical Segment.

† RULE I. FROM 3 times the diameter of the ſphere, take double the height of the ſegment; then multiply the remainder

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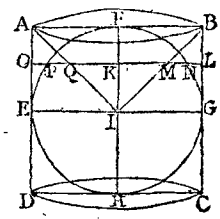
* For, put $d =$ the diameter, $c =$ the circumference, and $s =$ the ſurface, of the ſphere, or of its circumſcribing cylinder; alſo, $a =$ the number 3·1416.

Then, $\frac{1}{4}s$ is $=$ the baſe of the cylinder, or one great circle of the ſphere; and d is the height of the cylinder; therefore $\frac{1}{4}ds$ is the content of the cylinder. But $\frac{2}{3}$ of the cylinder is the ſphere; that is, $\frac{2}{3}$ of $\frac{1}{4}ds$, or $\frac{1}{6}ds$ is the ſphere; which is the firſt rule.

Again, becauſe the ſurface s is $= dc$, and $c = ad$, therefore $s = ad^2$; and conſeq. $\frac{1}{6}ds = \frac{1}{6}ad^3 = ·5236d^3$, the content, as in the 2d rule.

† By the demonſtr. of theor. 117, Geom. it appears that the ſpheric ſegment PFN, is equal to the difference between the cylinder ABLO, and the conic fruſtum ABMQ.



But, putting $d =$ AB or FH the diameter of the ſphere or cylinder, $b =$ FK the height of the ſegment, $r =$ PK the radius of its baſe, and $a = 3·1416$; then the cone ABI is $= \frac{1}{4}ad^2 \times \frac{1}{3}\text{FI} = \frac{1}{12}ad^3$; and by the ſimilar cones ABI, QMI, as $\text{FI}^3 : \text{KI}^3 :: \frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - b}{\frac{1}{2}d})^3 =$ the cone QMI; therefore the cone ABI

the

remainder by the ſquare of the height, and the product by the decimal ·5236, for the content.

RULE II. To three times the ſquare of the radius of the ſegment's baſe, add the ſquare of its height; then multiply the ſum by the height, and the product by ·5236, for the content.

Ex. 1. To find the content of a ſpherical ſegment, of 2 feet in height, cut from a ſphere of 8 feet diameter.

Anſ. 41·888.

Ex. 2. What is the ſolidity of the ſegment of a ſphere, its height being 9, and the diameter of its baſe 20?

Anſ. 1795·4244.

Note. The general rules for meaſuring all ſorts of figures having been now delivered, we may next proceed to apply them to the ſeveral practical uſes in life, as follows.

— the cone QMI $= \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - b}{\frac{1}{2}d})^3 = \frac{1}{4}ad^2b - \frac{1}{2}adb^2 + \frac{1}{3}ab^3$ is = the conic fruſtum ABMQ.

And $\frac{1}{4}ad^2b$ is = the cylinder ABLO.

Then the difference of theſe two is $\frac{1}{2}adb^2 - \frac{1}{3}ab^3 = \frac{1}{6}ab^2 \times (3d - 2b)$, for the ſpheric ſegment PFN; which is the firſt rule.

Again, becauſe $PR^2 = FR \times RH$, or $r^2 = b\,(d - b)$, therefore $d = \frac{r^2}{b} + b$, and $3d - 2b = \frac{3r^2}{b} + b = \frac{3r^2 + b^2}{b}$; which being ſubſtituted in the former rule, it becomes $\frac{1}{6}ab^2 \times \frac{3r^2 + b^2}{b} = \frac{1}{6}ab \times (3r^2 + b^2)$, which is the 2d rule.

Note. By ſubtracting a ſegment from a half ſphere, or from another ſegment, the content of any fruſtum or zone may be found.

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

I. OF THE CHAIN.

LAND is meaſured with a chain, called Gunter's Chain, from its inventor, of 4 poles or 22 yards, or 66 feet in length. It conſiſts of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is eſtimated in acres, roods, and perches. An acre is equal to 10 ſquare chains, that is, 10 chains in length and 1 chain in breadth. Or it is 220 × 22 = 4840 ſquare yards. Or it is 40 × 4 = 160 ſquare poles. Or it is 1000 × 100 = 100000 ſquare links. Theſe being all the ſame quantity.

Alſo, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are ſquare poles, or the ſquare of a pole of $5\frac{1}{2}$ yards long, or the ſquare of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 ſquare links. So that the diviſions of land meaſure will be thus:

625 ſq links = 1 pole or perch
40 perches = 1 rood
4 roods = 1 acre.

The length of lines, meaſured with a chain, are beſt ſet down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore after the content is found, it will be in ſquare links; then cut off five of the figures on the right-hand for decimals, and the reſt will be acres. Theſe decimals are then multiplied by 4 for roods, and the decimals of theſe again by 40 for perches.

EXAM. Suppoſe the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

792

792	3·04920
385	4
3960	·19680
6336	40
2376	
	7·87200
3·04920	

Anſ. 3 acres, 0 roods, 7 perches.

2. OF THE PLAIN TABLE.

THIS inſtrument conſiſts of a plain rectangular board, of any convenient ſize: the centre of which, when uſed, is fixed by means of ſcrews to a three-legged ſtand, having a ball and ſocket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a ſheet of paper upon the table. The one ſide of this frame is uſually divided into equal parts, for drawing lines acroſs the table, parallel or perpendicular to the ſides; and the other ſide of the frame is divided into 360 degrees from a centre which is in the middle of the table; by means of which the table is to be uſed as a theodolite, &c.

2. A needle and compaſs ſcrewed into the ſide of the table, to point out the directions, and to be a check upon the ſights.

3. An index, which is a braſs two-foot ſcale, with either a ſmall teleſcope, or open ſights erected perpendicularly upon the ends. Theſe ſights and one edge of the index are in the ſame plane, and that edge is called the fiducial edge of the index.

To uſe this inſtrument, take a ſheet of paper which will cover it, and wet it to make it expand; then ſpread it flat on the table, preſſing down the frame upon the edges, to ſtretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, ſtretch itſelf ſmooth and flat from any cramps and unevenneſs. On this paper is to be drawn the plan or form of the thing meaſured.

Then, begin at any part of the ground the moſt proper,

and make a point on a convenient part of the paper or table, to repreſent that point of the ground; then fix in that point one leg of the compaſſes, or a fine ſteel pin, and apply to it the fiducial edge of the index, moving it round till through the ſights you perceive ſome remarkable object, as the corner of a field, &c.; and from the ſtation point draw a line with the point of the compaſſes along the fiducial edge of the index; then ſet another object or corner, and draw its line; do the ſame by another, and ſo on, till as many objects are ſet as may be thought fit. Then meaſure from the ſtation, towards as many of the objects as may be neceſſary, and no more, taking the requiſite offsets to corners or crooks in the hedges, laying the meaſures down on their reſpective lines on the table. Then, at any convenient place, meaſured to, fix the table in the ſame poſition, and ſet the objects which appear from thence, &c. as before; and thus continue till the work is finiſhed, meaſuring ſuch lines as are neceſſary, and determining as many as may be, by interſecting lines of direction drawn from different ſtations.

Of ſhifting the Paper on the Plain Table.

When one paper is full, and you have occaſion for more; draw a line in any manner through the fartheſt point of the laſt ſtation line, to which the work can be conveniently laid down; then take the ſheet off the table, and fix another on, drawing a line upon it, in a part the moſt convenient for the reſt of the work; then fold or cut the old ſheet by the line drawn on it, applying the edge to the line on the new ſheet, and as they lie in that poſition, continue the laſt ſtation line on the new paper, placing on it the reſt of the meaſure, beginning at where the old ſheet left off. And ſo on from ſheet to ſheet.

When the work is done, and you would faſten all the ſheets together into one piece, or rough plan, the aforeſaid lines are to be accurately joined together, in the ſame manner as when the lines were transferred from the old ſheets to the new ones.

But it is to be noted, that if the ſaid joining lines, on the old and new ſheets, have not the ſame inclination to the ſide of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to reſpect ſtill the ſame degree of the compaſs, the eaſieſt way of drawing the lines in the ſame poſition, is to draw them both parallel to the ſame ſides of the table, by means of the equal diviſions marked on the other two ſides.

3. OF

3. OF THE THEODOLITE.

THE theodolite is a brazen circular ring, divided into 360 degrees, and having an index with ſights, or a teleſcope, placed on the centre, about which the index is moveable; alſo a compaſs fixed t the centre, to point out courſes and check the ſights; the whole being fixed by the centre on a ſtand of a convenient height for uſe.

In uſing this inſtrument, an exact account, or field-book, of all meaſures and things neceſſary to be remarked in the plan, muſt be kept, from which to make out the plan on returning home from the ground.

Begin at ſuch part of the ground, and meaſure in ſuch directions, as you judge moſt convenient; taking angles or directions to objects, and meaſuring ſuch diſtances as appear neceſſary, under the ſame reſtrictions as in the uſe of the plain table. And it is ſafeſt to fix the theodolite in the original poſition at every ſtation by means of fore and back objects, and the compaſs, exactly as in uſing the plain table; regiſtering the number of degrees cut off by the index when directed to each object; and, at any ſtation, placing the index at the ſame degree as when the direction towards that ſtation was taken from the laſt preceding one, to fix the theodolite there in the original poſition.

The beſt method of laying down the aforeſaid lines of direction, is to deſcribe a pretty large circle; then quarter it, and lay on it the ſeveral numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all theſe marked points in the circle. Then, by means of a parallel ruler, draw, from ſtation to ſtation, lines parallel to the aforeſaid lines drawn from the centre to the reſpective points in the circumference.

4. OF THE CROSS.

THE croſs conſiſts of two pair of ſights ſet at right angles to each other, upon a ſtaff having a ſharp point at the bottom to ſtick in the ground.

The croſs is very uſeful to meaſure ſmall and crooked pieces of ground. The method is to meaſure a baſe or chief line, uſually in the longeſt direction of the piece, from corner to corner; and while meaſuring it, finding the places where perpendiculars would fall on this line, from the ſeveral corners and bends in the boundary of the piece, with the croſs, by fixing it, by trials, on ſuch parts of the line, ſo that through one pair of the ſights both ends of the line may appear, and through the other pair you can perceive the

correſponding bends or corners; and then meaſuring the lengths of the ſaid perpendiculars.

REMARKS.

Beſides the fore-mentioned inſtruments, which are moſt commonly uſed, there are ſome others; as the circumferentor, which reſembles the theodolite in ſhape and uſe; and the ſemicircle, for taking angles, &c.

The perambulator is uſed for meaſuring roads, and other great diſtances on level ground, and by the ſides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole in circumference, upon which the machine turns; and the diſtance meaſured, is pointed out by an index, which is moved round by clock work.

Levels, with teleſcopic or other ſights, are uſed to find the level between place and place, or how much one place is higher or lower than another. And in meaſuring any ſloping or oblique line, either aſcending or deſcending, a ſmall pocket level is uſeful for ſhowing how many links for each chain are to be deducted, to reduce the line to the true horizontal length.

An offset-ſtaff is a very uſeful and neceſſary inſtrument for meaſuring the offsets and other ſhort diſtances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten ſmall arrows, or rods of iron or wood, are uſed to mark the end of every chain length, in meaſuring lines. And ſometimes pickets, or ſtaves with flags, are ſet up as marks or objects of direction.

Various ſcales are alſo uſed in protracting and meaſuring on the plan or paper; ſuch as plane ſcales, line of chords, protractor, compaſſes, reducing ſcale, parallel and perpendicular rules, &c. Of plane ſcales, there ſhould be ſeveral ſizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of theſe, the beſt for uſe are thoſe that are laid on the very edges of the ivory ſcale, to prick off diſtances by, without compaſſes.

5. OF THE FIELD-BOOK.

In ſurveying with the plane table, a field-book is not uſed, as every thing is drawn on the table immediately when it is meaſured. But in ſurveying with the theodolite, or any other inſtrument, ſome ſort of a field-book muſt be uſed, to write down in it a regiſter or account of all that is done and occurs relative to the ſurvey in hand.

This book every one contrives and rules as he thinks fitteſt for himſelf. The following is a ſpecimen of a form which has been formerly uſed. It is ruled into 3 columns: the middle, or principal column, is for the ſtations, angles, bearings,

bearings, diſtances meaſured, &c.; and thoſe on the right and left are for the offsets on the right and left, which are ſet againſt their correſponding diſtances in the middle column; as alſo for ſuch remarks as may occur, and may be proper to note in drawing the plan, &c.

Here ⊙ 1 is the firſt ſtation, where the angle or bearing is 105° 25′. On the left, at 73 links in the diſtance or principal line, is an offset of 92; and at 610 an offset of 24 to a croſs hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the firſt line, the o denotes its terminating in the hedge. And ſo on for the other ſtations.

A line is drawn under the work, at the end of every ſtation line, to prevent confuſion.

Form of this Field-Book.

Offsets and Remarks on the left.	Stations, Bearings, and Diſtances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
	00	25 corner
92	73	
	248	Brown's hedge
croſs a hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
	00	00
houſe corner 51	25	21
	120	29 a tree
34	734	40 a ſtyle
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a ſpring
foot-path 16	810	
croſs hedge 18	973	20 a pond

But ſome ſkilful ſurveyors now make uſe of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; by which they ſketch a neat boundary on either hand, as they paſs along; an example of which will be given further on, in the method of ſurveying a large eſtate.

In ſmaller ſurveys and meaſurements, a good way of ſetting down the work, is, to draw, by the eye on a piece of paper, a figure reſembling that which is to be meaſured; and ſo writing the dimenſions, as they are found, againſt the correſponding parts of the figure. And this method may be practiſed to a conſiderable extent, even in the larger ſurveys.

SECTION II.

THE PRACTICE OF SURVEYING.

THIS part contains the ſeveral works proper to be done in the field, or the ways of meaſuring by all the inſtruments, and in all ſituations.

PROBLEM I.

To Meaſure a Line or Diſtance.

To meaſure a line on the ground with the chain: Having provided a chain, with 10 ſmall arrows, or rods, to ſtick one into the ground, as a mark, at the end of every chain; two perſons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremoſt, and is called the leader; the other being called the follower, for diſtinction's ſake.

A picket, or ſtation-ſtaff, being ſet up in the direction of the line to be meaſured, if there do not appear ſome marks naturally in that direction; they meaſure ſtraight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, till all the ten arrows are uſed. They are then all returned to the leader, to uſe over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finiſhed; then the number of changes of the arrows ſhews the number of tens, to which the follower adds the arrows he

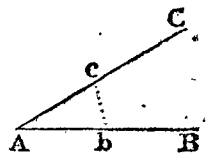
he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is on a declivity, ascending or descending; at every chain length, lay the offset staff, or link-staff down in the slope of the chain, upon which lay the small pocket level, to shew how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take Angles and Bearings.

LET B and C be two objects, or two pickets set up perpendicular, and let it be required to take their bearings, or the angle formed between them at any station A.



1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand: plant it at the station A, and fix a fine pin, or a point of the compasses, in a proper point of the paper, to represent the point A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the very same manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index about till, through its sights, you see the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the instrument, shew the quantity of the angle.

3. *With the Magnetic Needle and Compass.*

Turn the instrument, or compass, so, that the north end of the needle-point to the flower-de-luce. Then direct the sights

ſights to one mark as B, and note the degrees cut by the needle. Next direct the ſights to the other mark C, and note again the degrees cut by the needle. Then their ſum or difference, as the caſe is, will give the quantity of the angle BAC.

4. *By Meaſurement with the Chain, &c.*

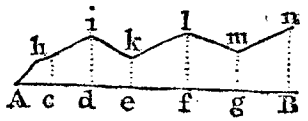
Meaſure one chain length, or any other length, along both directions, as to b and c. Then meaſure the diſtance b c, and it is done.—This is eaſily transferred to paper, by making a triangle A b c with theſe three lengths, and then meaſuring the angle A.

PROBLEM III.

To meaſure the Offsets.

A h i k l m n being a crooked hedge, or river, &c. From A meaſure in a ſtraight direction along the ſide of it to B. And in meaſuring along this line A B, obſerve when you are directly oppoſite any bends or corners of the hedge, as at c, d, e, &c.; and from thence meaſure the perpendicular offsets c h, d i, &c. with the offset-ſtaff, if they are not very large, otherwiſe with the chain itſelf. And the work is done. The regiſter, or field-book, may be as follows:

Offs. left.	Baſe line A B.
0	⊙ A
ch 62	45 Ac
di 84	220 Ad
ek 70	34 A
fl 98	510 A
gm 57	634 Ag
kn 91	785 AB

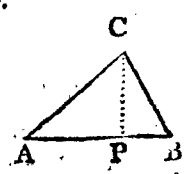


PROBLEM IV.

To ſurvey a Triangular Field ABC.

1. *By the Chain.*

AP 794
AB 1121
PC 826



Having

Having ſet up marks at the corners, which is to be done in all caſes where there are not marks naturally; meaſure with the chain from A to P, where a perpendicular would fall from the angle C, and ſet up a mark at P, noting down the diſtance AP. Then complete the diſtance AB by meaſuring from P to B. Having ſet down this meaſure, return to P, and meaſure the perpendicular PC. And thus, having the baſe and perpendicular, the area from them is eaſily found. Or having the place P of the perpendicular, the triangle is eaſily conſtructed.

Or, meaſure all the three ſides with the chain, and note them down. From which the content is eaſily found, or the figure conſtructed.

2. *By taking one or more of the Angles.*

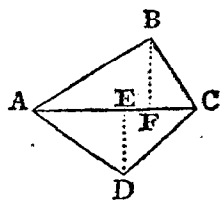
Meaſure two ſides AB, AC, and the angle A between them. Or meaſure one ſide AB, and the two adjacent angles A and B. From either of theſe ways the figure is eaſily planned; then by meaſuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

PROBLEM V.

To meaſure a Four-ſided Field.

1. *By the Chain.*

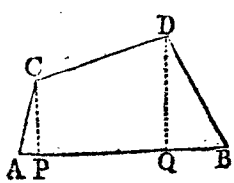
AE 214	210 DE
AF 362	306 BF
AC 592	



Meaſure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the laſt problem; or elſe the ſides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Otherwiſe by the Chain.

AP 110	352 PC
AQ 745	595 QD
AB 1110	



Meaſure

Meaſure on the longeſt ſide, the diſtances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking one or more of the Angles.*

Meaſure the diagonal AC (ſee the laſt fig. but one), and the angles CAB, CAD, ACB, ACD.—Or meaſure the four ſides, and any one of the angles as BAD.

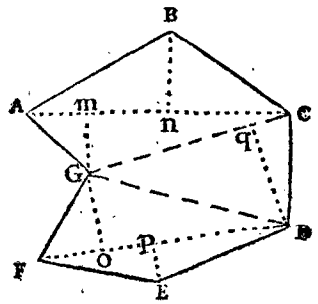
Thus		Or thus	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′

PROBLEM VI.

To ſurvey any Field by the Chain only.

HAVING ſet up marks at the corners, where neceſſary, of the propoſed field ABCDEFG. Walk over the ground, and conſider how it can beſt be divided in triangles and trapeziums; and meaſure them ſeparately as in the laſt two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the firſt trapezium, beginning at A, meaſure the diagonal AC, and the two perpendiculars G m, B n. Then the baſe GC, and the perpendicular D q. Laſtly, the diagonal DF, and the two perpendiculars p E, o G. All which meaſures write againſt the correſponding parts of a rough figure drawn to reſemble the figure to be ſurveyed, or ſet them down in any other form you chooſe.

Thus		Thus
A m	135	130 m G
A n	410	180 n B
A c	550	
C q	152	230 q D
C G	440	
F o	206	120 o G
F p	288	80 p E
F D	520	



Or

Or thus.

Meaſure all the ſides AB, BC, CD, DE, EF, FG, and GA; and the diagonals AC, CG, GD, DF.

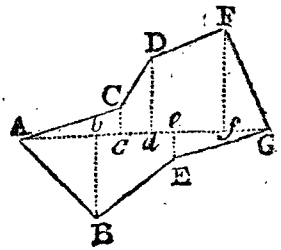
Otherwiſe.

Many pieces of land may be very well ſurveyed, by meaſuring any baſe line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by thoſe means divided into ſeveral triangles and trapezoids, all whoſe parallel ſides are perpendicular to the baſe line; and the ſum of theſe triangles and trapeziums will be equal to the figure propoſed if the baſe line fall within it; if not, the ſum of the parts which are without being taken from the ſum of the whole which are both within and without, will leave the area of the figure propoſed.

In pieces that are not very large, it will be ſufficiently exact to find the points, in the baſe line, where the ſeveral perpendiculars will fall, by means of the *croſs*, and from thence meaſuring to the corners for the lengths of the perpendiculars.—And it will be moſt convenient to draw the line ſo as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and meaſuring along the line AG, the diſtances and perpendiculars, on the right and left, are as below.

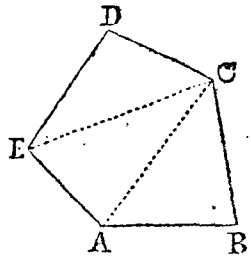
A b	315	350	b B
A c	440	70	c C
A d	585	320	d D
A e	610	50	e E
A f	990	470	f F
A G	1020	0	



PROBLEM VII.

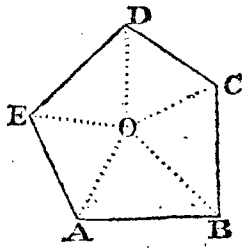
To ſurvey any Field with the Plain Table.

1. *From one Station.*



PLANT the table at any angle, as C, from whence all the other angles, or marks ſet up, can be ſeen; turn the table about till the needle point to the flower-de-luce; and there ſcrew it faſt. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the ſights you ſee the mark D; and by the edge of the index draw a dry or obſcure line: then meaſure the diſtance CD, and lay that diſtance down on the line CD. Then turn the index about the point C, till the mark E be ſeen through the ſights, by which draw a line, and meaſure the diſtance to E, laying it on the line from C to E. In like manner determine the poſitions of CA and CB, by turning the ſights ſucceſſively to A and B; and lay the lengths of thoſe lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2. *From a Station within the Field.*



When all the other parts cannot be ſeen from one angle, chooſe ſome place O within; or even without, if more convenient, from whence the other parts can be ſeen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index ſucceſſively to O, turning it round with the ſights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index; then meaſuring the diſtances OA, OB, &c. and laying them down upon thoſe lines. Laſtly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the Figure.*

When the figure is a wood, or water, or from ſome other obſtruction you cannot meaſure lines acroſs it; begin at any point

point A, and meaſure round it, either within or without the figure, and draw the directions of all the ſides thus: Plant the table at A, turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can ſee the point E, there draw a line; then the point B, and there draw a line: then meaſure theſe lines, and lay them down from A to E and B. Next, move the table to B, lay the index along the line AB, and turn the table about till you can ſee the mark A, and ſcrew faſt the table; in which poſition alſo the needle will again point to the flower-de-luce, as it will do indeed at every ſtation when the table is in the right poſition. Here turn the index about B till through the ſights you ſee the mark C; there draw a line, meaſure BC, and lay the diſtance upon that line after you have ſet down the table at C. Turn it then again into its proper poſition, and in like manner find the next line CD. And ſo on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the laſt direction EA on the ground, will paſs exactly through the point A on the paper; and the meaſured diſtance will alſo reach exactly to A. If theſe do not coincide, or nearly ſo, ſome error has been committed, and the work muſt be examined over again.

PROBLEM VIII.

To ſurvey a Field with the Theodolite, &c.

1. *From one Point or Station.*

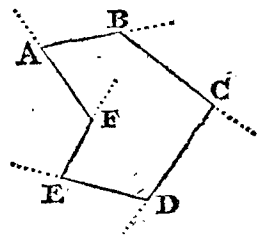
When all the angles can be ſeen from one point, as the angle E (firſt fig. to laſt prob.) place the inſtrument at E, and turn it about till, through the fixed ſights, you ſee the mark B, and there fix it. Then turn the moveable index about till the mark A is ſeen through the ſights, and note the degrees cut on the inſtrument. Next turn the index ſucceſſively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Laſtly, meaſure the lines CB, CA, CE, CD; and enter the meaſures in a field-book, or rather againſt the correſponding parts of a rough figure drawn by gueſs to reſemble the field.

2. *From a Point within or without.*

Plant the inſtrument at O (laſt fig.), and turn it about till the fixed ſights point to any object, as A; and there ſcrew it faſt. Then turn the moveable index round till the ſights

ſights point ſucceſſively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Laſtly meaſure the diſtances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. *By going round the Field.*



By meaſuring round, either within or without the field, proceed thus. Having ſet up marks at B, C, &c. near the corners as uſual, plant the inſtrument at any point A, and turn it till the fixed index be in the direction AB, and there ſcrew it faſt: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Meaſure the line AB, and plant the inſtrument at B, and there in the ſame manner obſerve the angle A. Then meaſure BC, and obſerve the angle C. Then meaſure the diſtance CD, and take the angle D. Then meaſure DE, and take the angle E. Then meaſure EF, and take the angle F. And laſtly meaſure the diſtance FA.

To prove the work; add all the inward angles A, B, C, &c. together, for when the work is right, their ſum will be equal to twice as many right angles as the figure has ſides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you meaſure the external angle, which is leſs than two right angles, ſubtract it from four right angles, or 360 degrees, to give the internal angle greater than a ſemicircle or 180 degrees.

Otherwiſe.

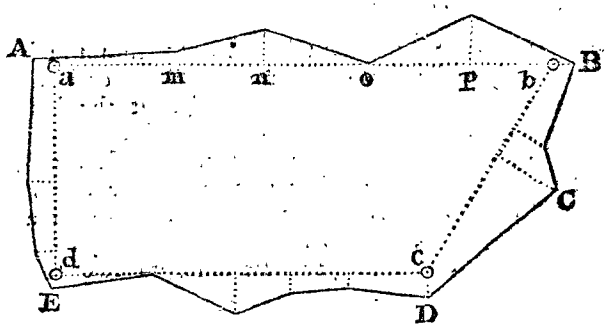
Inſtead of obſerving the internal angles, you may take the external angles, formed without the figure by producing the ſides further out. And in this caſe, when the work is right, their ſum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, ſubtract it from the ſum of the reſt, to leave 360 degrees.

PROBLEM IX.

To ſurvey a Field with Crooked Hedges, &c.

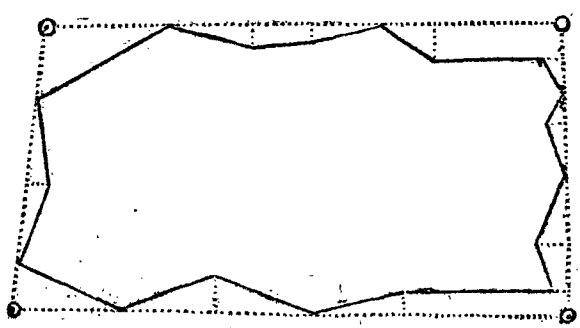
With any of the inſtruments, meaſure the lengths and poſitions of imaginary lines running as near the ſides of the field as

as you can; and, in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in surveying the piece ABCDE, set up marks a, b, c, d, dividing it into as few sides as may be. Then begin at any station a, and measure the lines ab, bc, cd, da, taking their positions, or the angles a, b, c, d; and, in going along the lines, measure all the offsets, as at m, n, o, p, &c. along every station line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c. then measure without, as in the figure here below.



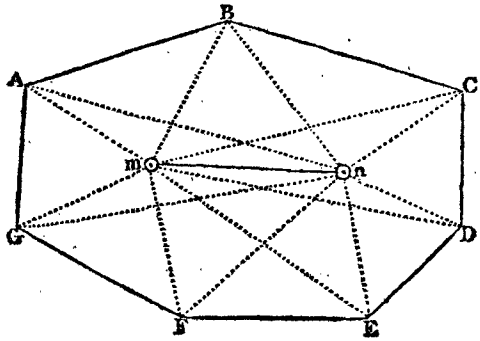
PROBLEM X.

To ſurvey a Field, or any other Thing, by two Stations.

THIS is performed by chooſing two ſtations, from whence all the marks and objects can be ſeen; then meaſuring the diſtance between the ſtations, and at each ſtation taking the angles formed by every object, from the ſtation line or diſtance.

The two ſtations may be taken either within the bounds, or in one of the ſides, or in the direction of two of the objects, or quite at a diſtance and without the bounds of the objects, or part to be ſurveyed.

In this manner, not only grounds may be ſurveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coaſt ſurveyed, or any other inacceſſible objects; by taking two ſtations, on two towers, or two hills, or ſuch like.



PROBLEM XI.

To ſurvey a Large Eſtate.

IF the eſtate be very large, and contain a great number of fields, it cannot well be done by ſurveying all the fields ſingly, and then putting them together; nor can it be done by taking all the angles and boundaries that incloſe it. For in theſe caſes, any ſmall errors will be ſo multiplied, as to render it very much diſtorted.

I. Walk

1. Walk over the eſtate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably well in your head And to help your memory, draw an eye draught of it on paper, or at leaſt of the principal parts of it, to guide you; ſetting the names within the fields in that draught.

2 Chooſe two or more eminent places in the eſtate, for ſtations, from whence all the principal parts of it can be ſeen: and let theſe ſtations be as far diſtant from one another as poſſible.

3. Take ſuch angles, between the ſtations, as you think neceſſary, and meaſure the diſtances from ſtation to ſtation, always in a right line: theſe things muſt be done, till you get as many angles and lines as are ſufficient for determining all the points of ſtation. And in meaſuring any of theſe ſtation diſtances, mark accurately where theſe lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c.; and where any remarkable object is placed, by meaſuring its diſtance from the ſtation line; and where a perpendicular from it cuts that line. And thus as you go along any main ſtation line, take offsets to the ends of all hedges, and to any pond, houſe, mill, bridge, &c. omitting nothing that is remarkable, and noting every thing down.

4. As to the inner parts of the eſtate, they muſt be determined in like manner, by new ſtation lines: for, after the main ſtations are determined, and every thing adjoining to them, then the eſtate muſt be ſubdivided into two or three parts by new ſtation lines; taking inner ſtations at proper places, where you can have the beſt view Meaſure theſe ſtation lines as you did the firſt, and all their interſections with hedges, and offsets to ſuch objects as appear. Then proceed to ſurvey the adjoining fields, by taking the angles that the ſides make with the ſtation line, at the interſections, and meaſuring the diſtances to each corner, from the interſections. For the ſtation lines will be the baſes to all the future operations; the ſituation of all parts being entirely dependent upon them; and therefore they ſhould be taken of as great length as poſſible; and it is beſt for them to run along ſome of the hedges or boundaries of one or more fields, or to paſs through ſome of their angles. All things being determined for theſe ſtations, you muſt take more inner ſtations, and continue to divide and ſubdivide till at laſt you come to ſingle fields; repeating the ſame work for the inner ſtations, as for the outer ones, till all is done; and cloſe the work as often as you can, and in as few lines as poſſible.

5. An eſtate may be ſo ſituated, that the whole cannot be ſurveyed together; becauſe one part of the eſtate cannot be ſeen from another. In this caſe, you may divide it into three or four parts, and ſurvey the parts ſeparately, as if they were lands belonging to different perſons; and at laſt join them together.

6. As it is neceſſary to protract or lay down the work as you proceed in it, you muſt have a ſcale of a due length to do it by. To get ſuch a ſcale, meaſure the whole length of the eſtate in chains; then conſider how many inches long the map is to be; and from theſe will be known how many chains you muſt have in an inch; then make the ſcale accordingly, or chooſe one already made.

The New Method of Surveying.

In the former method of meaſuring a large eſtate, the accuracy of it depends on the correctneſs of the inſtruments uſed in taking the angles. To avoid the errors incident to ſuch a multitude of angles, other methods have of late years been uſed by ſome few ſkilful ſurveyors: the moſt practical, expeditious, and correct, ſeems to be the following.

Chooſe two or more eminences, as grand ſtations, and meaſure a principal baſe line from one ſtation to the other, noting every hedge, brook, or other remarkable object as you paſs by it; meaſuring alſo ſuch ſhort perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this baſe line, or from any convenient parts of the ſame, go off with other lines to ſome remarkable object ſituated towards the ſides of the eſtate, without regarding the angles they make with the baſe line or with one another; ſtill remembering to note every hedge, brook or other object that you paſs by. Theſe lines, when laid down by interſections, will with the baſe line form a grand triangle on the eſtate; ſeveral of which, if need be, being thus laid down, you may proceed to form other ſmaller triangles and trapezoids on the ſides of the former: and ſo on, until you finiſh with the encloſures individually.

This grand triangle being completed, and laid down on the rough plan paper, the parts, exterior as well as interior, are to be completed by ſmaller triangles and trapezoids.

In countries where the lands are encloſed with high hedges, and where many lanes paſs through an eſtate, a theodolite may be uſed to advantage, in meaſuring the angles of ſuch lands;

lands; by which means, a kind of ſkeleton of the eſtate may be obtained, and the lane-lines ſerve as the baſes of ſuch triangles and trapezoids as are neceſſary to fill up the interior parts.

The field-book is ruled into three columns. In the middle one are ſet down the diſtances on the chain line at which any mark, offset, or other obſervation is made; and in the right and left hand columns are entered the offsets and obſervations made on the right and left hand reſpectively of the chain line.

It is of great advantage, both for brevity and perſpicuity, to begin at the bottom of the leaf and write upwards; denoting the croſſing of fences, by lines drawn acroſs the middle column, or only a part of ſuch a line on the right and left oppoſite the figures, to avoid confuſion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do, as will be beſt ſeen by comparing the book with the plan annexed to the field-book following, p. 74.

The letter in the left-hand corner at the beginning of every line, is the mark or place meaſured *from*; and, that at the right hand corner at the end, is the mark meaſured *to*: But when it is not convenient to go exactly from a mark, the place meaſured from, is deſcribed *ſuch a diſtance* from *one mark* towards *another*; and where a mark is not meaſured to, the exact place is aſcertained by ſaying, turn to the right or left hand, *ſuch a diſtance* to *ſuch a mark*, it being always underſtood that thoſe diſtances are taken in the chain line.

The characters uſed, are ⌜ for *turn to the right hand*, ⌝ for *turn to the left hand*, and ∧ placed over an offset, to ſhew that it is not taken at right angles with the chain line, but in the line with ſome ſtraight fence; being chiefly uſed when croſſing their directions, and it is a better way of obtaining their true places than by offsets at right angles.

When a line is meaſured whoſe poſition is determined, either by former work (as in the caſe of producing a given line, or meaſuring from one known place or mark to another) or by itſelf (as in the third ſide of a triangle) it is called a *faſt line*, and a double line acroſs the book is drawn at the concluſion of it; but if its poſition is not determined (as in the ſecond ſide of a triangle) it is called a *looſe line*, and a ſingle line is drawn acroſs the book. When a line becomes determined in poſition, and is afterwards continued, a double line half through the book is drawn.

When a looſe line is meaſured, it becomes abſolutely neceſſary to meaſure ſome line that will determine its poſition. Thus, the firſt line *ab*, being the baſe of a triangle, is always determined; but the poſition of the ſecond ſide *bj*, does not become determined, till the third ſide *jb* is meaſured; then the triangle may be conſtructed, and the poſition of both is determined.

At the beginning of a line, to fix a looſe line to the mark or place meaſured from, the ſign of turning to the right or left hand muſt be added (as at *j* in the third line); otherwiſe a ſtranger, when laying down the work; may as eaſily conſtruct the triangle *bjb* on the wrong ſide of the line *ab*, as on the right one: but this error cannot be fallen into, if the ſign above named be carefully obſerved.

In chooſing a line to fix a looſe one, care muſt be taken that it does not make a very acute or obtuſe angle; as in the triangle *p*B*r*, by the angle at B being very obtuſe, a ſmall deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when conſtructed, very conſiderable; but by conſtructing the triangle *p*B*q*, ſuch a deviation is of no conſequence.

Where the words *leave off* are written in the field-book, it is to ſignify that the taking of offsets is from thence diſcontinued; and of courſe ſomething is wanting between that and the next offset.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper-plates.

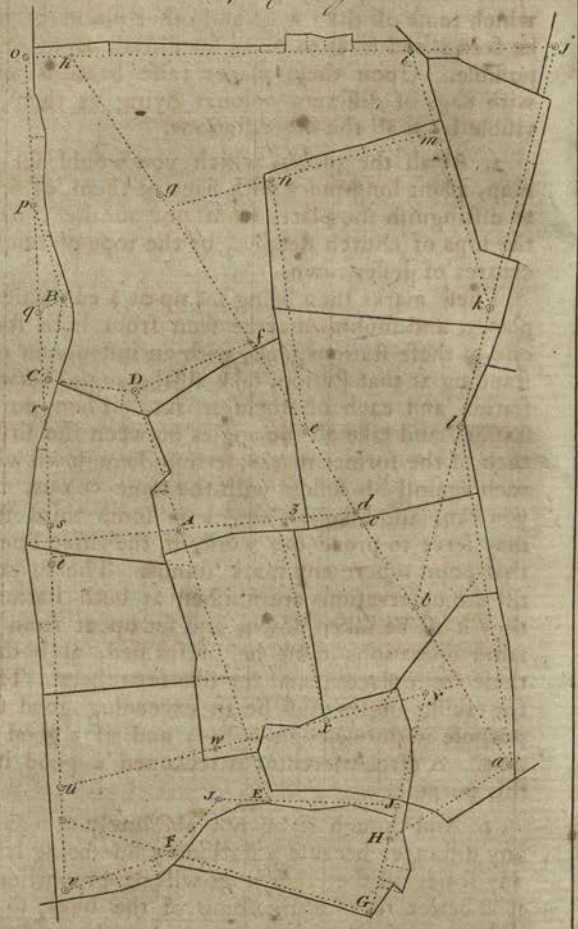
	1310	56. to c
	836	56
n	684	50
	1480	90 to g
	960	24
	930	n
	700	48
m	400	30
	1430	to i
	1290	40
	1004	36
	980	m
	610	24
k	280	32
	1794	to l
	1464	22
49°	1050	
	920	32
	630	60
	350	48
a	0	14
	3074	to b
	2494	
	2100	l
0	2072	
54	1730	
80	1530	
	1420	k
56+30	1170	
52	020	
j 32	280	40
	2574	j
	2494	
	2000	44
	1880	50
	1840	
50	1794	i
34+ 50	1464	
76	1328	
96	1240	
52 + 34	1130	
34	860	
h 66	190	
	4450	h
	3570	g
	2020	f
	2500	
	2210	
	2080	e
	1574	d
	1550	
	1510	c
	990	b
a	806	

		768	to A
		526	70
		496	
	70	460	
	40	124	
D		100	
		455	D
		400	76
C		48	10
		600	to r
		432	C
	56	160	
B	44	36	
B		152	to q
		480	B
p	24	160	
	0	1750	
		1600	44 to s
		1028	
		940	A
	44	666	
	70	310	s
d	60	236	
		2148	480 to b
		1950	y
		1836	
	128	1724	
	60	1600	
	30	1480	x
	0	1320	
	50	1110	
		1080	
		990	w
u		750	50
		4440	36
		4420	v
		3884	u
		3380	60
		2992	90
		2692	t
	120	2624	
		2592	
		2500	s leave off
		2070	56
		1900	
		1840	r
	60	1770	
		1320	q
		808	p
leave off	40	650	
	80	360	
o	20	170	
		220	o
h produce by t		190	46

		580	to V
	40	500	
	76	300	
F	76	100	
		360	to F
J	20	150	
		954	J
	15	850	
		730	to E
	30	490	
		340	60
	0	280	
I	20	170	30
		744	to H
		672	0
a	70	450	0
		1160	to y
	32	1000	
		890	
		780	32
		590	40
		570	I
		530	40
		376	H
		256	150 leave off
		190	64
G		144	130
		1076	G
		1070	30
		896	24
		632	
		620	30
180 from u towards v		588	F
		644	to f
D		488	32
		2280	
		2270	E
	20	2230	
	36	2050	
		2030	
		1940	180 to W
		1552	180
		1380	96
		950	110
		860	

Plan of the Field-Book.

Plan from the foregoing Field-book.



PROBLEM XII.

To ſurvey a County, or large Tract of Land.

1. CHOOSE two, three, or four eminent places, for ſtations; ſuch as the tops of high hills or mountains, towers, or church ſteeples, which may be ſeen from one another; from which moſt of the towns and other places of note may alſo be ſeen; and ſo as to be as far diſtant from one another as poſſible. Upon theſe places raiſe beacons, or long poles, with flags of different colours flying at them, ſo as to be viſible from all the other ſtations.

2. At all the places, which you would ſet down in the map, plant long poles with flags at them, of ſeveral colours, to diſtinguiſh the places from one another; fixing them on the tops of church ſteeples, or the tops of houſes, or in the centres of leſſer towns.

Theſe marks then being ſet up at a convenient number of places, and ſuch as may be ſeen from both ſtations; go to one of theſe ſtations, and, with an inſtrument to take angles, ſtanding at that ſtation, take all the angles between the other ſtation and each of theſe marks. Then go to the other ſtation, and take all the angles between the firſt ſtation and each of the former marks, ſetting them down with the others, each againſt his fellow with the ſame colour. You may, if you can, alſo take the angles at ſome third ſtation, which may ſerve to prove the work, if the three lines interſect in that point where any mark ſtands. The marks muſt ſtand till the obſervations are finiſhed at both ſtations; and then they muſt be taken down, and ſet up at freſh places. The ſame operations muſt be performed, at both ſtations, for theſe freſh places; and the like for others. The inſtrument for taking angles muſt be an exceeding good one, made on purpoſe with teleſcopic ſights; and of a good length of radius. A circumferentor is reckoned a good inſtrument for this purpoſe.

3. And though it be not abſolutely neceſſary to meaſure any diſtance, becauſe a ſtationary line being laid down from any ſcale, all the other lines will be proportional to it; yet it is better to meaſure ſome of the lines, to aſcertain the diſtances of places in miles, and to know how many geometrical miles there are in any length; as alſo from thence to make a ſcale to meaſure any diſtance in miles. In meaſuring any diſtance, it will not be exact enough to go along the high roads; by reaſon of their turnings and windings,

hardly ever lying in a right line between the ſtations, which muſt cauſe infinite reductions, and create endleſs trouble to make it a right line; for which reaſon it can never be exact. But a better way is to meaſure in a right line with a chain, between ſtation and ſtation, over hills and dales or level fields, and all obſtacles. Only in caſe of water, woods, towns, rocks, banks, &c. where one cannot paſs, ſuch parts of the line muſt be meaſured by the methods of inacceſſible diſtances; and beſides, allowing for aſcents and deſcents, when they are met with. A good compaſs that ſhews the bearing of the two ſtations, will always direct you to go ſtraight, when you do not ſee the two ſtations; and in the progreſs, if you can go ſtraight, offsets may be taken to any remarkable places, likewiſe noting the interſection of the ſtation line with all roads, rivers, &c.

4. From all the ſtations, and in the whole progreſs, be very particular in obſerving ſea-coaſts, river-mouths, towns, caſtles, houſes, churches, mills, trees, rocks, ſands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, ſluices, floodgates, locks, &c. and in general every thing that is remarkable.

5. After you have done with the firſt and main ſtation lines, which command the whole county; you muſt then take inner ſtations, at ſome places already determined; which will divide the whole into ſeveral partitions: and from theſe ſtations you muſt determine the places of as many of the remaining towns as you can. And if any remain in that part, you muſt take more ſtations, at ſome places already determined; from which you may determine the reſt. And thus go through all the parts of the county, taking ſtation after ſtation, till we have determined all we want. And in general the ſtation diſtances muſt always paſs through ſuch remarkable points as have been determined before, by the former ſtations.

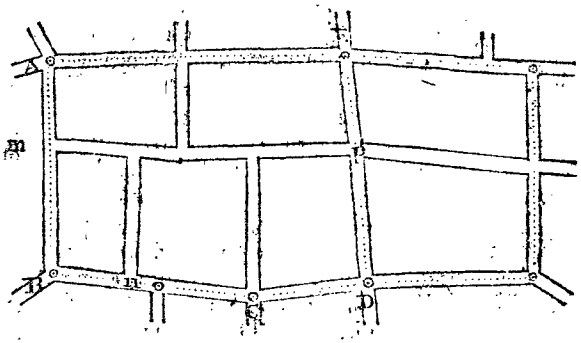
PROBLEM XIII.

To ſurvey a Town or City.

This may be done with any of the inſtruments for taking angles, but beſt of all with the plain table, where every minute part is drawn while in ſight. It is beſt alſo to have a chain of 50 feet long, divided into 50 links of one foot each, and an offset-ſtaff of 10 feet long.

Begin

Begin at the meeting of two or more of the principal ſtreets, through which you can have the longeſt proſpects, to get the longeſt ſtation lines: There having fixed the inſtrument, draw lines of direction along thoſe ſtreets, uſing two men as marks, or poles ſet in wooden pedeſtals; or perhaps ſome remarkable places in the houſes at the further ends, as windows, doors, corners, &c. Meaſure theſe lines with the chain, taking offsets with the ſtaff, at all corners of ſtreets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houſes, &c. Then remove the inſtrument to another ſtation, along one of theſe lines; and there repeat the ſame proceſs as before. And ſo on till the whole is finiſhed.



Thus, fix the inſtrument at A, and draw lines in the direction of all the ſtreets meeting there; then meaſure AB, noting the ſtreet on the left at m. At the ſecond ſtation B, draw the directions of the ſtreets meeting there; and meaſure from B to C, noting the places of the ſtreets at n and o as you paſs by them. At the 3d ſtation C, take the direction of all the ſtreets meeting there, and meaſure CD. At D do the ſame, and meaſure DE, noting the place of the croſs ſtreets at p. And in this manner go through all the principal ſtreets. This done, proceed to the ſmaller and intermediate ſtreets; and laſtly, to the lanes, alleys, courts, yards, and every part that it may be thought proper to repreſent in the plan.

SECTION III.

Of PLANNING, COMPUTING, and DIVIDING.

PROBLEM I.

To lay down the Plan of any Survey.

If the ſurvey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the ſurvey was with any other inſtrument, a plan of it is to be drawn from the meaſures that were taken in the ſurvey, and firſt of all a rough plan on paper.

To do this, you muſt have a ſet of proper inſtruments, for laying down both lines and angles, &c. as ſcales of various ſizes, the more of them, and the more accurate, the better; ſcales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal ſcales are beſt for the lines, becauſe they extend to three figures, or chains and links, which are hundredth parts of chains. But in uſing the diagonal ſcale, a pair of compaſſes muſt be employed to take off the lengths of the principal lines very accurately. But a ſcale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of thoſe offsets upon the ſtation line; which is done at only one application of the edge of the ſcale to that line, and then pricking off all at once the diſtances along it. Angles are to be laid down either with a good ſcale of chords, which is perhaps the moſt accurate way; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles muſt be laid down on the plan in the ſame order in which they were meaſured in the field, and in which they are written in the field-book; laying down firſt the angles for the poſition of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themſelves, all with dry or obſcure lines; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or cloſe properly, proceed next to the ſmaller objects, till you have entered every thing that ought to appear in the plan, as houſes, brooks, trees, hills, gates, ſtiles, roads, lanes, mills, bridges, woodlands, &c. &c.

The

The north ſide of a map or plan is commonly placed uppermoſt, and a meridian ſomewhere drawn, with the compaſs or flower-de-luce pointing north. Alſo, in a vacant part, a ſcale of equal parts or chains is drawn, with the title of the map in conſpicuous characters, and embelliſhed with a compartment. Hills are ſhadowed, to diſtinguiſh them in the map. Colour the hedges with different colours; repreſent hilly grounds by broken hills and valleys; draw ſingle dotted lines for foot-paths, and double ones for horſe or carriage roads. Write the name of each field and remarkable place within it, and, if you chooſe, its content in acres, roods, and perches.

In a very large eſtate, or a county, draw vertical and horizontal lines through the map, denoting the ſpaces between them by letters placed at the top, and bottom, and ſides, for readily finding any field or other object mentioned in a table.

In mapping counties, and eſtates that have uneven grounds of hills and valleys, reduce all oblique lines, meaſured uphill and down-hill, to horizontal ſtraight lines, if that was not done during the ſurvey, before they were entered in the field-book, by making a proper allowance to ſhorten them. For which purpoſe there is commonly a ſmall table engraven on ſome of the inſtruments for ſurveying.

PROBLEM II.

To compute the Contents of Fields.

1. COMPUTE the contents of the figures, whether triangles, or trapeziums, &c, by the proper rules for the ſeveral figures laid down in meaſuring; multiplying the lengths by the breadths, both in links, and divide by 2; the quotient is acres, after you have cut off five figures on the right for decimals. Then bring theſe decimals to roods and perches, by multiplying firſt by 4, and then by 40. An example of which is given in the deſcription of the chain, pag. 50.

2. In ſmall and ſeparate pieces, it is uſual to caſt up their contents from the meaſures of the lines taken in ſurveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, meaſured by offsets, all the parts between the offsets are moſt accurately meaſured ſeparately as ſmall trapezoids.

4. Sometimes ſuch pieces as that laſt mentioned, are computed by finding a mean breadth, by dividing the ſum of the offsets

offsets by the number of them, accounting that for one of them where the boundary meets the ſtation line; then multiply the length by that mean breadth.

But this method is commonly in ſome degree erroneous.

5. But in larger pieces, and whole eſtates, conſiſting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the meaſures of the lines and angles that were taken in ſurveying. For then new lines are drawn in the fields in the plan, ſo as to divide them into trapeziums and triangles, the baſes and perpendiculars of which are meaſured on the plan by means of the ſcale from which it was drawn, and ſo multiplied together for the contents. In this way, the work is very expeditiouſly done, and ſufficiently correct; for ſuch dimenſions are taken as afford the moſt eaſy method of calculation; and, among a number of parts, thus taken and applied to a ſcale, it is likely that ſome of the parts will be taken a ſmall matter too little, and others too great; ſo that they will, upon the whole, in all probability, very nearly balance one another. After all the fields, and particular parts, are thus computed ſeparately, and added all together into one ſum; calculate the whole eſtate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add theſe alſo together. Then if this ſum be equal to the former, or nearly ſo, the work is right; but if the ſums have any conſiderable difference, it is wrong, and they muſt be examined, and recomputed, till they nearly agree.

6. But the chief ſecret in computing, conſiſts in finding the contents of pieces bounded by curved or very irregular lines, or in reducing ſuch crooked ſides of fields or boundaries to ſtraight lines, that ſhall incloſe the ſame or equal area with thoſe crooked ſides, and ſo obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked ſides to ſtraight ones, is very eaſily and accurately performed in this manner:—Apply the ſtraight edge of a thin, clear piece of lanthorn-horn to the crooked line which is to be reduced, in ſuch a manner, that the ſmall parts cut off from the crooked figure by it, may be equal to thoſe which are taken in: which equality of the parts included and excluded you will preſently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the ſtraight edge of the horn. Do the ſame by the other ſides of the field or figure. So ſhall you have a ſtraight

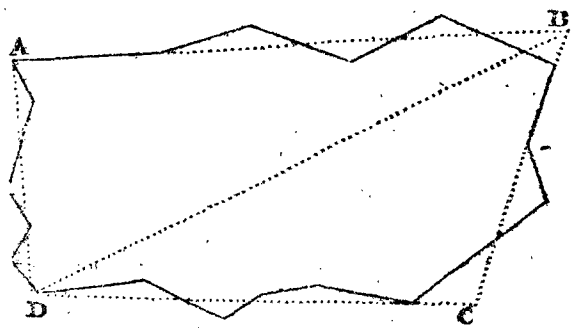
ſided

ſided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure propoſed.

Or, inſtead of the ſtraight edge of the horn, a horſe-hair may be applied acroſs the crooked ſides in the ſame manner; and the eaſieſt way of uſing the hair, is to ſtring a ſmall ſlender bow with it, either of wire, or cane, or whale-bone, or ſuchlike ſlender elaſtic matter; for, the bow keeping it always ſtretched, it can be eaſily and neatly applied with one hand, while the other is at liberty to make two marks by the ſide of it, to draw the ſtraight line by.

EXAMPLE.

Thus, let it be required to find the contents of the ſame figure as in prob. IX of the laſt ſection, pag. 65, to a ſcale of 4 chains to an inch.



Draw the four dotted ſtraight lines AB, BC, CD, DA, cutting off equal quantities on both ſides of them, which they do as near as the eye can judge: ſo is the crooked figure reduced to an equivalent right-lined one of four ſides ABCD. Then draw the diagonal BD, which, by applying a proper ſcale to it, meaſures 1256. Alſo the perpendicular, or neareſt diſtance, from A to this diagonal, meaſures 456; and the diſtance of C from it, is 428.

Then, half the ſum of 456 and 428, multiplied by the diagonal 1256, gives 555152 ſquare links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

PROBLEM III.

To transfer a Plan to another Paper, &c.

AFTER the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

Firſt Method.—Lay the rough plan on the clean paper, keeping them always preſſed flat and cloſe together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them aſunder, and connect the pricked points, on the clean paper, with lines; and it is done. This method is only to be practiſed in plans of ſuch figures as are ſmall and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black lead powder; and lay the ſaid black part on the clean paper on which the plan is to be copied, and in the proper poſition. Then with the blunt point of ſome hard ſubſtance, as braſs, or ſuch like, trace over the lines of the whole plan; preſſing the tracer ſo much as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink —Or, inſtead of blacking the rough plan, you may keep conſtantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of ſquares. This is performed by dividing both ends and ſides of the plan which is to be copied, into any convenient number of equal parts, and connecting the correſponding points of diviſion with lines; which will divide the plan into a number of ſmall ſquares. Then divide the paper, upon which the plan is to be copied, into the ſame number of ſquares, each equal to the former when the plan is to be copied of the ſame ſize, but greater or leſs than the others, in the proportion in which the plan is to be increaſed or diminiſhed, when of a different ſize. Laſtly, copy into the clean ſquares the parts contained in the correſponding ſquares of the old plan; and you will have the copy, either of the ſame ſize, or greater or leſs in any proportion.

Fourth Method.—A fourth method is by the inſtrument called a pentagraph, which alſo copies the plan in any ſize required.

Fifth Method.—But the neateſt method of any is this. Procure a copying frame or glaſs, made in this manner; namely, a large ſquare of the beſt window glaſs, ſet in a broad frame of wood, which can be raiſed up to any angle, when the lower ſide of it reſts on a table. Set this frame up to any angle before you, facing a ſtrong light; fix the old plan and clean paper together with ſeveral pins quite around, to keep them together, the clean paper being laid uppermoſt, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glaſs, namely, that part which you intend to begin at to copy firſt; and, by means of the light ſhining through the papers, you will very diſtinctly perceive every line of the plan through the clean paper. In this ſtate then trace all the lines on the paper with a pencil. Having drawn that part which covers the glaſs, ſlide another part over the glaſs, and copy it in the ſame manner. Then another part. And ſo on, till the whole is copied.

Then, take them aſunder, and trace all the pencil-lines over with a fine pen and Indian ink, or with common ink.

And thus you may copy the fineſt plan, without injuring it in the leaſt.

When the lines are copied on the clean paper, the next buſineſs is to write ſuch names, remarks, or explanations as may be judged neceſſary; laying down the ſcale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; illuſtrating or colouring every part in the manner that ſhall ſeem moſt natural, ſuch as ſhading rivers or brooks with crooked lines; drawing the repreſentations of trees, buſhes, hills, woods, hedges, houſes, gates, roads, &c, in their proper places; running a ſingle dotted line for a foot-path, and a double one for a carriage road; and either repreſenting the baſes or the elevations of buildings, &c.

OF ARTIFICERS' WORKS AND TIMBER MEASURING.

I. OF THE CARPENTER'S OR SLIDING RULE.

THE Carpenter's or Sliding Rule, is an inſtrument much uſed in meaſuring of timber and artificers' works, both for taking the dimenſions, and computing the contents.

The inſtrument conſiſts of two equal pieces, each a foot in length, which are connected together by a folding joint.

One ſide or face of the rule, is divided into inches, and eighths, or half quarters. On the ſame face alſo are ſeveral plane ſcales, divided into twelfth parts by diagonal lines; which are uſed in planning dimenſions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into 10 equal parts, and each of theſe into 10 parts again: ſo that by means of this laſt ſcale, dimenſions are taken in feet, tenths and hundredths, and multiplied as common decimal numbers, which is the beſt way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a ſlider, which runs in a groove made in the ſtock. The ſame numbers ſerve for both theſe two middle lines, the one being above the numbers, and the other below.

Theſe four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or loweſt line D, is a ſingle one, proceeding from 4 to 40. It is alſo called the girt line, from its uſe in computing the contents of trees and timber; and upon it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this inſtrument ſerve the purpoſe of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 ſhillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is accounted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and ſo on. And all the ſmaller diviſions are altered proportionally.

II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by ſeveral different meaſures. As,

Glazing and maſonry by the foot;

Painting, plaſtering, paving, &c, by the yard, of 9 ſquare feet;

Flooring, partitioning, roofing, tiling, &c, by the ſquare, of 100 ſquare feet.

And brickwork, either by the yard of 9 ſquare feet, or by the perch, or ſquare rod or pole, containing $272\frac{1}{4}$ ſquare feet, or $30\frac{1}{4}$ ſquare yards, being the ſquare of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is troubleſome to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272. But when the exact diviſor $272\frac{1}{4}$ is to be uſed, it will be eaſier to multiply the feet by 4, and then divide ſucceſſively by 9, 11, and 11. Alſo to divide ſquare yards by $30\frac{1}{4}$, firſt multiply them by 4, and then divide twice by 11.

All works, whether ſuperficial or ſolid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

III. BRICKLAYERS' WORK.

BRICKWORK is eſtimated at the rate of a brick and a half thick. So that, if a wall be more or leſs than this ſtandard thickneſs, it muſt be reduced to it, as follows:

Multiply the ſuperficial content of the wall by the number of half bricks in the thickneſs, and divide the product by 3.

The dimenſions of a building are uſually taken by meaſuring half round on the outſide and half round it on the inſide; the ſum of theſe two gives the compaſs of the wall, to be multiplied by the height, for the content of the materials.

Chimneys are by ſome meaſured as if they were ſolid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or meaſured round for their breadth, and the height of the ſtory is their height, taking the depth of the jambs for their thickneſs. And in this caſe, no deduction is made for the vacuity from the floor to the mantle-tree, becauſe of the gathering of the breaſt and wings, to make room for the hearth in the next ſtory.

To meaſure the chimney ſhafts, which appear above the building; girt them about with a line for the breadth, to multiply by their height. And account their thickneſs half a brick more than it really is, in conſideration of the plaſtering and ſcaffolding.

All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole meaſure is taken for workmanſhip, and that all outſide meaſure too, namely, meaſuring quite round the outſide of the building, being in conſideration of the trouble of the returns or angles. There are alſo ſome other allowances, ſuch as double meaſure for feathered gable ends, &c.

EXAMPLES.

EXAM. I. How many yards and rods of ſtandard brickwork are in a wall whoſe length or compaſs is 57 feet 3 inches, and height 24 feet 6 inches; the walls being $2\frac{1}{2}$ bricks or 5 half bricks thick? Anſ. 8 rods, $17\frac{1}{3}$ yards.

EXAM. II. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick?

Anſ. 169·753 yards.

EXAM. III. A triangular gable is raiſed $17\frac{1}{2}$ feet high, on an end wall whoſe length is 24 feet 9 inches, the thickneſs being 2 bricks; required the reduced content? Anſ. 32·08$\frac{1}{3}$ yards.

EXAM. IV. The end wall of a houſe is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, which riſes 42 courſes of bricks, of which every 4 courſes make a foot. What is the whole content in ſtandard meaſure? Anſ. 253·626 yards.

IV. MA-

IV. MASONS' WORK.

To maſonry belong all ſorts of ſtone-work; and the meaſure made uſe of is a foot, either ſuperficial or ſolid.

Walls, columns, blocks of ſtone or marble, &c, are meaſured by the cubic foot; and pavements, ſlabs, chimney-pieces, &c, by the ſuperficial or ſquare foot.

Cubic or ſolid meaſure is uſed for the materials, and ſquare meaſure for the workmanſhip.

In the ſolid meaſure, the true length, breadth, and thickneſs, are taken, and multiplied continually together. In the ſuperficial, there muſt be taken the length and breadth of every part of the projection, which is ſeen without the general upright face of the building.

EXAMPLES.

EXAM. I. Required the ſolid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick?

Anſ. $1310\frac{3}{4}$ feet.

EXAM. II. What is the ſolid content of a wall, the length being 12 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Anſ. 521·375 feet.

EXAM. III. Required the value of a marble ſlab, at 8s. per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches? Anſ. 4l. 1s. $10\frac{1}{2}$d.

EXAM. IV. In a chimney-piece, ſuppoſe the

length of the mantle and ſlab, each	4 feet	6 inches
breadth of both together -	3	2
length of each jamb - -	4	4
breadth of both together -	1	9

Required the ſuperficial content? Anſ. 21 feet, 10 inches.

V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a houſe, ſuch as flooring, partitioning, roofing, &c.

Large and plain articles are uſually meaſured by the ſquare foot or yard, &c; but enriched mouldings, and ſome other articles, are often eſtimated by running or lineal meaſure, and ſome things are rated by the piece.

In meaſuring of joiſts, it is to be obſerved, that only one of their dimenſions is the ſame with that of the floor; for the other exceeds the length of the room by the thickneſs of the wall, and $\frac{1}{3}$ of the ſame, becauſe each end is let into the wall about $\frac{2}{3}$ of its thickneſs.

No deductions are made for hearths, on account of the additional trouble and waſte of materials.

Partitions are meaſured from wall to wall for one dimenſion, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In meaſuring of joiners' work, the ſtring is made to ply cloſe to every part of the work over which it paſſes.

The meaſure of centering for cellars is found, by making a ſtring paſs over the ſurface of the arch for the breadth, and taking the length of the cellar for the length; but in groin centering, it is uſual to allow double meaſure, on account of their extraordinary trouble.

In roofing, the length of the houſe in the inſide, together with $\frac{2}{3}$ of the thickneſs of one gable, is to be conſidered as the length; and the breadth is equal to double the length of a ſtring which is ſtretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

For ſtair-caſes, take the breadth of all the ſteps, by making a line ply cloſe over them, from the top to the bottom, and multiply the length of this line by the length of a ſtep, for the whole area.—By the length of a ſtep is meant the length of the front and the returns at the two ends; and by the breadth, is to be underſtood the girt of its two outer ſurfaces, or the tread and riſer.

For the baluſtrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel poſt, for the length; and twice the length of the baluſter upon the landing, with the girt of the hand-rail, for the breadth.

For wainſcotting, take the compaſs of the room for the length; and the height from the floor to the ceiling, making the ſtring ply cloſe into all the mouldings, for the breadth.—Out of this muſt be made deductions for windows, doors, and chimneys, &c; but workmanſhip is counted for the whole, on account of the extraordinary trouble.

For doors, it is uſual to allow for their thickneſs, by adding it into both the dimenſions of length and breadth, and then multiply

multiply them together for the area.—If the door be panneled on both ſides, take double its meaſure for the workmanſhip; but if one ſide only be panneled, take the area and its half for the workmanſhip.—*For the ſurrounding architrave*, gird it about the outermoſt part for its length; and meaſure over it, as far as it can be ſeen when the door is open, for the breadth.

Window-ſhutters, *baſes*, &c. are meaſured in the ſame manner.

In the meaſuring of roofing for workmanſhip alone, holes for chimney ſhafts and ſky-lights are generally deducted.

But in meaſuring for work and materials, they commonly meaſure in all ſky-lights, luthern-lights, and holes for the chimney ſhafts, on account of their trouble and waſte of materials.

EXAMPLES.

EXAM. I. Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad? Anſ. 11 ſq. 76 feet.

EXAM. II. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many ſquares are in it?

Anſ. 5 ſq. 98$\frac{1}{8}$ feet.

EXAM. III. How many ſquares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Anſ. 18·3972 ſquares.

EXAM. IV. What coſt the roofing of a houſe at 10s. 6d. a-ſquare; the length, within the walls, being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Anſ. 12l. 12s. 11$\frac{3}{4}$.

EXAM. V. To how much at 6s. per ſquare yard, amounts the wainſcotting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compaſs 83 feet 8 inches; alſo the three window ſhutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and ſhutters, being worked on both ſides, are reckoned work and half work?

Anſ. 36l. 12s. 2$\frac{1}{2}$.

VI. SLATERS' AND TILERS' WORK.

IN theſe articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of ſlates at the bottom, or for how much one row of ſlates or tiles is laid over another.

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building with its half added, is the girt over both ſides.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney ſhafts or window holes.

EXAMPLES.

EXAM. I. Required the content of a ſlated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches?

Anſ. 174$\frac{1}{16}$ yards.

EXAM. II. To how much amounts the tiling of a houſe, at 25s. 6d. per ſquare; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, alſo the eaves projecting 16 inches on each ſide, and the roof of a true pitch?

Anſ. 24l. 9s. 5$\frac{1}{2}$d.

VII. PLASTERERS' WORK.

PLASTERERS' work is of two kinds, namely, ceiling, which is plaſtering upon laths; and rendering, which is plaſtering upon walls: which are meaſured ſeparately.

The contents are eſtimated either by the foot or yard, or ſquare of 100 feet. Inriched mouldings, &c, are rated by running or lineal meaſure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are ſeldom deducted, as the plaſtered returns at the top and ſides are allowed to compenſate for the window opening.

EXAMPLES.

EXAM. I. How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Anſ. 122$\frac{1}{2}$.

EXAM. II. To how much amounts the ceiling of a room, at 10d. per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches?

Anſ. 1l. 9s. 8$\frac{3}{4}$d.

EXAM. III. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8d. and the latter at 3d. per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet ſquare?

Anſ. 8l. 5s. 6$\frac{1}{2}$d.

EXAM.

EXAM. IV. Required the quantity of plaſtering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under ſide of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Anſ. 53 yards 5 feet 3 inches of rendering
18 5 6 of ceiling
39 $0\frac{11}{12}$ of cornice.

VIII. PAINTERS' WORK.

PAINTERS' work is computed in ſquare yards. Every part is meaſured where the colour lies; and the meaſuring line is forced into all the mouldings, and corners.

Windows are done at ſo much a piece. And it is uſual to allow double meaſure for carved mouldings, &c.

EXAMPLES.

EXAM. I. How many yards of painting contains the room which is 65 feet 6 inches in compaſs, and 12 feet 4 inches high? Anſ. $89\frac{3}{4}$ yards.

EXAM. II. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Anſ. $73\frac{2}{27}$ yards.

EXAM. III. What coſt the painting of a room, at 6d. per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; alſo the door is 7 feet by 3 feet 6, and the window ſhutters to two windows each 7 feet 9 by 3 feet 6, but the breaks of the windows themſelves are 8 feet 6 inches high, and 1 foot 3 inches deep: deducting the fire-place of 5 feet by 5 feet 6?

Anſ. 3l. 3s. $10\frac{1}{2}$d.

IX. GLAZIERS' WORK.

GLAZIERS take their dimenſions, either in feet, inches and parts, or feet, tenths, and hundredths. And they compute their work in ſquare feet.

In taking the length and breadth of a window, the croſs bars between the ſquares are included. Alſo windows of round or oval forms are meaſured as ſquare, meaſuring them to their greateſt length and breadth, on account of the waſte in cutting the glaſs.

EXAMPLES.

EXAM. I. How many ſquare feet contains the window which is 4·25 feet long, and 2·75 feet broad? Anſ. $11\frac{2}{3}$.

EXAM. II. What will the glazing a triangular ſky-light come to, at 10d. per foot; the baſe being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Anſ. 4l. 7s. $2\frac{1}{4}$d.

EXAM. III. There is a houſe with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches;

now the height of the firſt tier is	7 feet	10 inches
of the ſecond	6	8
of the third	5	4

Required the expence of glazing at 14d. per foot?

Anſ. 13l. 11s. $10\frac{1}{2}$d.

EXAM. IV. Required the expence of glazing the windows of a houſe at 13d. a foot; there being three ſtories, and three windows in each ſtory;

the height of the lower tier is	7 feet	9 inches
of the middle	6	6
of the upper	5	$3\frac{1}{4}$
and of an oval window over the door	1	$10\frac{1}{2}$

the common breadth of all the windows being 3 feet 9 inches? Anſ. 12l. 5s. 6d.

X. PAVERS' WORK.

PAVERS' work is done by the ſquare yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

EXAM. I. What coſt the paving a foot-path at 3s. 4d. a-yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Anſ. 5l. 7s. $11\frac{1}{2}$d.

EXAM. II. What coſt the paving a court, at 3s 2d. per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Anſ. 7l. 4s. $4\frac{1}{2}$d.

EXAM. III.

EXAM. III. What will be the expence of paving a rectangular court-yard, whoſe length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad ſtones, at 3s. a-yard; the reſt being paved with pebbles at 2s. 6d. a-yard?

Anſ. 40l. 5s. $10\frac{1}{2}$d.

XI. PLUMBERS' WORK.

PLUMBERS' work is rated at ſo much a pound, or elſe by the hundred weight, of 112 pounds

Sheet lead uſed in roofing, guttering, &c, is from 7 to 12lb. to the ſquare foot. And a pipe of an inch bore, is commonly 13 or 14lb. to the yard in length.

EXAMPLES.

EXAM. I. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb. to the ſquare foot?

Anſ. $1091\frac{2}{11}$lb.

EXAM. II. What coſt the covering and guttering a roof with lead, at 18s the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb, and the latter 7·373lb. to the ſquare foot?

Anſ. 115l. 9s. $1\frac{1}{2}$d.

XII. TIMBER MEASURING.

PROBLEM I.

To find the Area, or Superficial Content, of a Board or Plank.

MULTIPLY the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the ſum for the mean breadth.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then againſt the length in feet on B, is the content on A, in feet and fractional parts.

EXAM-

EXAMPLES.

EXAM. I. What is the value of a plank, at $1\frac{1}{2}$d. per foot, whoſe length is 12 feet 6 inches, and mean breadth 11 inches? Anſ. 1s. 5d.

EXAM. II. Required the content of a board, whoſe length is 11 feet 2 inches, and breadth 1 foot 10 inches? Anſ. 20 feet 5 inches 8″.

EXAM. III. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}$d. a foot? Anſ. 3s. $3\frac{3}{4}$d.

EXAM. IV. Required the value of 5 oaken planks at 3d. per foot, each of them being $17\frac{1}{2}$ feet long; and their ſeveral breadths are as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower. Anſ. 1l. 5s. $8\frac{1}{4}$d.

PROBLEM II.

To find the Solid Content of Squared or Four-ſided Timber.

MULTIPLY the mean breadth by the mean thickneſs, and the product again by the length, and the laſt product will give the content.

By the Sliding Rule.

C D D C

As length : 12 or 10 :: quarter girt : ſolidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; ſo is the quarter girt on D, to the content on C.

Note, 1. If the tree taper regularly from the one end to the other; either take the mean breadth and thickneſs in the middle, or take the dimenſions at the two ends, and half their ſum will be the mean dimenſions.

2. If the piece do not taper regularly, but is unequally thick in ſome parts and ſmall in others, take ſeveral different dimenſions, add them all together, and divide their ſum by the number of them, for the mean dimenſions.

3. The quarter girt is a geometrical mean proportional between the mean breadth and thickneſs, that is the ſquare root of their product. Sometimes unſkilful meaſurers uſe the arithmetical mean inſtead of it, that is half their ſum; but

but this is always attended with error, and the more ſo as the breadth and depth differ the more from each other.

EXAMPLES.

EXAM. I. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and leſs end 1 foot 6 inches and 1 foot 3 inches, and the thickneſs at the greater and leſs end 1 foot 3 inches and 1 foot: required the ſolid content? Anſ. 28 feet 7 inches.

EXAM. II. What is the content of the piece of timber, whoſe length is 24½ feet, and the mean breadth and thickneſs each 1·04 feet? Anſ. 26½ feet.

EXAM. III. Required the content of a piece of timber, whoſe length is 20·38 feet, and its ends unequal ſquares, the ſide of the greater being 19⅛, and the ſide of the leſs 9⅞? Anſ. 29·838 feet.

EXAM. IV. Required the content of the piece of timber, whoſe length is 27·36 feet; at the greater end the breadth is 1·78, and thickneſs 1·23; and at the leſs end the breadth is 1·04, and thickneſs 0·91? Anſ. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unſquared Timber.

MULTIPLY the ſquare of the quarter girt, or of ¼ of the mean circumference, by the length, for the content.

By the Sliding Rule.

As the length upon C : 12 or 10 upon D :: quarter girt, in 12ths or 10ths, on D : content on C.

Notes; 1. When the tree is tapering, take the mean dimenſions as in the former problems, either by gitting it in the middle, for the mean girt, or at the two ends, and take half the ſum of the two. But when the tree is very irregular, divide it into ſeveral lengths, and find the content of each part ſeparately.

2. This rule, which is commonly uſed, gives the anſwer about ¼ leſs than the true quantity in the tree, or nearly what the quantity would be after the tree is hewed ſquare in the uſual way: ſo that it ſeems intended to make an allowance for the ſquaring of the tree.

EXAM-

EXAMPLES.

EXAM. I. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Anſ. $116\frac{1}{3}$ feet.

EXAM. II. The length of a tree is 24 feet, its girt at the thicker end 6 feet, and at the ſmaller end 2 feet; required the content? Anſ. 96 feet.

EXAM. III. What is the content of a tree, whoſe mean girt is 3·15 feet, and length 14 feet 6 inches?
Anſ. 8·9929 feet.

EXAM. IV. Required the content of a tree, whoſe length is $17\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the firſt place 9·43 feet, in the ſecond 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16?
Anſ. 42·6075.

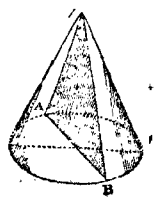
CONIC SECTIONS.

DEFINITIONS.

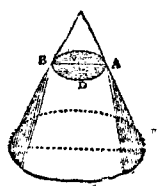
1. CONIC SECTIONS are the figures made by the mutual interſection of a cone and a plane.

2. According to the different poſitions of the cutting plane, there ariſe five different figures or ſections, namely, a triangle, a circle, an ellipſe, an hyperbola, and a parabola: the three laſt of which only are peculiarly called Conic Sections.

3. If the cutting plane paſs through the vertex of the cone, and any part of the baſe, the ſection will evidently be a triangle; as VAB.

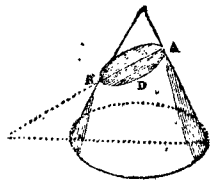


4. If the plane cut the cone parallel to the baſe, or make no angle with it, the ſection will be a circle; as ABD.

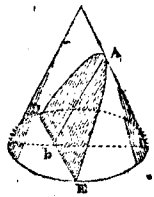


5. The

5. The ſection DAB is an ellipſe, when the cone is cut obliquely through both ſides, or when the plane is inclined to the baſe in a leſs angle than the ſide of the cone is.

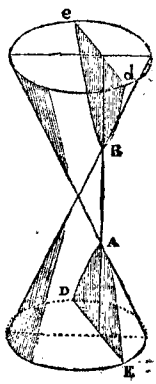


6. The ſection is a parabola, when the cone is cut by a plane parallel to the ſide, or when the cutting plane and the ſide of the cone make equal angles with the baſe.



7. The ſection is an hyperbola, when the cutting plane makes a greater angle with the baſe than the ſide of the cone makes.

8. And if all the ſides of the cone be continued through the vertex, forming an oppoſite equal cone, and the plane be alſo continued to cut the oppoſite cone, this latter ſection will be the oppoſite hyperbola to the former; as dBe.



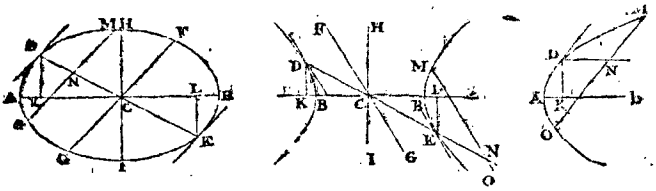
9. The vertices of any ſection, are the points where the cutting plane meets the oppoſite ſides of the cone, or the ſides of the vertical triangular ſection; as A and B.

Hence the ellipſe and the oppoſite hyperbolas, have each two vertices; but the parabola only one; unleſs we conſider the other as at an infinite diſtance.

10. The Axis, or Tranſverſe Diameter, of a conic ſection, is the line or diſtance AB between the vertices.

Hence the axis of a parabola is infinite in length, Ab being only a part of it.

Ellipſe. Oppoſ. Hyperb. Parabola.



11. The Centre C is the middle of the axis.

Hence the centre of a parabola is infinitely diſtant from the vertex. And of an ellipſe, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as AB or DE, drawn through the centre, and terminated on each ſide by the curve; and the extremities of the diameter, or its interſections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence alſo every diameter of the ellipſe and hyperbola have two vertices; but of the parabola, only one; unleſs we conſider the other as at an infinite diſtance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL are ordinates to the axis AB; and MN, NO ordinates to the diameter DE.

Hence the ordinates of the axis are perpendicular to it.

15. An Abſciſs is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

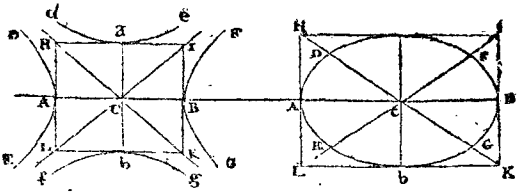
Hence, in the ellipſe and hyperbola, every ordinate has two abſciſſes; but in the parabola, only one; the other vertex of the diameter being infinitely diſtant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the ſemi-parameter.

Hence, the ellipſe and hyperbola have each two foci; but the parabola only one.



18. If DAE, FBG be two oppoſite hyperbolas having AB for their firſt or tranſverſe axis, and ab for their ſecond or conjugate axis. And if dae, fbg be two other oppoſite hyperbolas having the ſame axes, but in the contrary order, namely, ab their firſt axis, and AB their ſecond; then theſe two latter curves dae, fbg, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of oppoſite curves mutually conjugate to the other.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inſcribed rectangle HIKL; the diagonals HCK, ICL of this rectangle, are called the aſymptotes of the curves. And if theſe aſymptotes interſect at right angles, or the inſcribed rectangle be a ſquare, or the two axes AB and ab be equal, then the hyperbolas are ſaid to be right-angled, or equilateral.

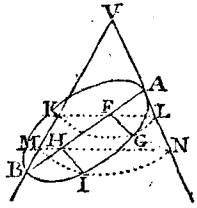
SCHOLIUM.

The rectangle inſcribed between the four conjugate hyperbolas, is ſimilar to a rectangle circumſcribed about an ellipſe by drawing tangents, in like manner, to the four extremities of the two axes; and the aſymptotes or diagonals in the hyperbola, are analagous to thoſe in the ellipſe, cutting this curve in ſimilar points, and making the pair of equal conjugate diameters. Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipſe turned inſide out, cut open at the extremities D, E, F, G, of the ſaid equal conjugate diameters, and thoſe four points drawn out to an infinite diſtance, the curvature being turned the contrary way, but the axes, and the rectangle paſſing through their extremities, continuing fixed.

OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis's are to each other as the Rectangles of their Abſciſſes.



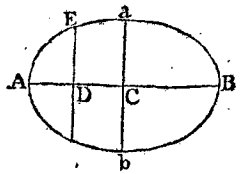
LET AVB be a plane paſſing through the axis of the cone; AGIH another ſection of the cone perpendicular to the plane of the former; AB the axis of this elliptic ſection; and FG, HI ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF . FB : AH . HB$.

For, through the ordinates FG, HI draw the circular ſections KGL, MIN parallel to the baſe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the ellipſe.

Now, by the ſimilar triangles AFL, AHN, and BFK, BHM,
it is AF : AH :: FL : HN,
and FB : HB :: KF : MH;
hence, taking the rectangles of the correſponding terms,
it is, the rect. AF . FB : AH . HB :: KF . FL : MH . HN.
But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;
Therefore the rect. $AF . FB : AH . HB :: FG^2 : HI^2$. Q. E. D.

THEOREM II.

As the Square of the Tranſverſe Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abſciſſes :
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD . DB : DE^2$

For, by theor. 1, $AC : CB : AD . DB :: Ca^2 : DE^2$;
But, if c be the centre, then $AC . CB = AC^2$, and ca is the ſemi-conj.

Therefore

Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

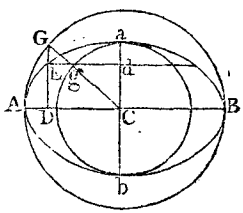
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$,
that is, $AB : p :: AD \cdot DB$ or $CA^2 - CB^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the tranſverſe,
Is to its parameter,
So is the rectangle of the abſciſſes,
To the ſquare of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :
Is to the Square of the Tranſverſe Axis ::
So is the Rectangle of the Abſciſſes of the Conjugate, or the Difference of the Squares of the Semi-conjugate and Diſtance of the Centre from any Ordinate of that Axis :
To the Square of their Ordinate.

That is,
$ca^2 : CB^2 :: ad \cdot db$ or $ca^2 - cd^2 : DE^2$.



For, draw the ordinate ED to the tranſverſe AB.
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CA^2 - CD^2$,
or - - $ca^2 : CA^2 :: cd^2 : CA^2 - dE^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by diviſion, $ca^2 : CA^2 :: ca^2 - cd^2$ or $ad \cdot db : dE^2$.
Q. E. D.

Corol. 1. If two circles be deſcribed on the two axes as diameters, the one inſcribed within the ellipſe, and the other circumſcribed about it; then an ordinate in the circle will be to the correſponding ordinate in the ellipſe, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

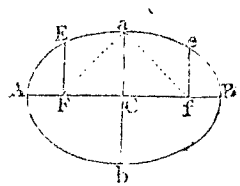
For, by the nature of the circle, AD . DB = DG²; therefore by the nature of the ellipſe, CA² : Ca² :: AD . DB or DG² : DE²,
or CA : Ca :: DG : DE.
In like manner - Ca : CA :: dg : dE.
Moreover, by equality, DG : DE or Cd :: dE or DC : dg.
Therefore cgG is a continued ſtraight line.

Corol. 2. Hence alſo, as the ellipſe and circle are made up of the ſame number of correſponding ordinates, which are all in the ſame proportion of the two axes, it follows that the areas of the whole circle and ellipſe, as alſo of any like parts of them, are in the ſame proportion of the two axes, or as the ſquare of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipſe, are as the ſquare of each axis and the rectangle of the two, and therefore the ellipſe is a mean proportional between the two circles.

THEOREM IV.

The Square of the Diſtance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes;
Or, the Square of the Diſtance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, CF² = CA² — Ca²,
or Ff² = AB² — ab².



For, to the focus F draw the ordinate FE; which, by the definition, will be the ſemi-parameter. Then, by the nature of the curve - CA² : Ca² :: CA² — CF² : FE²;
and by the def. of the para. CA² : Ca² :: Ca² : FE²;
therefore - - Ca² = CA² — CF²;
and by addit. and ſubtr. CF² = CA² — Ca²;
or, by doubling, - Ff² = AB² — ab². Q. E. D.

Corol. 1. The two ſemi-axes, and the focal diſtance from the centre, are the ſides of a right-angled triangle CFa; and the diſtance Fa from the focus to the extremity of the conjugate axis, is = AC the ſemi-tranſverſe.

Corol. 2.

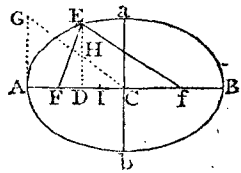
Corol. 2. The conjugate ſemi-axis ca is a mean proportional between AF, FB, or between Af, fB, the diſtances of either focus from the two vertices.

For $ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB$.

THEOREM V.

The Sum of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Tranſverſe Axis.

That is,
FE + fE = AB.



For, draw AG parallel and equal to ca the ſemi-conjugate; and join CG meeting the ordinate DE in H; alſo take CI a 4th proportional to CA, CF, CD.

Then, by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by ſim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
conſequently $DE^2 = AG - DH^2 = Ca^2 - DH^2$.

Alſo $FD = CF \mp CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
And, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But by theor. 4, $CF^2 + Ca^2 = CA^2$,
and, by ſuppoſition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

Again, by ſupp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by ſim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore $CI^2 = CD^2 - DH^2$;
conſequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

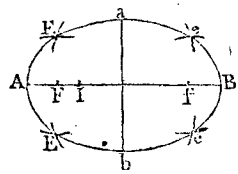
And the root or ſide of this ſquare is $FE = CA - CI = AI$.

In the ſame manner it is found that $fE = CA + CI = BI$.
Conſeq. by addit. $FE + fE = AI + BI = AB$. Q. E. D.

Corol. 1. Hence CI or CA — FE is a 4th proportional to CA, CF, CD.

Corol. 2. And fE — FE = 2CI; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

Corol 2. Hence is derived the common method of deſcribing this curve mechanically by points, or with a thread, thus:



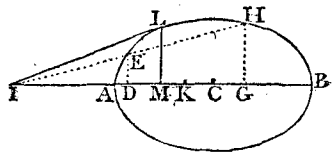
In the tranſverſe take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, deſcribe arcs interſecting in E, which will be a point in the curve. In like manner, aſſuming other points I, as many other points will be found in the curve. Then with a ſteady hand, the curve line may be drawn through all the points of interſection E.

Or, take a thread of the length of AB the tranſverſe axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always ſtretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point I in the Axis produced, a Line IL be drawn touching the curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or Middle of AB: Then ſhall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.



For, from the point I draw any line AEH to cut the curve in two points E and H; from which let fall the perps. ED and HG; and biſect DG in K.

Then, by theor. I, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by ſim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$.
or $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK$,
and, by comp. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA$,
or $CK : CI :: AD \cdot AG : AI^2$.

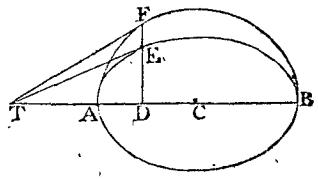
But,

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes CM : CI :: AM^2 : AI^2.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Centre.

That is,
CA is a mean proportional between CD and CT; or CD, CA, CT are continued proportionals.



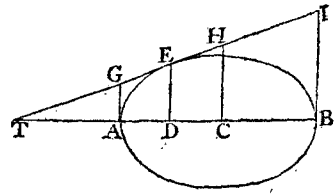
For, by theor. 6, CD : CT :: AD^2 : AT^2,
that is, CD : CT :: $(CA - CD)^2$: $(CT - CA)^2$,
or - CD : CT :: $CD^2 + CA^2$: $CA^2 + CT^2$,
and - CD : DT :: $CD^2 + CA^2$: $CT^2 - CD^2$,
or - CD : DT :: $CD^2 + CA^2$: (CT + CD) DT,
or - CD^2 : CD . CT :: $CD^2 + CA^2$: CD . DT + CT . DT,
hence - CD^2 : CA^2 :: CD . DT : CT . DT,
and - CD^2 : CA^2 :: CD : CT.
therefore (th. 78, Geom.) CD : CA :: CA : CT. Q. E. D.

Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipſe deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four Points, namely the Centre, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That is,
AG : DE :: CH : BI.



For, by theor. 7, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by ſim. tri. AG : DE :: CH : BI. Q. E. D.

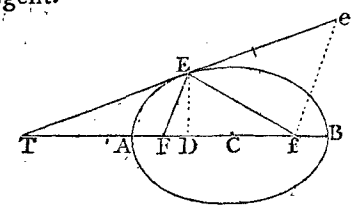
Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI } are alſo proportionals.

For theſe are as AG, DE, CH, BI, by ſimilar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; theſe two Lines will make equal Angles with the Tangent.

That is,
the ∠ FET = ∠ FEe.



For, draw the ordinate DE, and fe parallel to FE.
By cor. 1, theor. 5, CA : CD :: CF : CA — FE,
and by theor. 7, CA : CD :: CT : CA;
therefore, - CT : CF :: CA : CA — FE;
and by add. and ſub. TF : Tf :: FE : 2CA — FE or fE by th. 5.
But by ſim. tri. TF : Tf :: FE : fe;
therefore - fE = fe, and conſeq. ∠ e = ∠ fEe.
But, becauſe FE is parallel to fe, the ∠ e = ∠ FET;
therefore the ∠ FET = ∠ fEe. Q. E. D.

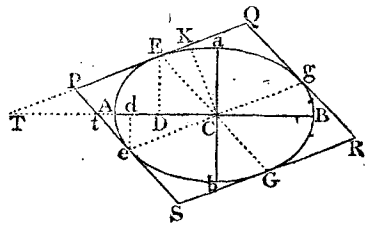
Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light iſſuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from thoſe points to the other focus. So the ray fE is reflected into FE. And this is the reaſon why the points F, f are called *foci*, or burning points.

THE-

THEOREM X.

All the Parallelograms circumſcribed about an Ellipſe are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the ſides of the parallelogram, and dividing it into four leſſer and equal parallelograms. Alſo, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis CA produced meet the ſides of the parallelogram, produced if neceſſary, in T and t.

Then, by theor. 7, CT : CA :: CA : CD,
and Ct : CA :: CA : Cd;
theref. by equality, CT : Ct :: Cd : CD;
but, by ſim. triangles, CT : Ct :: TD : Cd,
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and the rectangle TD . DC is = the ſquare Cd^2.

Again, by theor. 7, CD : CA :: CA : CT,
or, by diviſion, CD : CA :: DA : AT,
and by compoſition, CD : DB :: AD : DT;
conſeq. the rectangle CD . DT = Cd^2 = AD . DB *.

But, by theor. 1, $CA^2 : ca^2$ :: (AD . DB or) $Cd^2 : DE^2$,
therefore - CA : ca :: Cd : DE;
In like manner, CA : ca :: CD : de,
or - - ca : de :: CA : CD.

But, by theor. 7, CT : CA :: CA : CD;
theref. by equality, CT : CA :: ca : de.

But, by ſim. tri. CT : CK :: ce : de;
theref. by equality, CK : CA :: ca : ce,
and the rectangle CK . ce = CA . ca.

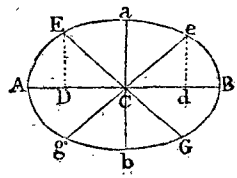
But the rect. CK . ce = the parallelogram CEPe,
theref. the rect. CA . ce = the parallelogram CEPe,
and conſeq. the rect. AB . ab = the paral. PQRS. Q. E. D.

* *Corol.* Becauſe Cd^2 = AD . DB = $CA^2 - CD^2$,
therefore $CA^2 = CD^2 + Cd^2$.
In like manner, $ca^2 = DE^2 + de^2$.

THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the ſame conſtant Quantity, namely, the Sum of the Squares of the two Axes.

That is,
$AB^2 + ab^2 = EG^2 + eg^2$,
where EG, eg are any conjugate diameters.



For, draw the ordinates ED, ed.

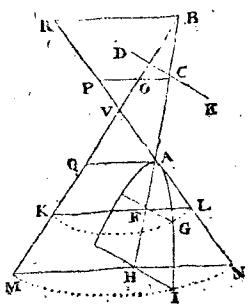
Then, by cor. to theor. 10, $CA^2 = CD^2 + cd^2$,
and - - - $ca^2 = DE^2 + de^2$;
therefore the ſum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and - - - $ce^2 = cd^2 + de^2$;
therefore the ſum $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$,
conſequently - $CA^2 + ca^2 = CE^2 + ce^2$;
or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$. Q. E. D.

OF THE HYPERBOLA.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abſciſſes.

LET AVB be a plane paſſing through the vertex and axis of the oppoſite cones; AGIH another ſection of them perpendicular to the plane of the former; AB the axis of the hyperbolic ſections; and FG, HI ordinates perpendicular to it. Then it will be, as
$FG^2 : HI^2 :: AF . FB : AH . HB$.



For, through the ordinates FG, HI draw the circular ſections KGL, MIN

MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the hyperbola.

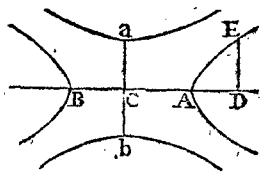
Now, by the similar triangles AFL, AHN, and BFK, BHM,
it is AF : AH :: FL : HN,
and FB : HB :: KF : MH;
hence, taking the rectangles of the corresponding terms,
it is, the rect. AF . FB : AH . HB :: KF FL : MH . HN.
But, by the circle, KF . FL = FG^2, and MH . HN = HI^2;
Therefore the rect. AF . FB : AH . HB :: FG^2 : HI^2.

Q. E. D.

THEOREM II.

As the Square of the Tranſverſe Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abſciſſes :
To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or
$AC^2 : ac^2 :: AD, DB : DE^2$.



For, by theor. 1, AC . CB : AD . DB :: Ca^2 : DE^2;
But, if c be the centre, then AC . CB = AC^2, and ca is the ſemi-conj.
Therefore AC^2 : AD . DB :: ac^2 : DE^2;
or, by permutation, AC^2 : ac^2 :: AD . DB : DE^2;
or, by doubling, AB^2 : ab^2 :: AD . DB : DE^2. Q. E. D.

Corol. Or, by div. $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$,
that is, $AB : p :: AD . DB$ or $CD^2 - CA^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

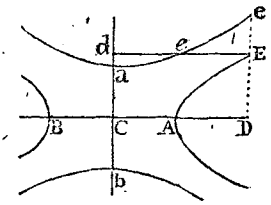
That is, As the tranſverſe,
Is to its parameter,
So is the rectangle of the abſciſſes,
To the ſquare of their ordinate.

THE-

THEOREM III.

As the Square of the Conjugate Axis :
To the Square of the Tranſverſe Axis ::
So is the Sum of the Squares of the Semi-conjugate, and Diſtance of the Centre from any Ordinate of this Axis :
To the Square of their Ordinate.

That is,
$ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.



For, draw the ordinate ED to the tranſverſe AB.

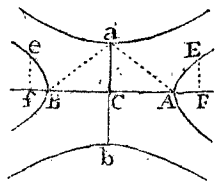
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CD^2 - CA^2$,
or - - $ca^2 : CA^2 :: cd^2 : dE^2 - CA^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$,
therefore by compoſ. $ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.
In like manner, $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q. E. D.

Corol. By the laſt theor. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
and by this theor. $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$,
therefore - $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner, $dE^2 : dE^2 :: cd^2 - ca^2 : cd^2 + ca^2$.

THEOREM IV.

The Square of the Diſtance of the Focus from the Centre, is equal to the Sum of the Squares of the Semi-Axes.
Or, the Square of the Diſtance between the Foci, is equal to the Sum of the Squares of the two Axes.

That is,
$cF^2 = CA^2 + Ca^2$, or
$Ff^2 = AB^2 + ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the ſemi-parameter. Then, by the nature

nature of the curve - $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: \quad ca^2 \quad : FE^2$;
therefore - - $ca^2 = CF^2 - CA^2$;
and by addition, - $CF^2 = CA^2 + ca^2$;
or, by doubling, - $Ff^2 = AB^2 + ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CAa; and the distance Aa is = CF the focal distance.

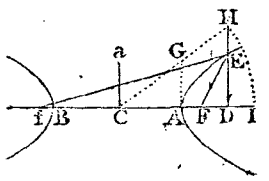
Corol. 2. The conjugate semi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CF^2 - CA^2 = CF + CA \cdot CF - CA = AF \cdot FB$.

THEOREM V.

The Difference of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Tranſverſe Axis.

That is,
$fE - FE = AB$.



FOR, draw AG parallel and equal to Ca the ſemi-conjugate; and join CG meeting the ordinate DE produced in H; alſo take CI a 4th proportional to CA, CF, CD.

Then, by th. 2, $CA^2 : AG :: CD^2 - CA^2 : DE^2$;
and, by ſim. △s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
conſequently $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.
Alſo, $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
and, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 - Ca^2 - 2CF \cdot CD + CD^2 + DH^2$.
But, by theor. 4, $CF^2 - Ca^2 = CA^2$
and, by ſuppoſition, $2CF \cdot CD = 2CA \cdot CI$;
therefore $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$;
Again, by ſuppoſ. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by ſim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore - $CI^2 = CD^2 + DH^2 = CH^2$;
conſequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or ſide of this ſquare is $FE = CI - CA = AI$.
In the ſame manner, it is found that $fE = CI + CA = BI$.
Conſeq. by ſubtract. $fE - FE = BI - AI = AB$. Q. E. D.

Corol. 1.

Corol. 1. Hence CH = CI is a 4th proportional to CA, CF, CD.

Corol. 2. And fE + FE = 2CH or 2CI; or FE, CH, fE are in continued arithmetical progreſſion, the common difference being CA the ſemi-tranſverſe.

Corol. 3. Hence is derived the common method of deſcribing this curve mechanically by points, thus:

In the tranſverſe AB, produced, take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, deſcribe arcs interſecting in E, which will be a point in the curve. In like manner, aſſuming other points I, as many other points will be found in the curve.

Then with a ſteady hand, the curve line may be drawn through all the points of interſection E.

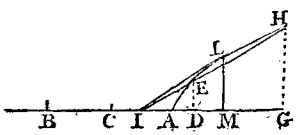
In the ſame manner are conſtructed the other two hyperbolas, uſing the axis ab inſtead of AB.

THEOREM VI.

If from any Point I in the Axis, a Line IL be drawn touching the Curve in one Point L; and the ordinate LM be drawn; and if C be the Centre or the Middle of AB: Then ſhall CM be to CI as the Square of AM to the Square of AI.

That is,

$CK : CI :: AD \cdot AG : AI^2$.



For, from the point I draw any line AEH to cut the curve in two points E and H; from which let fall the perps. ED, HG; and biſect DG in K.

Then, by theor. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by ſim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.
But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
theref. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,
or $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$,
and, by div. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA^2$,
or $CK : CI :: AD \cdot AG : AI^2$.

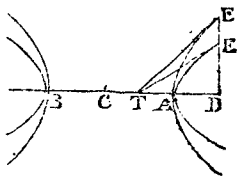
But,

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the laſt proportion becomes $\text{EM} : \text{CI} :: \text{AM}^2 : \text{AI}^2$. Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Centre.

That is,
CA is a mean proportion between CD and CT; or CD, CA, CT are continued proportionals.



For, by th. 6, $\text{CD} : \text{CT} :: \text{AD}^2 : \text{AT}^2$,
that is, $\text{CD} : \text{CT} :: (\text{CD} - \text{CA})^2 : (\text{CA} - \text{CT})^2$,
or $\text{CD} : \text{CT} :: \text{CD}^2 + \text{CA}^2 : \text{CA}^2 + \text{CT}^2$,
and $\text{CD} : \text{DT} :: \text{CD}^2 + \text{CA}^2 : \text{CD}^2 - \text{CT}^2$,
or $\text{CD} : \text{DT} :: \text{CD}^2 + \text{CA}^2 : (\text{CD} + \text{CT})\,\text{DT}$,
or $\text{CD}^2 : \text{CD}.\text{DT} :: \text{CD}^2 + \text{CA}^2 : \text{CD}.\text{DT} + \text{CT}.\text{TD}$;
hence $\text{CD}^2 : \text{CA}^2 :: \text{CD}.\text{DT} : \text{CT}.\text{TD}$,
and $\text{CD}^2 : \text{CA}^2 :: \text{CD} : \text{CT}$,
theref. (th. 78, Geom.) $\text{CD} : \text{CA} :: \text{CA} : \text{CD}$. Q. E. D.

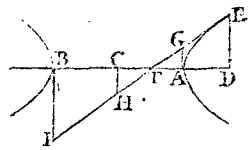
Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That

That is,
AG : DE :: CH : BI.



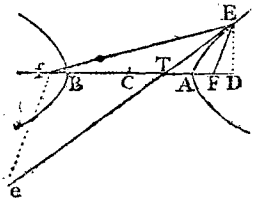
For, by theor. 7, TC · AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by fim. tri. AG : DE :: CH : BI. Q. E. D.

Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI } are alſo proportionals.
For theſe are as AG, DE, CH, BI, by ſimilar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; theſe two Lines will make equal Angles with the Tangent.

That is,
the ∠ FET = ∠ fEe.



For, draw the ordinate DE, and fe parallel to FE.

By cor. 1, theor. 5, CA : CD :: CF : CA + FE,
and by th. 7, CA : CD :: CT : CA;
therefore - CT : CF :: CA : CA + FE;
and by add. and ſub. TF : Tf :: FE : 2CA + FE or fE by th. 5.
But by ſim. tri. TF : Tf :: FE : fE;
therefore - fE = fe, and conſeq. ∠ e = ∠ fEe.
But, becauſe FE is parallel to fe, the ∠ e = ∠ FET;
therefore the ∠ FET = ∠ fEe. Q. E. D.

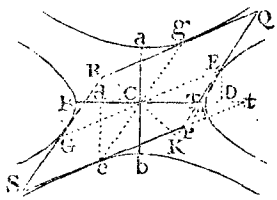
Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this propoſition, that rays of light iſſuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reaſon why the points F, f are called *foci*, or burning points.

THE-

THEOREM X.

All the Parallelograms inſcribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS = the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the ſides of the parallelogram, and dividing it into four leſſer and equal parallelograms. Alſo, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis produced meet the ſides of the parallelogram, produced, if neceſſary, in T and t.

Then, by theor. 7, CT : CA :: CA : CD,
and Ct : CA :: CA : Cd;
theref. by equality, CT : Ct :: Cd : CD;
but, by ſim. triangles, CT : Ct :: TD : Cd,
theref. by equality, TD : Cd :: Cd : CD,
and the rectangle TD . DC is = the ſquare Cd^2.

Again, by theor. 7, CD : CA :: CA : CT,
or, by diviſion, CD : CA :: DA : AT,
and, by compoſition, CD : DB :: DA : DT;
conſeq. the rectangle CD . DT = Cd^2 = AD . DB *.

But, by theor. 1, $CA^2 : Ca^2$ :: (AD . DB or) $Cd^2 : DE^2$,
therefore CA : Cd :: Cd : DE;

In like manner, CA : Ca :: CD : de,
or Ca : de :: CA : CD.

But, by theor. 7, CT : CA :: CA : CD;
theref. by equality, CT : CA :: Ca : de.

But, by ſim. tri. CT : CK :: Ce : de;
theref. by equality, CK : CA :: Ca : Ce,
and the rectangle CK . Ce = CA . Ca.

But the rect. CK . Ce = the parallelogram CEPe,
theref. the rect. CA . Ce = the parallelogram CEPe,
and conſeq. the rect. AB . ab = the paral. PQRS. Q. E. D.

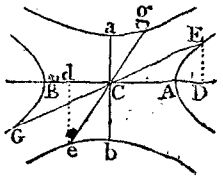
* *Corol.* Becauſe Cd^2 = AD . DB = $CD^2 - CA^2$,
therefore $CA^2 = CD^2 - Cd^2$.
In like manner, $Ca^2 = de^2 - DE^2$.

THE-

THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Difference of the Squares of the two Axes.

That is,
$AB^2 - ab^2 = EG^2 - eg^2$,
where EG, eg are any conjugate diameters.



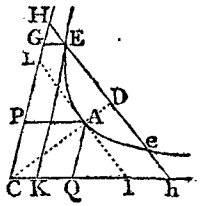
For, draw the ordinates ED, ed.

Then, by cor. to theor. 10, $CA^2 = CD^2 - cd^2$,
and - - $ca^2 = de^2 - DE^2$;
theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$,
But, by right-angled △s, $CE^2 = CD^2 + DE^2$,
and - - $ce^2 = cd^2 + de^2$;
therefore - $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$.
consequently - $CA^2 - ca^2 = CE^2 - ce^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines drawn Parallel to the Asymptotes.

That is, the lines GE, EK, AP, AQ being parallel to the asymptotes CH, cl; then the paral. CGEK = paral. CPAQ.



For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perp. to which draw AL or Al, which will be equal to the semi-conjugate, by definition 19. Also, draw HEDeh parallel to Ll.

Then, by theor. 2, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
theref. by subtract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. HE . Eh;
conseq. the square AL^2 = the rect. HE . Eh.

But,

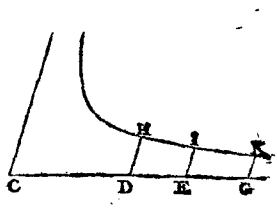
But, by ſim. tri. PA : AL :: GE : EH,
and, by the ſame, QA : Al :: EK : Eh;
theref. by comp. PA . AQ : AL^2 :: GE . EK : HE . Eh;
and, becauſe AL^2 = HE . Eh, theref. PA . AQ = GE . EK.

But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles GE . EK and PA . AQ.

And therefore the parallelogram GK = the paral. PQ.

That is, all the inſcribed parallelograms are equal to one another. Q. E. D.

Corol. 1. Becauſe the rectangle GEK or CGE is conſtant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the aſymptote continually approaches towards the curve, but never meets it: for GE decreaſes continually as CG increaſes; and it is always of *ſome* magnitude, except when CG is ſuppoſed to be infinitely great, for then GE is infinitely ſmall, or nothing. So that the aſymptote CG may be conſidered as a tangent to the curve at a point infinitely diſtant from C.

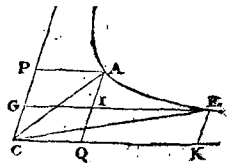


Corol. 2. If the abſciſſes CD, CE, CG, &c, taken on the one aſymptote, be in geometrical progreſſion increaſing; then ſhall the ordinates DH, EI, GK, &c, parallel to the other aſymptote, be a decreaſing geometrical progreſſion, having the ſame ratio. For, all the rectangles CDH, CEI, CGK, &c, being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abſciſſes CD, CE, CG, &c, which are geometricals. And the reciprocals of geometricals are alſo geometricals, and in the ſame ratio, but decreaſing, or in converſe order.

THEOREM XIII.

The three following Spaces, between the Aſymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Centre; and each of the two Quadrilaterals, contained by the ſaid Arc, and two Lines drawn from its Extremities parallel to one Aſymptote, and the intercepted Part of the other Aſymptote.

That is,
The ſector CAE = PAEG = QAEK,
all ſtanding on the ſame arc AE.



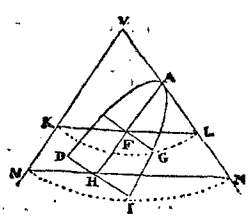
For, by theor. 12, CPAQ = CGEK;
ſubtract the common ſpace CGIQ,
ſo ſhall the paral. PI = the par. IK;
to each add the trilineal IAE,
then is the quadr. PAEG = QAEK.
Again, from the quadrilateral CAEK
take the equal triangles CAQ, CEK,
and there remains the ſector CAE = QAEK.
Therefore CAE = QAEK = PAEG. Q. E. D.

Of the PARABOLA.

THEOREM I.

The Abſciſſes are Proportional to the Squares of their Ordinates.

LET AVM be a ſection through the axis of the cone, and AGIH a parabolic ſection by a plane perpendicular to the former, and parallel to the ſide VM of the cone; alſo let AFH be the common interſection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.



Then it will be, as AF : AH :: FG^2 : HI^2.

For, through the ordinates FG, HI draw the circular ſections KGL, MIN, parallel to the baſe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

Then,

Then, by ſimilar triangles, AF : AH :: FL : HN;
but, becauſe of the parallels, KF = MH;
therefore AF : AH :: KF . FL : MH . HN.
But, by the circle, KF . FL = FG^2, and MH . HN = HI^2;
Therefore AF : AH :: FG^2 : HI^2. Q. E. D.

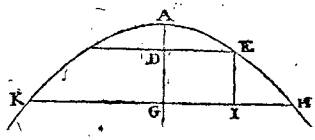
Corol. Hence the third proportional $\frac{FG^2}{AF}$ or $\frac{HI^2}{AH}$ is a conſtant quantity, and is equal to the parameter of the axis by defin. 15.

Or AF : FG :: FG : P the parameter.
Or the rectangle P . AF = FG^2.

THEOREM II.

As the Parameter of the Axis:
Is to the Sum of any Two Ordinates:
So is the Difference of thoſe Ordinates:
To the Difference of their Abſciſſes:

That is,
P : GH + DE :: GH — DE : DG,
Or,
P : KI :: IH : IE.



For, by cor. theor. 1, P . AG = GH^2,
and P . AD = DE^2;
theref. by ſubtraction, P . DG = $GH^2 - DE^2$.
Or P . DG = KI . IH,
therefore P : KI :: IH : DG or EI. Q. E. D.

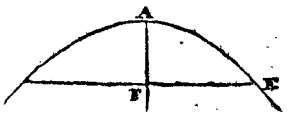
Corol. Hence, becauſe P . EI = KI . IH,
and, by cor. theor. 1, P . AG = GH^2,
therefore AG : EI :: GH^2 : KI . IH.

So that any diameter EI is as the rectangle of the ſegments KI, IH of the double ordinate KH.

THEOREM III.

The Diſtance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is,
AF = $\frac{1}{2}$FE = $\frac{1}{4}$P,
where F is the focus.

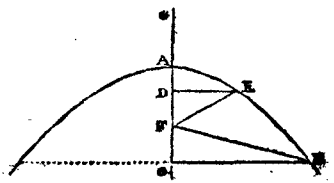


For, the general property is $AF : FE :: FE : P$.
But, by definition 17, - $FE = \frac{1}{2}P$;
therefore alſo - - $AF = \frac{1}{2}FE = \frac{1}{4}P$. Q. E. D.

THEOREM IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Diſtance and the Abſciſs of the Ordinate to that Point.

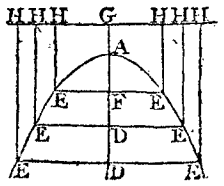
That is,
$FE = FA + AD = GD$,
taking $AG = AF$.



For, ſince $FD = AD \sim AF$,
theref. by ſquaring, $FD^2 = AF^2 - 2AF \cdot AD + AD^2$,
But, by cor. theor. 1, $DE^2 = P \cdot AD = 4AF \cdot AD$;
theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$,
But, by right-ang. tri. $FD^2 + DE^2 = FE^2$;
therefore - $FE^2 = AF^2 + 2AF \cdot AD + AD^2$,
and the root or ſide is $FE = AF + AD$,
or - - $FE = GD$, by taking $AG = AF$.
Q. E. D.

Corol. 1. If, through the point G, the line HG be drawn, it is called the directrix of the parabola. The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the diſtance of the focus from the point E.

Corol. 2. Hence alſo the curve is eaſily deſcribed by points. Namely, in the axis produced take $AG = AF$ the focal diſtance, and draw a number of lines EE perpendicular to the axis AD; then with the diſtances GD, GD, GD, &c, as radii, and the centre F, draw arcs croſſing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

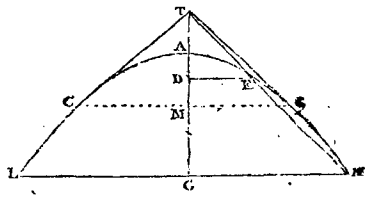


THE

THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the Point of Contact; then the Abſciſs of that Ordinate will be equal to the External Part of the Axis.

That is,
if TC touch the curve
at the point C;
then is AT = AM.



FOR, from the point T, draw any line cutting the curve in the two points E, H; to which draw the ordinates DE, GH; alſo draw the ordinate MC to the point of contact C.

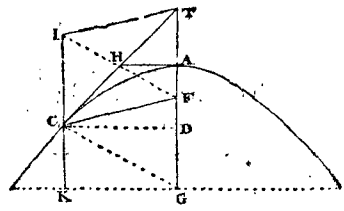
Then, by th. 1, $AD : AG :: DE^2 : GH^2$;
and, by ſim. tri. $TD^2 : TG^2 :: DE^2 : GH^2$;
theref. by equality, $AD : AG :: TD^2 : TG^2$;
and, by diviſion, $AD : DG :: TD^2 : TG^2 - TD^2$ or $DG.\overline{TD + TG}$,
or - $AD : TD :: TD : TD + TG$;
and, by diviſion, $AD : AT :: TD : TG$,
and again by div. $AD : AT :: AT : AG$;
or - AT is a mean propor. between AD, AG.

Now, if the line TH be ſuppoſed to revolve about the point T; then, as it recedes farther from the axis, the points E and H approach towards each other, the point E deſcending, and the point H aſcending, till at laſt they meet in the point C, when the line becomes a tangent to the curve at C. And then the points D and G meet in the point M, and the ordinates DE, GH in the ordinate CM. Conſequently AD, AG, becoming each equal to AM, their mean proportional AT will be equal to the abſciſs AM. That is, the external part of the axis, cut off by a tangent, is equal to the abſciſs of the ordinate to the point of contact. Q. E. D.

THEOREM VI.

If a Tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Diſtance of the Focus from the Interſection of the Tangent and Axis.

That is,
FC = FT.



For, draw the ordinate DC to the point of contact C.

Then, by theor. 5, AT = AD;
therefore - FT = AF + AD.
But, by theor. 4, FC = AF + AD;
theref. by equality, FC = FT. Q. E. D.

Corol. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C; then shall FG = FC = FT.

For, draw FH perpendicular to TC, which will also bisect TC, because FT = FC; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently FG = FT = FC.

So that F is the centre of a circle passing through T, C, G.

Corol. 2. The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle,
therefore - AH is a mean between AF, AT,
or between AF, AD, because AD = AT.
Likewise, FH is a mean between FA, FT,
or between FA, FC.

Corol. 3. The tangent TC makes equal angles with FC and the axis FT.

For, because FT = FC,
therefore the ∠ FCT = ∠ FTC.
Also, the angle GCF = the angle GCK,
drawing ICK parallel to the axis AG.

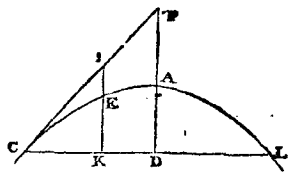
Corol. 4. And because the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THE

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and alſo any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then ſhall the Curve divide that Line in the ſame Ratio, as the Line divides the Double Ordinate.

That is,
IE : EK :: CK : KL.

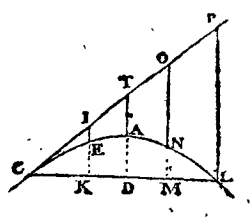


FOR, by ſim. triangles, CK : KI :: CD : DT or 2DA;
but, by the def. the param. P : CL :: CD : 2DA;
therefore, by equality, P : CK :: CL : KL.
But, by theor. 2, - P : CK :: KL : KE;
therefore, by equality, CL : KL :: KI : KE;
and, by diviſion, - CK : KL :: IE : EK. Q. E. D.

THEOREM VIII.

The ſame being ſuppoſed as in theor. 7; then ſhall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2.
and IE, TA, ON, PL, &c,
are as CI^2, CT^2, CO^2, CP^2, &c,
or as CK^2, CD^2, CM^2, CL^2, &c.



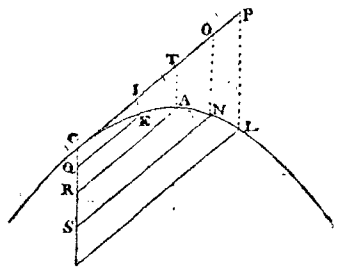
FOR, by theor. 7, IE : EK :: CK : KL,
or, by equality, IE : EK :: CK^2 : CK . KL.
But, by cor. th. 2, EK is as the rect. CK . KL,
and therefore IE is as CK^2, or as CI^2. Q. E. D.

Corol. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c, be appended to the points I, T, O, &c, and moveable about them, and of such lengths as that their extremities E, A, N, &c, be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c, will always form the curve of some parabola, in every position of the tangent.

THEOREM IX.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c,
are as QE^2, RA^2, SN^2, &c.
Or - CQ : CR :: QE^2 : RA^2,
&c.



For, draw the tangent CT, and the externals EI, AT, NO, &c, parallel to the axis, or to the diameter CS.

Then, because the ordinates QE, RA, SN, &c, are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c, are parallelograms, whose opposite sides are equal,

namely,	IE,	TA,	ON,	&c,
are equal to	CQ,	CR,	CS,	&c.
Therefore, by theor. 8,	CQ,	CR,	CS,	&c,
are as - -	CI^2,	CT^2,	CO^2,	&c,
or as their equals	QE^2,	RA^2,	SN^2,	&c. Q. E. D.

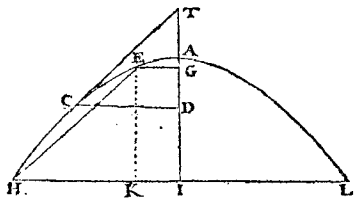
Corol. Here, like as in theor. 2, the difference of the abscisses, are as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle under the sum and difference of the ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any abscis and its ordinate.

THE-

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Interſections, and a third to the Point of Contact, theſe three Ordinates will be in Arithmetical Progreſſion, or the Sum of the Extremes will be equal to Double the Mean.

That is,
EG + HI = 2CD.



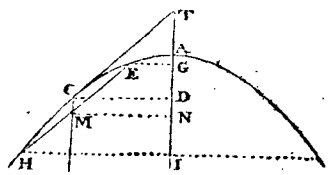
FOR, draw EK parallel to the axis, and produce HI to L.

Then, by ſim. triangles, EK : HK :: TD or 2AD : CD;
but, by theor. 2, - EK : HK :: KL : P the param.
theref. by equality, - 2AD : KL :: CD : P.
But, by the defin. - 2AD : 2CD :: CD : P;
theref. the 2d terms are equal, KL = 2CD,
that is, - - EG + HI = 2CD. Q. E. D.

THEOREM XI.

Any Diameter biſects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,
ME = MH.



FOR, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by theor. 10, 2MN or 2CD = EG + HI,
therefore M is the middle of EH.

And, for the ſame reaſon, all its parallels are biſected.

Q. E. D.

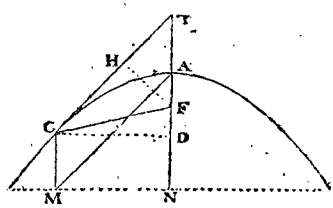
SCHOL.

SCHOL. Hence, as the abſciſſes of any diameter and their ordinates have the ſame relations as thoſe of the axis, namely, that the ordinates are biſected by the diameter, and their ſquares proportional to the abſciſſes; ſo all the other properties of the axis and its ordinates and abſciſſes, before demonſtrated, will likewiſe hold good for any diameter and its ordinates and abſciſſes. And alſo thoſe of the parameters, underſtanding the parameter of any diameter, as a third proportional to any abſciſs and its ordinate. Some of the moſt material of which are demonſtrated in the following theorems:

THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, $4FC = p$,
the param. of the diam. CM.



FOR, draw the ordinate MA parallel to the tangent CT; as alſo CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abſciſſes AD, CM or AT being equal, by theor. 5, the parameters will be as the ſquares of the ordinates CD, MA or CT, by the definition;

that is, $P : p :: CD^2 : CT^2$,
But, by ſim. tri. $FH : FT :: CD : CT$;
therefore $P : p :: FH^2 : FT^2$.
But, by cor. 2, th. 6, $FH^2 = FA \cdot FT$;
therefore $P : p :: FA \cdot FT : FT^2$,
or, by equality, $P : p :: FA : FT$ or FC.
But, by theor. 3, $P = 4FA$,
and therefore $p = 4FT$ or $4FC$. Q. E. D.

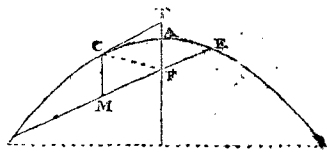
Corol. Hence the parameter p of the diameter CM is equal to $4FA + 4AD$, or to $P + 4AD$, that is, the parameter of the axis added to $4AD$.

THE-

THEOREM XIII.

If an Ordinate to any Diameter, paſs through the Focus, it will be equal to Half its Parameter; and its Abſciſs equal to One Fourth of the ſame Parameter.

That is, CM $= \frac{1}{4}$p,
and ME $= \frac{1}{2}$p.

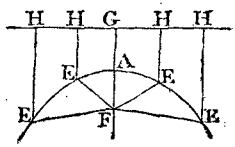


FOR, join FC, and draw the tangent CT.

By the parallels, CM = FT;
and, by theor. 6, FC = FT;
alſo, by theor. 12, FC $= \frac{1}{4}$p;
therefore - - CM $= \frac{1}{4}$p.
Again, by the defin. CM or $\frac{1}{4}$p : ME :: ME : p,
and conſequently - ME $= \frac{1}{2}$p $=$ 2CM.

Corol. 1. Hence, of any diameter, the double ordinate which paſſeth through the focus, is equal to the parameter, or to quadruple its abſciſs.

Corol. 2. Hence, and from cor. 1 to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.

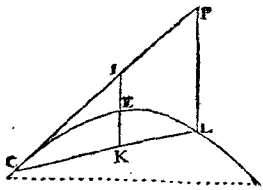


THEOREM XIV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in ſome other Point, as alſo another Line parallel to the Axis, and limited by the Firſt Line and the Tangent: then ſhall the Curve divide this Second Line in the ſame Ratio, as the Second Line divides the Firſt Line.

That

That is,
IE : EK :: CK : KL.



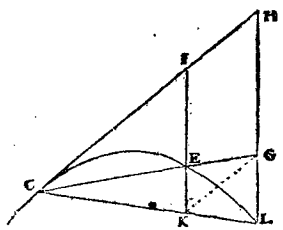
FOR, draw LP parallel to IK, or to the axis.

Then, by theor. 8, IE : PL :: CI² : CP²,
or, by ſim. tri. IE : PL :: CK² : CL².
Alſo, by ſim. tri. IK : PL :: CK : CL,
or - IK : PL :: CK² : CK . CL;
therefore by equality, IE : IK :: CK . CL : CL²,
or - IE : IK :: CK : CL;
and, by diviſion, IE : EK :: CK : KL. Q. E. D.

Corol. When CK = KL, then IE = EK = ½IK.

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent and alſo Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Interſection E and L, meeting thoſe Two Right Lines in two other Points G and K: Then will the Line KG joining theſe laſt Two Points be parallel to the Tangent.

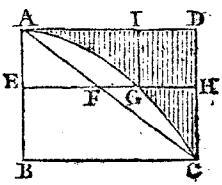


FOR, by theor. 14, CK : KL :: EI : EK;
and by comp. - CK : CL :: EI : KI;
and by the parallels :: GH : LH.
But, by ſim. tri. CK : CL :: KI : LH;
theref. by equal. KI : LH :: GH : LH;
conſequently - KI = GH,
and therefore - KG is parallel and equal to IH. Q. E. D.

THEOREM XVI.

If a Rectangle be deſcribed about a Parabola, having the ſame Baſe and Altitude; and a Line be drawn from the Vertex to the Extremity of the Baſe of the Parabola, forming a right-angled Triangle, of the ſame Baſe and Altitude alſo; then any Line or Ordinate drawn acroſs the three Figures, perpendicular to the Axis, will be cut in Continual Proportion by the Sides of thoſe Figures.

That is,
EF : EG :: EG : EH,
Or, EF, EG, EH, are in continued proportion.



For, by theor. 1, AB : AE :: BC^2 : EG^2,
and, by ſim. tri. - AB : AE :: BC : EF,
theref. of equality, EF : BC :: EG^2 : BC^2,
that is, - EF : EH :: EG^2 : EH^2,
theref. by Geom. th. 78, EF, EG, EH are proportionals,
or - - EF : EG :: EG : EH. Q. E. D.

THEOREM XVII.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumſcribing Parallelogram.

That is, the ſpace ABCGA = $\frac{2}{3}$ABCD;
or, the ſpace ADCGA = $\frac{1}{3}$ABCD.

FOR, conceive the ſpace ADCGA to be compoſed of, or divided into, indefinitively ſmall parts, by lines parallel to DC or AB, ſuch as IG, which divide AD into like ſmall and equal parts, the number or ſum of which is expreſſed by the line AD. Then,

by the parabola, BC^2 : EG^2 :: AB : AE,
that is, - AD^2 : AI^2 :: DC : IG.

Hence it follows, that any one of theſe narrow parts, as IG, is = $\frac{DC}{AD^2} \times AI^2$, whence, AD and DC being given or conſtant quantities, it appears that the ſaid parts IG, &c, are proportional to AI^2, &c, or proportional to a ſeries of ſquare numbers,

numbers, whoſe roots are in arithmetical progreſſion, and the area ADCGA equal to $\frac{DC}{AD^2}$ drawn into the ſum of ſuch a ſeries of arithmeticals, the number of which is expreſſed by AD.

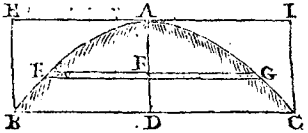
Now, by the remark at pag. 217, vol. i, the ſum of the ſquares of ſuch a ſeries of arithmeticals, is expreſſed by $\frac{1}{6}n \cdot n+1 \cdot 2n+1$, where n denotes the number of them. In the preſent caſe, n repreſents an infinite number, and then the ſubfactors $n+1$, $2n+1$, become only n and $2n$, omitting the 1 as inconſiderable in reſpect of the infinite number n: hence the expreſſion above becomes barely $\frac{1}{6}n \cdot n \cdot 2n = \frac{1}{3}n^3$.

To apply this to the caſe above: n will denote AD or BC; and the ſum of all the AI^2's becomes $\frac{1}{3}AD^3$ or $\frac{1}{3}BC^3$; conſequently the ſum of all the $\frac{DC}{AD^2} \times$ AI^2's, is $\frac{DC}{AD^2} \times \frac{1}{3}AD^3 = \frac{1}{3}AD \cdot DC = \frac{1}{3}BD$, which is the area of the exterior part ADCGA. That is, the ſaid exterior part ADCGA, is $\frac{1}{3}$ of the parallelogram ABCD; and conſequently the interior part ABCGA is $\frac{2}{3}$ of the ſame parallelogram. Q. E. D.

Corol. The part AFCGA, incloſed between the curve and the right line AFC, is $\frac{1}{6}$ of the ſame parallelogram.

THEOREM XVIII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is Half its Circumſcribing Cylinder.



LET ABC be a paraboloid, generated by the rotation of the parabola AC about its axis AD. Suppoſe the axis AD be divided into an infinite number of equal parts, through which let circular planes paſs, as EFG, all thoſe circles making up the whole ſolid paraboloid.

Then, if c = the number 3·1416, then $2c \times$ FG is the circumference of the circle EFG whoſe radius is FG; therefore $c \times$ FG^2 is the area of that circle.

But, by cor. theor. 1, Parabola, $p \times$ AF = FG^2, where p denotes the parameter of the parabola; conſequently $pc \times$ AF will

will alſo expreſs the ſame circular ſection EG, and therefore $pc \times$ the ſum of all the AF's will be the ſum of all thoſe circular ſections, or the whole content of the ſolid paraboloid.

But all the AF's form an arithmetical progreſſion, beginning at o or nothing, and having the greateſt term and the ſum of all the terms each expreſſed by the whole axis AD. And ſince the ſum of all the terms of ſuch a progreſſion, is equal to $\frac{1}{2}$AD $\times$ AD or $\frac{1}{2}$AD^2, half the product of the greateſt term and the number of terms; therefore $\frac{1}{2}$AD^2 is equal to the ſum of all the AF's, and conſequently, $pc \times \frac{1}{2}$AD^2, or $\frac{1}{2}c \times p \times$ AD^2, is the ſum of all the circular ſections, or the content of the paraboloid.

But, by the parabola, p : DC :: DC : AD, or $p = \frac{DC^2}{AD}$; conſequently $\frac{1}{2}c \times p \times$ AD^2 becomes $\frac{1}{2}c \times$ AD $\times$ DC^2 for the ſolid content of the paraboloid. But $c \times$ AD $\times$ DC^2 is equal to the cylinder BCIH; conſequently the paraboloid is the half of its circumſcribing cylinder. Q. E. D.

THEOREM XIX.

The Solidity of the Fruſtum BEGC of the Paraboloid, is equal to a Cylinder whoſe Height is DF, and its Baſe Half the Sum of the two Circular Baſes EG, BC.

For, by laſt theor. $\frac{1}{2}pc \times$ AD^2 = the ſolid ABC,
and $\frac{1}{2}pc \times$ AF^2 = the ſolid AEG,
theref. the dif. $\frac{1}{2}pc \times$ (AD^2 — AF^2) = the fruſt. BEGC.
But AD^2 — AF^2 = DF $\times$ (AD + AF),
theref. $\frac{1}{2}pc \times$ DF $\times$ (AD + AF) = the fruſt. BEGC.
But, by the parab. $p \times$ AD = DC^2, and $p \times$ AF = FG^2;
theref. $\frac{1}{2}c \times$ DF $\times$ (DC^2 + FG^2) = the fruſt. BEGC.

Q. E. D.

OF MOTION, FORCES, &c.

DEFINITIONS.

Art. 1. BODY is the maſs, or quantity of matter, in any material ſubſtance; and it is always proportional to its weight or gravity, whatever its figure may be.

2. Body is either Hard, Soft, or Elaſtic. A Hard Body is that whoſe parts do not yield to any ſtroke or percuſſion, but retains its figure unaltered. A Soft Body is that whoſe parts yield to any ſtroke or impreſſion, without reſtoring themſelves again; the figure of the body remaining altered. And an Elaſtic Body is that whoſe parts yield to any ſtroke, but which preſently reſtore themſelves again, and the body regains the ſame figure as before the ſtroke.

We know of no bodies that are abſolutely, or perfectly, either hard, ſoft, or elaſtic; but all partaking theſe properties, more or leſs, in ſome intermediate degree.

3. Bodies are alſo either Solid or Fluid. A Solid Body, is that whoſe parts are not eaſily moved amongſt one another, and which retains any figure you give it. But a Fluid Body is that whoſe parts yield to the ſlighteſt impreſſion, being eaſily moved amongſt one another; and its ſurface, when left to itſelf, is always obſerved to ſettle in a ſmooth plane at the top.

4. Denſity is the proportional weight or quantity of matter in any body.

So, of two ſpheres, or cubes, &c. of equal ſize or magnitude; if the one weigh only one pound, but the other two pounds; then the denſity of the latter is double the denſity of the former; if it weigh three pounds, its denſity is triple, and ſo on.

5. Motion is a continual and ſucceſſive change of place.—If the body move equally, or paſs over equal ſpaces in equal times, it is called Equable or Uniform Motion. But if it increaſe or decreaſe, it is Variable Motion; and it is called Accelerated Motion in the former caſe, and Retarded Motion in the latter.—Alſo, when the body moved is conſidered with reſpect

reſpect to ſome other body at reſt, it is ſaid to be Abſolute Motion. But when compared with others in motion, it is called Relative Motion.

6. Velocity, or Celerity, is an affection of motion, by which a body paſſes over a certain ſpace in a certain time. Thus, if a body in motion paſs uniformly over 40 feet in 4 ſeconds of time, it is ſaid to move with the velocity of 10 feet per ſecond; and ſo on.

7. Momentum, or Quantity of Motion, is the power or force incident to moving bodies, by which they continually tend from their preſent places, or with which they ſtrike any obſtacle that oppoſes their motion.

8. Force is a power exerted on a body to move it. If the force act conſtantly, or inceſſantly, it is a Permanent Force: like preſſure or the force of gravity. But if it act inſtantaneouſly, or but for an imperceptibly ſmall time, it is called Impulſe, or Percuſſion: like the ſmart blow of a hammer.

9. Forces are alſo diſtinguiſhed into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percuſſion, or for a certain time as a permanent force.

10. Accelerative, or Retardive Force, is commonly underſtood to be that which affects the velocity only; or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly and to the maſs or body moved inverſely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. And if the ſame force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter caſe is only 10; and ſo is but half the former.

11. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Abſolute Gravity, when the body is in empty ſpace; and Relative Gravity, when immerſed in a fluid.

12. Specific Gravity is the proportion of the weights of different bodies of equal magnitude; and ſo is proportional to the denſity of the body.

AXIOMS.

13. EVERY body naturally endeavours to continue in its preſent ſtate, whether it be at reſt, or moving uniformly in a right line.

14. The Change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

15. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and theſe changes are directed towards oppoſite or contrary parts.

GENERAL LAWS OF MOTION, &c.

PROPOSITION I.

16. *The Quantity of Matter, in all Bodies, is in the Compound Ratio of their Magnitudes and Denſities.*

THAT is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its denſity.

For, by art. 4, in bodies of equal magnitude, the maſs or quantity of matter is as the denſity. But, the denſities remaining, the maſs is as the magnitude; that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and ſo on. Therefore the maſs is in the compound ratio of the magnitude and denſity.

17. *Corol.* 1. In ſimilar bodies, the maſſes are as the denſities and cubes of the diameters, or of any like linear dimenſions.—For the magnitudes of bodies are as the cubes of the diameters, &c.

18. *Corol.* 2. The maſſes are as the magnitudes and ſpecific gravities.—For, by art. 4 and 12, the denſities of bodies are as the ſpecific gravities.

19. *Scho-*

19. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its denſity, g its ſpecific gravity, and a its diameter or other dimenſion; then, $\propto$ being the mark for general proportion, from this propoſition and its corollaries we have theſe general proportions;

$$b \propto md \propto mg \propto a^3 d,$$
$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$
$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$
$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

PROPOSITION II.

20. *The Momentum, or Quantity of Motion, generated by a Single Impulſe, or any Momentary Force, is as the Generating Force.*

THAT is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cauſe. So that a double force will impreſs a double quantity of motion; a triple force, a triple motion; and ſo on. That is, the motion impreſſed, is as the motive force which produces it.

PROPOSITION III.

21. *The Momenta, or Quantities of Motion, in Moving Bodies, are in the Compound Ratio of the Maſſes and Velocities.*

That is, m is as bv.

FOR, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the maſſes; for a double maſs will ſtrike with a double force; a triple maſs, with a triple force; and ſo on. Again, when the maſs is the ſame, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and ſo on; that is, the motive force is as the velocity; but the momentum impreſſed, is as the force which produces it, by prop. 2; and therefore the momentum is as the velocity when the maſs is the ſame. But the momentum was found to be as the maſs when the velocity is the ſame.

Conſequently, when neither are the ſame, the momentum is in the compound ratio of both the maſs and velocity.

PROPOSITION IV.

22. *In Uniform Motions, the Spaces deſcribed are in the Compound Ratio of the Velocities and the Times of their Deſcription.*

That is, s is as tv.

For, by the nature of uniform motion, the greater the velocity, the greater is the ſpace deſcribed in any one and the ſame time; that is, the ſpace is as the velocity, when the times are equal. And when the velocity is the ſame, the ſpace will be as the time; that is, in a double time a double ſpace will be deſcribed; in a triple time, a triple ſpace; and ſo on. Therefore univerſally, the ſpace is in the compound ratio of the velocity, and the time of deſcription.

23. *Corol.* 1. In uniform motions, the time is as the ſpace directly, and velocity reciprocally; or as the ſpace divided by the velocity. And when the velocity is the ſame, the time is as the ſpace. But when the ſpace is the ſame, the time is reciprocally as the velocity.

24. *Corol.* 2. The velocity is as the ſpace directly and the time reciprocally; or as the ſpace divided by the time. And when the time is the ſame, the velocity is as the ſpace. But when the ſpace is the ſame, the velocity is reciprocally as the time.

Scholium.

25. In uniform motions generated by momentary impulſe, let b = any body or quantity of matter to be moved,
f = force of impulſe acting on the body b,
v = the uniform velocity generated in b,
m = the momentum generated in b,
s = the ſpace deſcribed by the body b,
t = the time of deſcribing the ſpace s with the veloc. v.

Then from the laſt three propoſitions and corollaries, we have theſe three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of thoſe ſix quantities, in uniform motions, and impulſive or percuſſive forces:

$f \propto m$

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{tf}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating to uniform motions, and the effects of momentary or impulsive forces.

PROPOSITION V.

26. *The Momentum generated by a Constant and Uniform Force, acting for any Time, is in the Compound Ratio of the Force and Time of Acting.*

That is, m is as ft.

For, supposing the time divided into very small parts, by prop. 2, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But, by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

27. *Corol.* 1. The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose, the motion of bodies.

28. *Corol.* 2. The velocity generated, or destroyed, in any time, is directly as the force and time, and reciprocally as the body or mass of matter.—For, by this and the 3d prop. the compound ratio of the body and velocity, is as that of the force and time; and therefore th velocity is as the force and time divided by the body. And if the body and force be given, or constant, the velocity will be as the time.

PROPOSITION VI.

29. *The Spaces paſſed over by Bodies, urged by any Conſtant and Uniform Forces, acting during any Times, are in the Compound Ratio of the Forces and Squares of the Times directly, and the Body or Mass reciprocally.*

Or, the Spaces are as the Squares of the Times, when the Force and Body are given.

THAT is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given.

For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has paſſed over the ſpace s. Then, becauſe the velocity is as the time, by the laſt corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increaſe of velocity is uniform, the ſame ſpace s will be deſcribed in the ſame time t, by the velocity $\frac{1}{2}v$ uniformly continued from beginning to end. But, in uniform motions, the ſpace is in the compound ratio of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the laſt corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$, that is, the ſpace is as the force and ſquare of the time directly, and as the body reciprocally. Or s is as t^2, the ſquare of the time only, when b and f are given.

30. *Corol.* 1. The ſpace s is alſo as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the ſpace actually deſcribed. But $tv = s$ is the ſpace which might be deſcribed in the ſame time t, with the laſt velocity v, if it were uniformly continued for the ſame or an equal time. Therefore the ſpace s, or $\frac{1}{2}tv$, which is actually deſcribed, is juſt half the ſpace s, or tv, which would be deſcribed with the laſt or greateſt velocity, uniformly continued for an equal time t.

31. *Corol.* 2. The ſpace s is alſo as v^2, the ſquare of the velocity; becauſe the velocity v is as the time t.

Scholium.

32. Propoſitions 3, 4, 5, 6, give theorems for reſolving all queſtions relating to motions uniformly accelerated. Thus, put b = any body or quantity of matter,

f = the force conſtantly acting on it,

t = the

t = the time of its acting,
v = the velocity generating in the time t,
s = the space described in that time,
m = the momentum at the end of the time.

Then, from these fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}.$$

$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f \cdot t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$

$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$

$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$

$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$

$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv} \quad \&c.$$

33. And from these proportions those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if F be put $= \frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto \mathrm{F}t^2 \propto \frac{v^2}{\mathrm{F}}.$$

$$v \propto \frac{s}{t} \propto \mathrm{F}t \propto \sqrt{\mathrm{F}s}.$$

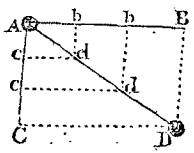
$$t \propto \frac{s}{v} \propto \frac{v}{\mathrm{F}} \propto \sqrt{\frac{s}{\mathrm{F}}}.$$

THE COMPOSITION AND RESOLUTION OF FORCES.

34. COMPOSITION of FORCES, is the uniting of two or more forces into one, which ſhall have the ſame effect; or the finding of one force that ſhall be equal to ſeveral others taken together, in any different directions. And the Reſolution of Forces, is the finding of two or more forces which, acting in any different directions, ſhall have the ſame effect as any given ſingle force.

PROPOSITION VII.

35. *If a Body at* A *be urged in the Directions* AB *and* AC, *by any two Similar Forces, ſuch that they would ſeparately cauſe the Body to paſs over the Spaces* AB, AC *in an equal time; then if both Forces act together, they will cauſe the Body to move, in the ſame time, through* AD *the Diagonal of the Parallelogram* ABDC.



DRAW cd parallel to AB, and bd parallel to AC. And while the body is carried over Ab or cd by the force in that direction, let it be carried over bd by the force in that direction; by which means it will be found at d. Now, if the forces be impulſive or momentary, the motions will be uniform, and the ſpaces deſcribed will be as the times of deſcription;

theref. Ab or cd : AB or CD :: time in Ab : time in AB,
and cd or Ac : BD or AC :: time in Ac : time in AC;

but the time in Ab = time in Ac, and time in AB = time in AC, therefore Ab : bd :: AB : BD by equality.

And as this is always the caſe in every point d, d, &c. therefore the path of the body is the ſtraight line AdD, or the diagonal of the parallelogram.

But if the ſimilar forces, by means of which the body is moved in the directions AB, AC, be uniformly accelerating ones, then the ſpaces will be as the ſquares of the times; in which caſe, call the time in bd or cd t, and the time in AB or AC, T; then

it will be Ab or cd : AB or CD :: t^2 : T^2,
and bd or AC : BD or AC :: t^2 : T^2,
theref. by equality, Ab : bd :: AB : BD;

and ſo the body is always found in the diagonal, as before.

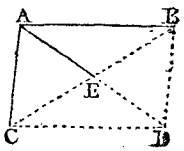
36. *Corol.*

36. *Corol.* 1. If the forces be not ſimilar, by which the body is urged in the directions AB, AC, it will move in ſome curve line, depending on the nature of the forces.

37. *Corol.* 2. Hence it appears, that the body moves over the diagonal AD, by the compound motion, in the very ſame time that it would move over the ſide AB, by the ſingle force impreſſed in that direction, or that it would move over the ſide AC by the force impreſſed in that direction.

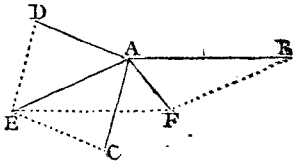
38. *Corol.* 3. The forces in the directions AB, AC, AD, are reſpectively proportional to the lines AB, AC, AD, and in theſe directions.

39. *Corol.* 4. The two oblique forces AB, AC, are equivalent to the ſingle direct force AD, which may be compounded of theſe two, by drawing the diagonal of the parallelogram. Or they are equivalent to the double of AE drawn to the middle of the line BC.

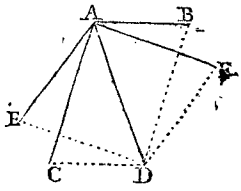


And thus any force may be compounded of two or more other forces; which is the meaning of the expreſſion, *compoſition of forces.*

40. *Exam.* Suppoſe it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one ſingle force, which ſhall be equivalent to, and have the ſame effect as if a body at A were acted on by three forces in the directions AB, AC, AD, and proportional to theſe three lines. Firſt reduce the two AC, AD to one AE, by completing the parallelogram ADEC. Then reduce the two AE, AB to one AF, by the parallelogram AEFB. So ſhall the ſingle force AF be the direction, and as the quantity, which ſhall of itſelf produce the ſame effect, as if all the three AB, AC, AD acted together.

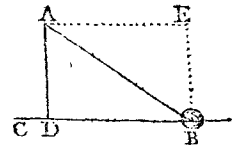


41. *Corol.* 5. Any ſingle direct force AD, may be reſolved into two oblique forces, whoſe quantities and directions are AB, AC, having the ſame effect, by deſcribing any parallelogram whoſe diagonal may be AD: and this is called the reſolution of forces. So the force AD may be reſolved into the two AB, AC, by the parallelogram ABDC;



ABDC; or into the two AE, AF, by the parallelogram AEDF; and so on, for any other two. And each of these may be resolved again into as many others as we please.

42. *Corol.* 6. Hence may be found the effect of any given force, in any other direction, besides that of the line in which it acts; as of the force AB in any other given direction CB. For draw AD perpendicular to CB; then shall DB be the effect of the force AB in the direction CB. For, the given force AB is equivalent to the two AD, DB, or AE; of which the former AD or EB, being perpendicular, does not alter the velocity in the direction CB; and therefore DB is the whole effect of AB in the direction CB. That is, a direct force expressed by the line DB acting in the direction DB, will produce the same effect or motion on a body B, in that direction, as the oblique force expressed by, and acting in, the direction AB, produces in the same direction CB. And hence a direct force DB is to an oblique force AB, as AB to DB, or as radius to the cosine of the angle ABD of inclination of those forces. For the same reason, the force or effect in the direction AB, is to the force or effect in the direction AD or EB, as AB to AD; or as radius to sine of the same angle ABD, or cosine of the angle DAB of those directions.



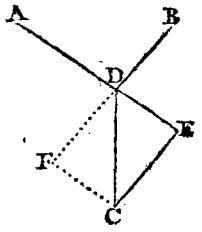
43. *Corol.* 7. Hence also, if the two given forces, to be compounded, act in the same line, either both the same way, or the one directly opposite to the other; then their joint or compounded force will act in the same line also, and will be equal to the sum of the two when they act the same way, or to the difference of them when they act in opposite directions; and the compound force, whether it be the sum or difference, will always act in the direction of the greater of the two.

PROPOSITION VIII.

44. *If Three Forces* A, B, C, *acting all together, keep one another in Equilibrio; they will be Proportional to the Three Sides* DE, CE, CD, *of a Triangle, which are drawn Parallel to the Directions of the Forces* AD, DB, CD.

PRODUCE AD, BD, and draw CF, CE parallel to them.

them. Then the force in CD is equivalent to the two AD, BD, by the ſuppoſition; but the force CD is equivalent to the two ED and CE or FD; therefore, if CD repreſent the force C, ED will repreſent its oppoſite force A, and CE or FD its oppoſite force B. Conſequently the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

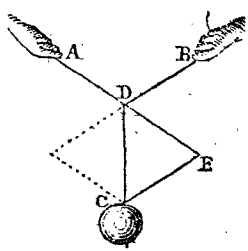


45. *Corol.* 1. Becauſe the three ſides CD, CE, DE, are proportional to the ſines of their oppoſite angles E, D, C, therefore the three forces, when in equilibrio, are proportional to the ſines of the angles of the triangle made of their lines of direction; namely, each force proportional to the ſine of the angle made by the directions of the other two.

46. *Corol.* 2. The three forces, acting againſt, and keeping one another in equilibrio, are alſo proportional to the ſides of a triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with thoſe directions. For ſuch a triangle is always ſimilar to the former, which is made by drawing lines parallel to the directions; and therefore their ſides are in the ſame proportion to one another.

47. *Corol.* 3. If any number of forces be kept in equilibrio by their actions againſt one another; they may be all reduced to two equal and oppoſite ones.—For, by cor. 4, prop. 7, any two of the forces may be reduced to one force acting in the ſame plane; then this laſt force and another may likewiſe be reduced to another force acting in their plane: and ſo on, till at laſt they be all reduced to the action of only two oppoſite forces, which will be equal, as well as oppoſite, becauſe the whole are in equilibrio by the ſuppoſition.

48. *Corol.* 4. If one of the forces, as C, be a weight, which is ſuſtained by two ſtrings drawing in the directions DA, DB; then the force or tenſion of the ſtring AD, is to the weight C, or tenſion of the ſtring DC, as DE to DC; and the force or tenſion of the ſtring BD, is to the weight C, or tenſion of CD, as CE to CD.



49. *Corol.*

49. *Corol.* 5. If three forces be in equilibrio by their mutual actions; the line of direction of each force, as DC, paſſes through the oppoſite angle C of the parallelogram formed by the directions of the other two.

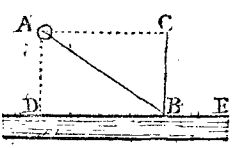
50. *Remark.* Theſe properties, in this propoſition and its corollaries, hold true of all ſimilar forces whatever, whether they be inſtantaneous or continual, or whether by percuſſion, drawing, puſhing, preſſing, or weighing; and is of the utmoſt importance in mechanics and the doctrine of forces.

Of the COLLISION of BODIES.

PROPOSITION IX.

51. *If a Body impinge or act Obliquely on a Plane Surface; the Force or Energy of the Stroke, or Action, is as the Sine of the Angle of Incidence.*

Or, the Force on the Surface is to the ſame if it had acted Perpendicularly, as the Sine of Incidence is to Radius.



Let AB expreſs the direction and the abſolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the ſine of the angle ABD, or as AB to BC, drawing BC perpendicular, and AC parallel to DE.

For, by prop. 7, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, becauſe it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the ſine of the angle BAC, or ABD, to radius.

52. *Corol.* 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the ſecond body, is made only in a direction perpendicular to the ſurface on which it acts.

For the force in AB acts on DE only by the force CB, and in that direction.

53. *Corol.* 2. If the plane DE be not abſolutely fixed, it will move, after the ſtroke, in the direction perpendicular to its ſurface. For it is in that direction that the force is exerted.

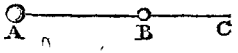
PRO-

PROPOSITION X.

54. *If one Body* A, *ſtrike another Body* B, *which is either at Reſt or moving towards the Body* A, *or moving from it, but with a leſs Velocity than that of* A; *then the Momenta, or Quantities of Motion, of the two Bodies, eſtimated in any one Direction, will be the very ſame after the Stroke that they were before it.*

For, becauſe action and re-action are always equal, and in contrary directions, whatever momentum the one body gains one way by the ſtroke, the other muſt juſt loſe as much in that ſame direction; and therefore the quantity of motion in that direction, reſulting from the motions of both the bodies, remains ſtill the ſame as it was before the ſtroke.

55. Thus, if A with a momentum of 10, ſtrike B at reſt, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up ſtill the ſame momentum between them as before, namely 10.



56. If B were in motion before the ſtroke, with a momentum of 5, in the ſame direction, and receive from A an additional momentum of 2. Then the motion of A after the ſtroke will be 8, and that of B, 7; which between them make 15, the ſame as 10 and 5, the motions before the ſtroke.

57. Laſtly, if the bodies move in oppoſite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the ſtroke, the whole motion from both, in the direction of AB, is $10 - 5$ or 5. But, after the ſtroke, the motion of A is 4 in the direction AB, and the motion of B is $6 - 5$ or 1 in the ſame direction AB; therefore the ſum $4 + 1$, or 5, is ſtill the ſame motion from both, as it was before.

PROPOSITION XI.

58. *The Motion of Bodies included in a Given Space, is the ſame, with regard to each other, whether that Space be at Reſt, or move uniformly in a Right Line.*

For, if any force be equally impreſſed both on the body and the line in which it moves, this will cauſe no change in the

the motion of the body along the right line. For the ſame reaſon, the motions of all the other bodies, in their ſeveral directions, will ſtill remain the ſame. Conſequently their motions among themſelves will continue the ſame, whether the including ſpace be at reſt, or be moved uniformly forward. And therefore their mutual actions on one another, muſt alſo remain the ſame in both caſes.

PROPOSITION XII.

59. *If a Hard and Fixed Plane be ſtruck by either a Soft or a Hard Unelaſtic Body, the Body will adhere to it. But if the Plane be ſtruck by a Perfectly Elaſtic Body, it will rebound from it again with the ſame Velocity with which it ſtruck the Plane.*

For, as the parts which are ſtruck, of the elaſtic body, ſuddenly yield and give way by the force of the blow, and as ſuddenly reſtore themſelves again with a force equal to the force which impreſſed them, by the definition of elaſtic bodies; the intenſity of the action of that reſtoring force on the plane, will be equal to the force or momentum with which the body ſtruck the plane. And, as action and reaction are equal and contrary, the plane will act with the ſame force on the body, and ſo cauſe it to rebound or move back again with the ſame velocity as it had before the ſtroke.

But hard or ſoft bodies, being devoid of elaſticity, by the definition, having no reſtoring force to throw them off again, they muſt neceſſarily adhere to the plane ſtruck.

60. *Corol.* 1. The effect of the blow of the elaſtic body, on the plane, is double to that of the unelaſtic one, the velocity and maſs being equal in each.

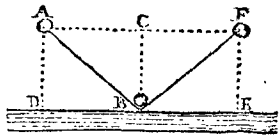
For the force of the blow from the unelaſtic body, is as its maſs and velocity, which is only deſtroyed by the reſiſtance of the plane. But in the elaſtic body, that force is not only deſtroyed and ſuſtained by the plane; but another alſo equal to it is ſuſtained by the plane, in conſequence of the reſtoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intenſity of the blow is doubled.

61. *Corol.* 2. Hence unelaſtic bodies loſe, by their colliſion, only half the motion loſt by elaſtic bodies; their maſs and velocities being equal.—For the latter communicate double the motion of the former.

PRO-

PROPOSITION XIII.

62. *If an Elaſtic Body* A *impinge on a Firm Plane* DE *at the Point* B, *it will rebound from it in an Angle equal to that in which it ſtruck it; or the Angle of Incidence will be Equal to the Angle of Reflection; namely, the Angle* ABD *equal to the Angle* FBE.

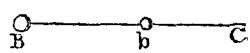


LET AB expreſs the force of the body A in the direction AB; which let be reſolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF. Now, action and re-action being equal, the plane will reſiſt the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on nor diminiſhed by it, but ſtill continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by compoſition. But, becauſe AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually ſimilar and equal; and conſequently the angles at A and F are equal, as alſo their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

PROPOSITION XIV.

63. *To determine the Motion of Non-elaſtic Bodies, when they ſtrike each other Directly, or in the Same Line of Direction.*

LET the non-elaſtic body B, moving with the velocity V in the direction B*b*, and the body *b* with the velocity *v*, ſtrike each other.



Then, becauſe the momentum of any moving body is as the maſs into the velocity, BV = M is the momentum of the body B, and *bv* = *m* the momentum of the body *b*, which let be the leſs powerful of the two motions. Then, by prop. 10, the bodies will both move together as one maſs in the direction BC after the ſtroke, whether before the ſtroke the body *b* moved towards C or towards B. Now, according as that motion of *b* was from or towards B, that is, whether the motions were in the ſame or contrary ways, the momentum after the ſtroke, in direction BC, will be

be the ſum or difference of the momentums before the ſtroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the ſame way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body b were at reſt.

Then divide each momentum by the common maſs of matter $B + b$, and the quotient will be the common velocity after the ſtroke in the direction BC; namely, the common velocity will be,

$\frac{BV + bv}{B + b}$ in the firſt caſe, $\frac{BV - bv}{B + b}$ in the 2nd, and $\frac{BV}{B + b}$ in the third.

64. For example, if the bodies, or weights, B and b be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the ſtroke; then 15 and 6 will be as their momentums, and 8 the ſum of their weights; conſequently after the ſtroke the common velocity will be as

$\frac{15 + 6}{8} = \frac{21}{8}$ or $2\frac{5}{8}$ in the firſt caſe,

$\frac{15 - 6}{8} = \frac{9}{8}$ or $1\frac{1}{8}$ in the ſecond, and

$\frac{15}{8}$ - - - or $1\frac{7}{8}$ in the third.

PROPOSITION XV.

65. *If two Perfectly Elaſtic Bodies impinge on one another; their Relative Velocity will be the ſame both Before and After the Impulſe; that is, they will recede from each other with the Same Velocity with which they approached and met.*

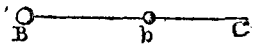
For the compreſſing force is as the intenſity of the ſtroke; which, in given bodies, is as the relative velocity with which they meet or ſtrike. But perfectly elaſtic bodies reſtore themſelves to their former figure by the ſame force by which they were compreſſed; that is, the reſtoring force is equal to the compreſſing force, or to the force with which the bodies approach each other before the impulſe. But the bodies are impelled from each other by this reſtoring force; and therefore this force, acting on the ſame bodies, will produce a relative velocity equal to that which they had before; or it will make the bodies recede from each other with the ſame velocity

velocity with which they before approached, or ſo as to be equally diſtant from one another at equal times before and after the impact.

66. *Remark.* It is not meant by this propoſition, that each body will have the ſame velocity after the impulſe as it had before; for that will be varied according to the relation of the maſſes of the two bodies; but that the velocity of the one will be, after the ſtroke, as much increaſed as that of the other is decreaſed, in one and the ſame direction. So if the elaſtic body B move with a velocity V, and overtake the elaſtic body *b* moving the ſame way with the velocity *v*; then their relative velocity, or that with which they ſtrike, is $V - v$, and it is with this ſame velocity that they ſeparate from each other after the ſtroke. But if they meet each other, or the body *b* move contrary to the body B; then they meet and ſtrike with the velocity $V + v$, and it is with the ſame velocity that they ſeparate and recede from each other after the ſtroke. But whether they move forwards or backwards after the impulſe, and with what particular velocities, are circumſtances that depend on the various maſſes and velocities of the bodies before the ſtroke, and which make the ſubject of the next propoſition.

PROPOSITION XVI.

67. *To determine the Motions of Elaſtic Bodies after Striking each other directly.*



LET the elaſtic body B move in the direction BC, with the velocity V; and let the velocity of the other body *b* be *v* in the ſame direction; which latter velocity *v* will be poſitive if *b* move the ſame way as B, but negative if *b* move in the oppoſite direction to B. Then their relative velocity in the direction BC is $V - v$; alſo the momenta before the ſtroke are BV and *bv*, the ſum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of *b*, in the ſame direction BC, after the ſtroke; then their relative velocity is $y - x$, and the ſum of their momenta $Bx + by$ in the ſame direction.

But the momenta before and after the colliſion, eſtimated in the ſame direction, are equal, by prop. 10, as alſo the relative velocities, by the laſt prop. Whence ariſe theſe two equations,

$$BV + bv = Bx + by,$$
$$V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b,$$

both in the direction BC, when V and v are both positive, or the bodies both moved towards C before the collision. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

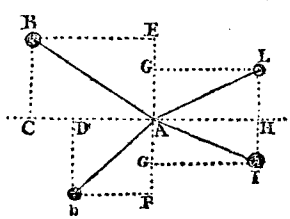
And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V,$$

for the velocities in this case.

PROPOSITION XVII.

68. *If Bodies strike one another Obliquely, it is proposed to determine their Motions after the Stroke.*



LET the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is resolved into the two BC, CA; and the motion in bA is resolved into the two bD, DA; of which the antecedents BC, bD are the velocities with which they directly meet, and the consequents CA, DA are parallel; therefore, by these the bodies do not impinge on each other, and consequently the motions, according to these directions,

rections, will not be changed by the impulſe; ſo that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions, therefore, of the bodies B, *b*, directly ſtriking each other with the velocities EA, FA, will be determined by prop. 16 or 14, according as the bodies are elaſtic or non-elaſtic; which being done, let AG be the velocity, ſo determined, of one of them, as A; and ſince there remains alſo in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the ſtroke. And after the ſame manner is the motion of the other body *b* determined after the impact.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROPOSITION XVIII.

69. *All the Properties of Motion delivered in Propoſition VI, its Corollaries and Scholium, for Conſtant Forces, are true in the Motions of Bodies freely deſcending by their own Gravity: namely, that the Velocities are as the Times, and the Spaces as the Squares of the Times, or as the Squares of the Velocities.*

FOR, ſince the force of gravity is uniform, and conſtantly the ſame, at all places near the earth's ſurface, or at nearly the ſame diſtance from the centre of the earth; and that this is the force by which bodies deſcend to the ſurface; they therefore deſcend by a force which acts conſtantly and equally; conſequently all the motions freely produced by gravity, are as above ſpecified, by that propoſition, &c.

SCHOLIUM.

70. Now it has been found, by numberleſs experiments, that gravity is a force of ſuch a nature, that all bodies, whether light or heavy, fall perpendicularly through equal ſpaces in the ſame time, abſtracting from the reſiſtance of the air; as lead or gold and a feather, which in an exhauſted receiver fall from the top to the bottom in the ſame time. It is alſo

found, that the velocities acquired by deſcending, are in the exact proportion of the times of deſcent: and farther, that the ſpaces deſcended are proportional to the ſquares of the times, and therefore to the ſquares of the velocities. And hence it follows, that the weights, or gravities, of bodies near the ſurface of the earth, are proportional to the quantities of matter contained in them; and that the ſpaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Moreover, as it is found, by accurate experiments, that a body, in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the firſt ſecond of time, and conſequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet, by corol. 1, prop. 6. So, if g denote $16\frac{1}{12}$ feet, the ſpace fallen through in one ſecond of time, or $2g$ the velocity generated in that time; then, becauſe the velocities are directly proportional to the times, and the ſpaces to the ſquares of the times;

$$\text{theref. } 1'' : t'' :: 2g : 2gt = v \text{ the velocity,}$$
$$\text{and } 1^2 : t^2 :: g : gt^2 = s \text{ the ſpace.}$$

And hence, for the deſcents of gravity, we have theſe general equations, namely,

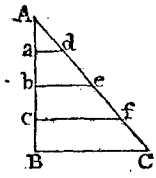
$$s = gt^2 = \frac{v^2}{4g} = \tfrac{1}{2}tv.$$
$$v = 2gt = \frac{2s}{t} = 2\sqrt{gs}.$$
$$t = \frac{v}{2g} = \frac{2s}{v} = \sqrt{\frac{s}{g}}.$$
$$g = \frac{v}{2t} = \frac{s}{t^2} = \frac{v^2}{4s}.$$

Hence, becauſe the times are as the velocities, and the ſpaces as the ſquares of either; therefore,

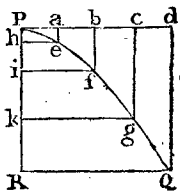
if the times be as the numbs. 1, 2, 3, 4, 5, &c,
the velocities will alſo be as 1, 2, 3, 4, 5, &c,
and the ſpaces as their ſquares 1, 4, 9, 16, 25, &c,
and the ſpace for each time as 1, 3, 5, 7, 9, &c;

namely, as the ſeries of the odd numbers, which are the differences of the ſquares denoting the whole ſpaces. So that, if the firſt ſeries of natural numbers be ſeconds of time,

namely, the times in ſeconds $1''$, $2''$, $3''$, $4''$, &c,
the velocity in feet will be $32\frac{1}{6}$, $64\frac{1}{3}$, $96\frac{1}{2}$, $128\frac{2}{3}$, &c,
the ſpaces in the whole times $16\frac{1}{12}$, $64\frac{1}{3}$, $144\frac{3}{4}$, $257\frac{1}{3}$, &c,
and the ſpace for each ſecond $16\frac{1}{12}$, $48\frac{1}{4}$, $80\frac{5}{12}$, $112\frac{7}{12}$, &c.



71. Theſe relations of times, velocities, and ſpaces, may be thus aptly repreſented by lines and geometrical figures. Namely, if the line AB denote the time of any body's deſcent, and BC the velocity gained at the end of that time, by joining AC, and dividing the time AB into any parts at the points a, b, c; then ſhall ad, be, cf, parallel to BC, be the velocities at the points of time a, b, c, or at the ends of the time Aa, Ab, Ac; becauſe theſe latter lines, by ſimilar triangles, are proportional to the former ad, be, cf, ſince the times are proportional to the velocities. Alſo, the area of the triangle ABC will repreſent the ſpace deſcended by the force of gravity in the time AB, in which it generates the velocity BC; becauſe that area is equal to $\frac{1}{2}AB \times BC$, and the ſpace deſcended is $s = \frac{1}{2}tv$, or half the product of the time and the laſt velocity. And, for the ſame reaſon, the leſs triangles Aad, Abe, Acf, will repreſent the ſeveral ſpaces deſcribed in the correſponding times Aa, Ab, Ac, and velocities ad, be, cf; thoſe triangles or ſpaces being alſo as the ſquares of their like ſides Aa, Ab, Ac, which repreſent the times, or of ad, be, cf, which repreſent the velocities.



72. But as areas are rather unnatural repreſentations of the ſpaces paſſed over by a body in motion, which are lines, the relations may better be repreſented by the abſciſſes and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c, parallel to RQ, repreſent the times from the beginning, or the velocities, then ae, bf, cg, &c, parallel to the axis PR, will repreſent the ſpaces deſcribed by a falling body in thoſe times; for, in a parabola, the abſciſſes Ph, Pi, Pk, &c, or ae, bf, cg, &c, which are the ſpaces deſcribed, are as the ſquares of the ordinates he, if, kg, &c, or Pa, Pb, Pc, &c, which repreſent the times or velocities.

73. And becauſe the laws for the deſtruction of motion, are the ſame as thoſe for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1ſt, A body thrown directly upwards, with any velocity, will loſe equal velocities in equal times.

2*d*, If a body be projected upwards, with the velocity it acquired in any time by deſcending freely; it will loſe all its velocity in an equal time, and will aſcend juſt to the ſame height from whence it fell, and will deſcribe equal ſpaces in equal times, both in riſing and falling, but in an inverſe order; and it will have equal velocities at any one and the ſame point of the line deſcribed, both in aſcending and deſcending.

3*d*, If bodies be projected upwards, with any velocities, the height aſcended to, will be as the ſquares of thoſe velocities, or as the ſquares of the times of aſcending, till they loſe all their velocities.

74. To illuſtrate now the rules for the natural deſcent of bodies by a few examples, let it be required,

1*ſt*, To find the ſpace deſcended by a body in 7 ſeconds of time, and the velocity acquired.

Anſ.

2*d*, To find the time of generating a velocity of 100 feet per ſecond, and the whole ſpace deſcended.

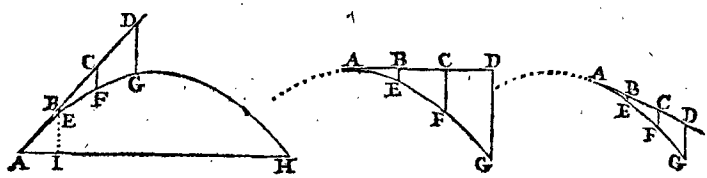
Anſ.

3*d*, To find the time of deſcending 400 feet, and the velocity at the end of that time.

Anſ.

PROPOSITION XIX.

75. *If a Body be projected in Free Space, either Parallel to the Horizon, or in an Oblique Direction, by the Force of Gun Powder, or any other Impulſe; it will, by this Motion, in Conjunction with the Action of Gravity, deſcribe the Curve Line of a Parabola.*



Let the body be projected from the point A, in the direction AD, with an uniform velocity; then, in any equal portions of time, it would, by prop. 4, deſcribe the equal ſpaces

ſpaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c, in the direction of gravity, or perpendicular to the horizon, and equal to the ſpaces through which the body would deſcend by its gravity, in the ſame times in which it would uniformly paſs over the correſponding ſpaces AB, AC, AD, &c, by the projectile motion. Then, ſince by theſe two motions, the body is carried over the ſpace AB, in the ſame time as over the ſpace BE, and the ſpace AC in the ſame time as the ſpace CF, and the ſpace AD in the ſame time as the ſpace DG, &c; therefore, by the compoſition of motions, at the end of thoſe times, the body will be found reſpectively in the points E, F, G, &c; and conſequently the real path of the projectile will be the curve line AEFG, &c. But the ſpaces AB, AC, AD, &c, deſcribed by uniform motion, are as the times of deſcription; and the ſpaces BE, CF, DG, &c, deſcribed in the ſame times by the accelerating force of gravity, are as the ſquares of the times; conſequently the perpendicular deſcents are as the ſquares of the ſpaces in AD, that is BE, CF, DG, &c, are reſpectively proportional to AB^2, AC^2, AD^2, &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG &c, to which AD is a tangent at the point A.

76. *Corol.* 1. The horizontal velocity of a projectile, is always the ſame conſtant quantity, in every point of the curve; becauſe the horizontal motion is in a conſtant ratio to the motion in AD, which is the uniform projectile motion. And the conſtant horizontal velocity, is in proportion to the projectile velocity, as radius to the coſine of the angle DAH, or angle of elevation or depreſſion of the piece above or below the horizontal line AH.

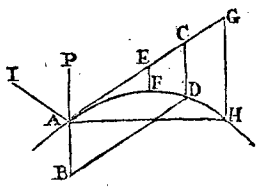
77. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the ſecant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is conſtant, and AI is to AB, as radius to the ſecant of the angle A; therefore the motion at A, in AB, is every where as the ſecant of the angle A.

78. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the firſt uniform projectile velocity at A, or point of contact of a tangent, as 2GD to AD. For, the times in AD and DG being equal, and the velocity acquired by freely

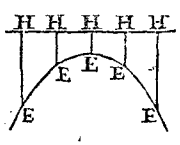
deſcending through DG being ſuch as would carry the body uniformly over twice DG in an equal time, and the ſpaces deſcribed with uniform motions being as the velocities, therefore the ſpace AD is to the ſpace 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

PROPOSITION XX.

79. *The Velocity in the Direction of the Curve, at any Point of it, as* A, *is equal to that which is generated by Gravity in freely deſcending through a Space which is equal to One-Fourth of the Parameter of the Diameter of the Parabola at that Point.*



LET PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is alſo the height due to the perpendicular velocity at D, which muſt be equal to the former: but, by the laſt corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as theſe velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the Parabola.



80. *Corol.* 1. Hence, and from cor. 2, theor. 13 of the Parabola, it appears, if from the directrix of the parabola which is the path of the projectile, ſeveral lines HE be drawn perpendicular to the directrix, or parallel to the axis; that the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

81. *Corol.* 2. If a body, after falling through the height PA (laſt fig. but one), which is equal to AB, and when it arrives at A, have its courſe changed, by reflection from a firm plane AI, or otherwiſe, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and

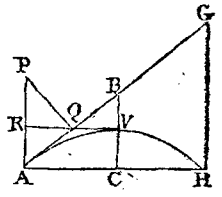
and the parallelogram be completed; the body will describe the parabola passing through the point D.

82. *Coröl.* 3. Because AC = 2AB or 2CD or 2AP, therefore $AC^2 = 2AP \times 2CD$ or AP . 4CD; and because all the perpendiculars EF, CD, GH are as AE^2, AC^2, AG^2; therefore also AP . 4EF = AE^2, and AP . 4GH = AG^2, &c; and, because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is AP : AE :: AE : 4EF,
and AP : AC :: AC : 4CD,
and AP : AG :: AG : 4GH,
and so on.

PROPOSITION XXI.

83. *Having given the Direction, and the Impetus, or Altitude due to the First Velocity of a Projectile; to determine the Greatest Height to which it will rise, and the Random or Horizontal Range.*

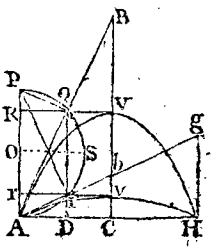


Let AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. Upon AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perp. to AG, take AG = 4AQ, and draw GH perp. to AH; then AH is the range.

For, by the last corollary, AP : AG :: AG : 4GH;
and, by similar triangles, AP : AG :: AQ : GH,
or AP : AG :: 4AQ : 4GH;
therefore AG = 4AQ; and, by similar triangles, AH = 4QR.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}$AG = 2AQ, or AQ = QB; consequently AR = BV, which is = CV by the property of the parabola.

84. *Corol.* 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, upon AP, as a diameter, a semicircle be described, it will pass through the point Q.



85. *Corol.* 2. If the horizontal range and the projectile velocity be

be given, the direction of the piece ſo as to hit the object H; will be thus eaſily found: Take AD = ¼AH. draw DQ perpendicular to AH, meeting the ſemicircle, deſcribed on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the ſame projectile velocity, give the very ſame horizontal range AH. And theſe two directions make equal angles qAD, QAP with AH and AP, becauſe the arc PQ = the arc Aq.

86. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = ¼AH, and erect the perpendicular DQ, meeting AB in Q; ſo ſhall DQ be equal to the greateſt altitude CV. Alſo, erect AP perpendicular to AH, and QP to AQ; ſo ſhall AP be the height due to the velocity.

87. *Corol.* 4. When the body is projected with the ſame velocity, but in different directions; the horizontal ranges AH will be as the ſines of double the angles of elevation.—Or, which is the ſame, as the rectangle of the ſine and coſine of elevation. For AD or RQ, which is ¼AH, is the ſine of the arc AQ, which meaſures double the angle QAD of elevation.

And when the direction is the ſame, but the velocities different; the horizontal ranges are as the ſquare of the velocities, or as the height AP, which is as the ſquare of the velocity; for the ſine AD or RQ or ¼AH is as the radius, or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the ſquares of the velocities, and the ſines of double the angles of elevation.

88. *Corol.* 5. The greateſt range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greateſt ſine. Or the radius OS, which is ¼ of the range, is the greateſt ſine.

And hence the greateſt range, or that at an elevation of 45°, is juſt double the altitude AP which is due to the velocity, or equal to 4VC. And conſequently, in that caſe, C is the focus of the parabola, and AH its parameter. Alſo, the ranges are equal, at angles equally above and below 45°.

89. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its ſine equal to half the radius; conſequently then its range will be equal to AP, or half the greateſt range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

90. *Corol.* 7,

90. *Corol.* 7. The greateſt altitude CV, being equal to AR, is as the verſed ſine of double the angle of elevation, and alſo as AP or the ſquare of the velocity. Or as the ſquare of the ſine of elevation, and the ſquare of the velocity; for the ſquare of the ſine is as the verſed ſine of the double angle.

91. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the ſquare root of the altitude, or as the projectile velocity and ſine of the elevation.

SCHOLIUM.

92. From the laſt propoſition, and its corollaries, may be deduced the following ſet of theorems, for finding all the circumſtances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t denote the ſine, coſine, and tangent of elevation; f, v the ſine and verſed ſine of the double elevation; R the horizontal range, T the time of flight, V the projectile velocity, H the greateſt height of the projectile, $g = 16\frac{1}{2}$ feet, and a the impetus, or the altitude due to the velocity V. Then,

$$R = 2af = 4asc = \frac{fV^2}{2g} = \frac{scV^2}{g} = \frac{gcT^2}{s} = \frac{gT^2}{t} = \frac{4H}{t},$$

$$V = \sqrt{4ag} = \sqrt{\frac{2gR}{f}} = \sqrt{\frac{gR}{sc}} = \frac{gT}{s} = \frac{2}{s}\sqrt{gH},$$

$$T = \frac{sV}{g} = 2s\sqrt{\frac{a}{g}} = \sqrt{\frac{tR}{g}} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}},$$

$$H = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}tR = \frac{sR}{4c} = \frac{s^2V^2}{4g} = \frac{vV^2}{8g} = \frac{g}{4}T^2.$$

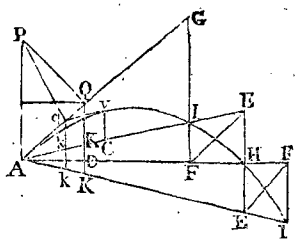
And from any of theſe, the angle of direction may be found. Alſo, in theſe theorems, g may, in many caſes, be taken $= 16$, without the ſmall fraction $\frac{1}{12}$; which will be near enough for common uſe.

PROPOSITION XXII.

93. *To determine the Range on an Oblique Plane; having given the Impetus or Velocity, and the Angle of Direction.*

LET AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of

of the piece, and AP the altitude due to the projectile velocity at A.



By the laſt propoſition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; ſo ſhall the projectile paſs through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

94. *Otherwiſe*, without the Horizontal Range.

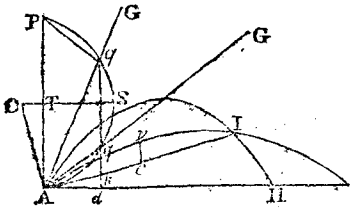
Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and FI parallel to AP or DQ; ſo ſhall AI be the range on the oblique plane. For, AH = 4AD, therefore EH is parallel to FI, and ſo on, as above.

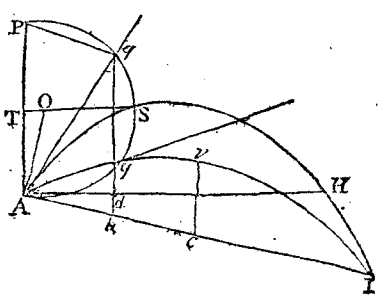
Otherwiſe.

95. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4Ak. Alſo, kq will be equal to cv the greateſt height above the plane.

For, by cor. 2, prop. 20, AP : AG :: AG : 4GI;
and, by ſim. triangles, AP : AG :: Aq : GI,
or AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and by ſim. triangles, AI = 4Ak.

Alſo, qk, or $\frac{1}{4}$GI, is = to cv by theor. 13 of the Parabola.





96. *Corol.* 1. If AO be drawn perp. to the plane AI, an

AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

97. *Corol.* 2. If there be given the range and velocity, or the impetus, the direction will hence be easily found thus: Take $Ak = \frac{1}{4}AI$, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears, that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

98. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take $Ak = \frac{1}{4}AI$, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the $\angle AqP = \angle Akq$; so shall AP be the impetus, or the altitude due to the projectile velocity.

99. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly as the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally as the square of the cosine of the angle of the plane above or below the horizon.

For, put s = sin. $\angle$ qAI or APq,
c = cos. $\angle$ qAH or sin. PAq,
C = cos. $\angle$ IAH or sin. Akd or Akq or AqP.
Then, in the triangle APq, $C : s :: AP : Aq$;
and in the triangle Akq, $C : c :: Aq : Ak$;
theref. by composition, $C^2 : cs :: AP : Ak = \frac{1}{4}AI$.

So that the oblique range $AI = \frac{cs}{C^2} \times 4AP$.

100. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

101. *Corol.* 5. The greatest height cv or kq of the projectile,

tile, above the plane, is equal to $\frac{s^2}{c^2} \times$ AP. And therefore it is as the impetus and ſquare of the ſine of direction above the plane directly, and ſquare of the coſine of the plane's inclination reciprocally.

For - C (ſin. AqP) : s (ſin. APq) :: AP : Aq,
and C (ſin. Akq) : s (ſin. kAq) :: Aq : kq,
therefore by comp. $C^2 : s^2 ::$ AP : kq.

102. *Corol.* 6. The time of flight in the curve AvI is $= \frac{2s}{C}\sqrt{\frac{AP}{g}}$, where $g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and ſine of direction above the plane directly, and coſine of the plane's inclination reciprocally. For the time of deſcribing the curve, is equal to the time of falling freely through GI or 4kq or $\frac{4s^2}{C^2} \times$ AP. Therefore, the time being as the ſquare root of the diſtance,

$\sqrt{g} : \frac{2s}{C}\sqrt{AP} :: 1'' : \frac{2s}{C}\sqrt{\frac{AP}{g}}$, the time of flight.

SCHOLIUM.

103. From the foregoing corollaries may be collected the following ſet of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

c = coſ. of direction above the horizon,
C = coſ. of inclination of the plane,
s = ſin. of direction above the plane,
R the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greateſt height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 16\frac{1}{12}$ feet. Then,

$$R = \frac{cs}{C^2} \times 4a = \frac{cs}{C^2 g} V^2 = \frac{gc}{s} T^2 = \frac{4c}{s} H.$$

$$H = \frac{s^2}{C^2} a = \frac{s^2 V^2}{4gC^2} = \frac{sR}{4c} = \frac{g}{4} T^2.$$

$$V = \sqrt{4ag} = C\sqrt{\frac{gR}{cs}} = \frac{gC}{s} T = \frac{2C}{s}\sqrt{gH}.$$

$$T = \frac{2s}{C}\sqrt{\frac{a}{g}} = \frac{sV}{gC} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}}.$$

And from any of theſe, the angle of direction may be found.

PRAC-

PRACTICAL GUNNERY.

104. THE two foregoing propoſitions contain the whole theory of projectiles, with theorems for all the caſes, regularly arranged for uſe, both for oblique and horizontal planes. But, before they can be applied to uſe in reſolving the ſeveral caſes in the practice of gunnery, it is neceſſary that ſome more data be laid down, as derived from good experiments made with balls or ſhells diſcharged from cannon or mortars, by gunpowder, under different circumſtances. For, without theſe, thoſe theorems can be of very little uſe in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulſion of balls from guns, but more eſpecially on account of the enormous reſiſtance of the air to all projectiles that are made with any velocities that are conſiderable. As to the caſes in which projectiles are made with ſmall velocities, or ſuch as do not exceed 200 or 300, or 400 feet per ſecond, they may be reſolved tolerably near the truth, eſpecially for the larger ſhells, by the parabolic theory laid down above. But, in caſes of great projectile velocities, it is quite inadequate, without the help of ſeveral data drawn from many and good experiments. For ſo great is the effect of the reſiſtance of the air to projectiles of conſiderable velocity, that ſome of thoſe which in the air range only between 2 and 3 miles at the moſt, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this reſiſtance are alſo various, both according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one ſize of ball or ſhell, will not ſerve for another ſize, though the velocity ſhould be the ſame; nor will the experiments with one velocity ſerve for other velocities, though the ball be the ſame. And therefore it is plain that, to form rules for practical gunnery, we ought to have good experiments made with each ſize of mortar, and with every variety of charge, from the leaſt to the greateſt. And not only ſo, but theſe ought alſo to be repeated at many different angles of elevation, namely, for every ſingle degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By ſuch a courſe of experiments, it will be found that the greateſt range, inſtead of being conſtantly that for an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45 and 30,

being more or leſs, both according to the velocity and the weight of the projectile; the ſmaller velocities and larger ſhells ranging fartheſt when projected almoſt at an elevation of 45°; while the greateſt velocities, eſpecially with the ſmaller ſhells, range fartheſt with an elevation of about 30°.

105. There have, at different times, been made certain ſmall parts of ſuch a courſe of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the ſizes of mortars were uſed, and a variety of ſmall charges of powder. But they were all at the elevation of 45°; and conſequently theſe are defective in the higher charges, and in all the other angles of elevation.

Other experiments were alſo carried on in the ſame place in the year 1784 and 1786, with various angles of elevation indeed, but with only one ſize of mortar, and one only charge of powder, and that but a ſmall one too: ſo that all thoſe nearly agree with the parabolic theory. Other experiments have alſo been carried on with the ballaſtic pendulum, at different times; from which have been obtained ſome of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the ſquare root of the charge directly, and as the ſquare root of the weight of ball reciprocally; and that ſome rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here ſet down in the following Table.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	14½
12	15	1680	6700	15½
2	45	860	5100	21

106. But as we are not yet provided with a ſufficient number and variety of experiments, on which to eſtabliſh true rules for practical gunnery, independent of the parabolic theory, we muſt content ourſelves with the data of ſome one certain

certain experimented range and time of flight, at a given angle of elevation; and then by help of these and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, as in the following examples:

107. *Example* 1. If a ball of 1 lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Anf. the velocities are -	485	477	462	566	566

108. *Exam.* 2. If a shell be found to range 1000 yards, when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Anf. 2612 feet or 871 yards.

109. *Exam.* 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder?

Anf. at 24° 16′, or at 65° 44′.

110. *Exam.* 4. With what impetus, velocity, and charge of powder, must a 13 inch shell be fired at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?

Anf. impetus 1802, veloc. 340, charge 4lb. $7\frac{1}{2}$ oz.

111. *Exam.* 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′, with the same charge of powder? Anf. 4332 feet.

112. *Exam.* 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

Anf. $6\frac{3}{4}$lb. of powder.

113. *Exam.* 7. What will be the time of flight for any given range, at the elevation of 45°?

Anf. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

114. *Exam.* 8. In what time will a shell range 3250 feet at an elevation of 32°? Anf. $11\frac{1}{4}$ sec. nearly.

115. *Exam.* 9. How far will a ſhot range on a plane which aſcends 8° 15′, and another which deſcends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′? Anſ. 4244 feet on the aſcent, and 6745 feet on the deſcent.

116. *Exam.* 10. How much powder will throw a 13 inch ſhell 4244 feet on an inclined plane which aſcends 8° 15′, the elevation of the mortar being 32° 30′?
Anſ. 7.3765lb. or 7lb. 6oz.

117. *Exam.* 11. At what elevation muſt a 13 inch mortar be pointed, to range 6745 feet, on a plane which deſcends 8° 15′; the charge being $7\frac{3}{8}$ lb. of powder? Anſ. 32° 30′.

118. *Exam.* 12. In what time will a 13 inch ſhell ſtrike a plane which riſes 8° 30′, when elevated 45°, and diſcharged with an impetus of 2304 feet? Anſ. $14\frac{2}{5}$ ſeconds.

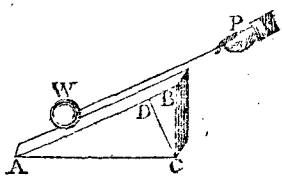
THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

PROPOSITION XXIII.

119. *If a Weight* W *be ſuſtained on an Inclined Plane* AB, *by a Power* P, *acting in a Direction* WP, *parallel to the Plane. Then*

The Weight of the Body, W,	*The Length* AB,
The Suſtaining Power, P, *and*	*The Height* BC, *and*
The Preſſure on the Plane, p,	*The Baſe* AC,
are reſpectively as	*of the Plane.*

FOR, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC,



the power acting parallel to DB, and the preſſure perp. to AB, or paral. to DC; but when three forces keep one another in equilibrio, they are proportional to the ſides of the triangle CBD, made by lines

lines in the direction of those forces by prop. 8; therefore those forces are to one another as CB, BD, DC. But the two triangles ABC, CBD are equiangular, and have their like sides proportional; therefore the three CB, BD, DC, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

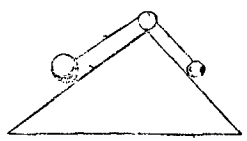
120. *Corol.* 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine, and cosine, of the plane's elevation BAC above the horizon.

For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC, are respectively as - - sin. C, sin. A, sin. B, or as - - - - radius, sine, cosine, of the angle A of elevation.

Or, the three forces are as AC, CD, AD, perp. to their directions.

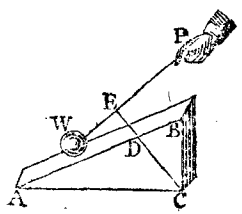
121. *Corol.* 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$, or the force with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

122. *Corol.* 3. Hence, if there be two planes of the same height, and two bodies be laid upon them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes.



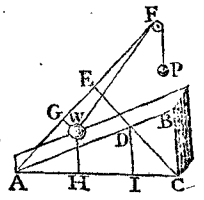
And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

123. *Corol.* 4. In the same manner, when the power P acts in any other direction whatever WP; by drawing CDE perp. to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively as AC, CD, AD, drawn perpendicular to the directions of those forces.



PROPOSITION XXIV.

124. *If a Weight* W *on an Inclined Plane* AB, *be in Equilibrio with another Weight* P *hanging freely; then if they be ſet a moving, their Perpendicular Velocities, in that Place, will be Reciprocally as thoſe Weights.*



LET the weight W deſcend a very ſmall ſpace, from W to A, along the plane, by which the ſtring PFW will come into the poſition PFA. Draw WH perpendicular to the horizon AC, and WG perpendicular to AF: then WH will be the ſpace perpendicularly deſcended by the weight W, and AG, or the difference between FA and FW, will be the ſpace perpendicularly aſcended by the weight P; and their perpendicular velocities are as thoſe ſpaces WH and AG paſſed over in thoſe directions, in the ſame time. Draw CDE perpendicular to AF, and DI perpendicular to AC.

Then, in the ſim. figs. AGWH and AEDI, -	AG : WH :: AE : DI;
and in the ſim. tri. AEC, DIC,	AC : CD :: AE : DI;
but, by cor. 4, prop. 23,	AC : CD :: W : P;
therefore, by equality; -	AG : WH :: W : P.

That is, their perpendicular ſpaces, or velocities, are reciprocally as their weights or maſſes.

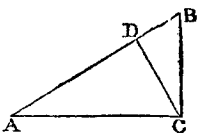
125. *Corol.* 1. Hence it follows, that if any two bodies be in equilibrio on two inclined planes, and if they be ſet a moving, their perpendicular velocities will be reciprocally as their weights. Becauſe the perpendicular weight which ſuſtains the one, would alſo ſuſtain the other.

126. *Corol.* 2. And hence alſo, if two bodies ſuſtain each other in equilibrio, on any planes, and they be put in motion; then each body multiplied by its perpendicular velocity, will give equal products.

PROPOSITION XXV.

127. *The Velocity acquired by a Body deſcending freely down an Inclined Plane* AB, *is to the Velocity acquired by a Body falling Perpendicularly, in the ſame Time; as the Height of the Plane* BC, *is to its Length* AB.

FOR the force of gravity, both perpendicularly and on the plane,



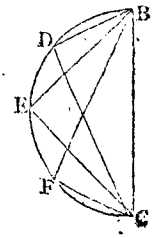
plane, is conſtant; and theſe two, by corol. 2, prop. 23, are to each other as AB to BC. But, by art. 28, the velocities generated by any conſtant forces, in the ſame time, are as thoſe forces. Therefore the velocity down BA is to the velocity down BC, in the ſame time, as the force on BA to the force on BC; that is, as BC to BA.

128. *Corol.* 1. Hence, as the motion down an inclined plane is produced by a conſtant force, it will be an uniformly accelerated motion; and therefore the laws before laid down for accelerated motions, hold good for motions on inclined planes; ſuch, for inſtance, as the following: That the velocities are as the times of deſcending from reſt; that the ſpaces deſcended are as the ſquares of the velocities, or ſquares of the times; and that if a body be thrown up an inclined plane, with the velocity it acquired in deſcending, it will loſe all its motion, and aſcend to the ſame height, in the ſame time, and will repaſs any point of the plane with the ſame velocity as it paſſed it in deſcending.

129. *Corol.* 2. Hence alſo, the ſpace deſcended down an inclined plane, is to the ſpace deſcended perpendicularly, in the ſame time, as the height of the plane CB, to its length AB, or as the ſine of inclination to radius. For the ſpaces deſcribed by any forces, in the ſame time, are as the forces, or as the velocities.

130. *Corol.* 3. Conſequently the velocities and ſpaces deſcended, by bodies down different inclined planes, are as the ſines of elevation of the planes.

131. *Corol.* 4. If CD be drawn perpendicular to AB; then, while a body falls freely through the perpendicular ſpace BC, another body will, in the ſame time, deſcend down the part of the plane BD. For, by ſimilar triangles, - - BC : BD :: BA : BC, that is, as the ſpaces deſcended by corol. 2.

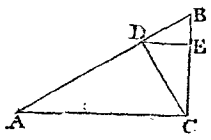


Or, in any right-angled triangle BDC, having its hypothenuſe BC perpendicular to the horizon, a body will deſcend down any of its three ſides BD, BC, DC, in the ſame time. And therefore, if upon the diameter BC a circle be deſcribed, the times of deſcending down any chords BD, BE, BF, DC, EC, FC, &c, will be all equal, and

each equal to the time of falling freely through the perpendicular diameter BC.

PROPOSITION XXVI.

132. *The Time of descending down the Inclined Plane* BA, *is to the Time of falling through the Height of the Plane* BC, *as the Length* BA *to the Height* BC.



DRAW CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time t, and the time of describing BA call T.

Now, because the spaces described by constant forces, are as the squares of the times; therefore $t^2 : T^2 :: BD : BA$.

But the three BD, BC, BA are in continual proportion; therefore $BD : BA :: BC^2 : BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or $t : T :: BC : BA$.

133. *Corol.* Hence the times of descending down different planes, of the same height, are to one another as the lengths of the planes.

PROPOSITION XXVII.

134. *A Body acquires the Same Velocity in descending down any Inclined Plane* BA, *as by falling perpendicularly through the Height of the Plane* BC.

FOR, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting. But if

F denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of describing BC, and
T the time of descending down AB;
then, by art. 119, $F : f :: BA : BC$;
and by art. 132, $t : T :: BC : BA$;
theref. by comp. $Ft : fT :: 1 : 1$.

That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

135. *Corol.* 1.

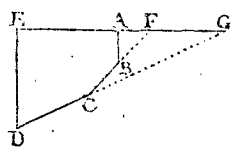
135. *Corol.* 1. Hence the velocities acquired, by bodies deſcending down any planes, from the ſame height, to the ſame horizontal line, are equal.

136. *Corol.* 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal, at all other equal altitudes A, C.

137. *Corol.* 3. Hence alſo, the velocities acquired by deſcending down any planes, are as the ſquare roots of the heights.

PROPOSITION XXVIII.

138. *If a Body deſcend down any Number of Contiguous Planes* AB, BC, CD; *it will at laſt acquire the Same Velocity, as a Body falling perpendicularly through the Same Height* ED, *ſuppoſing the Velocity not altered by changing from one Plane to another.*



PRODUCE the planes DC, CB to meet the horizontal line EA produced in F and G. Then, by art. 135, the velocity at B is the ſame, whether the body deſcend through AB or FB. And therefore the velocity at C will be the ſame, whether the body deſcend through ABC or through FC, which is alſo again, by art. 135, the ſame as by deſcending through GC. Conſequently it will have the ſame velocity at D, by deſcending through the planes AB, CD, as by deſcending through the plane GD; ſuppoſing no obſtruction to the motion by the body impinging on the planes at B and C; and this again is the ſame velocity as by deſcending through the ſame perpendicular height ED.

139. *Corol.* 1. If the lines ABCD, &c. be ſuppoſed indefinitely ſmall, they will form a curve line, which will be the path of the body: from which it appears, that a body acquires alſo the ſame velocity in deſcending along any curve, as in falling perpendicularly through the ſame height.

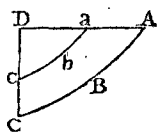
140. *Corol.* 2. Hence alſo, bodies acquire the ſame velocity, by deſcending from the ſame height, whether they deſcend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by deſcending down any lines or curves, are as the ſquare roots of the perpendicular heights.

141. *Corol.* 3. And a body, after its deſcent through any curve, will acquire a velocity which will carry it to the ſame height through an equal curve, or through any other curve, either by running up the ſmooth concave ſide, or by being retained in the curve by a ſtring, and vibrating like a pendulum: Alſo, the velocities will be equal, at all equal altitudes; and the aſcent and deſcent will be performed in the ſame time, if the curves be the ſame.

PROPOSITION XXIX.

142. *The Times in which Bodies deſcend through Similar Parts of Similar Curves,* ABC, abc, *placed alike, are as the Square Roots of their Lengths.*

THAT is, the time in AC is to the time in ac, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are ſimilar, they may be conſidered as made up of an equal number of correſponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the correſponding ſmall ſimilar parts will alſo be parallel to each other. But the time of deſcribing each of theſe pairs of correſponding parallel parts, by art. 128, are as the ſquare roots of their lengths, which, by the ſuppoſition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the ſame ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

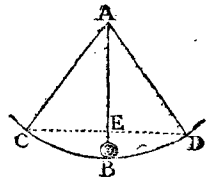
143. *Corol.* 1. Becauſe the axes DC, Dc, of ſimilar curves, are as the lengths of the ſimilar parts AC, ac, therefore the times of deſcent in the curves AC, ac, are as $\sqrt{DC}$ to $\sqrt{Dc}$, the ſquare roots of their axes.

144. *Corol.* 2. As it is the ſame thing, whether the bodies run down the ſmooth concave ſide of the curves, or be made to deſcribe thoſe curves by vibrating like a pendulum, the lengths being DC, Dc; therefore the times of the vibration of pendulums, in ſimilar arcs of any curves, are as the ſquare roots of the lengths of the pendulums.

SCHOLIUM.

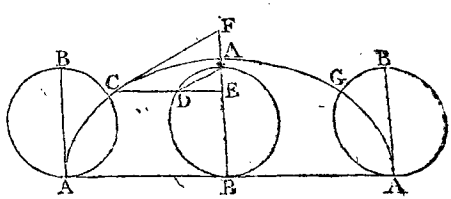
145. Having, in the laſt corollary, mentioned the pendulum, it may not be improper here to add ſome remarks concerning it.

A pen-



A pendulum conſiſts of a ball, or any other heavy body B, hung by a fine ſtring or thread, moveable about a centre A, and deſcribing the arc CBD; by which vibration the ſame motion happens to this heavy body, as would happen to any body deſcending by its gravity along the ſpherical ſuperficies CBD, if that ſuperficies was perfectly hard and ſmooth. If the pendulum be carried to the ſituation AC, and then let fall, the ball in deſcending will deſcribe the arc CB, and in the point B it will have that velocity which is acquired by deſcending through CB, or by a body falling freely through EB. This velocity will be ſufficient to cauſe the ball to aſcend through an equal arc BD, to the ſame height D from whence it fell at C: having there loſt all its motion, it will again begin to deſcend by its own gravity; and in the loweſt point B it will acquire the ſame velocity as before, which will cauſe it to reaſcend to C: and thus, by aſcending and deſcending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no reſiſtance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never ceaſe. But from thoſe obſtructions, though ſmall, it happens, that the velocity of the ball in the point B is a little diminiſhed in every vibration; and conſequently it does not return preciſely to the ſame points C or D, but the arcs deſcribed continually become ſhorter and ſhorter, till at length they grow inſenſible; unleſs the motion be aſſiſted by a mechanical contrivance, as in clocks, called a maintaining power.

DEFINITION.



146. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the ſame point A arrive at the line again, making juſt one revolution, and thereby meaſuring out a ſtraight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA; then this curve is called a cycloid; and ſome of its properties are contained in the following lemma:

LEM-

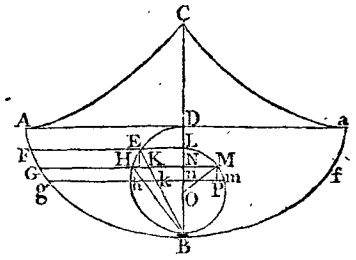
LEMMA.

147. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perp. to the axis, and the chord of the circle AD: Then the chief properties are these:

The right line CD = the circular arc AD;
The cycloidal arc AC = double the chord AD;
The semi-cycloid ACA = double the diameter AB, and
The tangent CF is parallel to the chord AD.

PROPOSITION XXX.

148. *When a Pendulum vibrates in a Cycloid; the Time of one Vibration, is to the Time in which a Body falls through Half the Length of the Pendulum, as the Circumference of a Circle is to its Diameter.*



LET ABa be the cycloid; DB its axis, or the diameter of the generating semicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball descend from F, and, in vibrating, describe the arc FBf. Divide FB into innumerable small parts, one of which is Gg; draw FEL, GM, gm perpendicular to DB. On LB describe the semicircle LMB, whose centre is O; draw Mp parallel to DB; also draw the chords BE, BH, EH, and the radius OM.

Now the triangles BEH, BHK are similar; therefore BK : BH :: BH : BE, or $BH^2 = BK \cdot BE$, or $BH = \sqrt{BK \cdot BE}$. Also, the sim. tri. Mmp, MON, give Mp : Mm :: MN : MO. And, by the nature of the cycloid, Hh is equal and parallel to Gg.

If another body descend down the chord EB, it will have the same velocity as the ball in the cycloid has at the same height. So that Kk and Gg are passed over with the same velocity, and consequently the time in passing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH to BK by similar triangles, or $\sqrt{BK \cdot BE}$ to BK, or $\sqrt{BE}$ to $\sqrt{BK}$, or as $\sqrt{BL}$ to $\sqrt{BN}$ by similar triangles.

That is, the time in Gg : time in Kk :: $\sqrt{BL}$: $\sqrt{BN}$.

Again,

Again, the time of deſcribing any ſpace with an uniform motion, is directly as the ſpace, and reciprocally as the velocity; alſo, the velocity in K or Kk, is to the velocity at B, as $\sqrt{EK}$ to $\sqrt{EB}$, or as $\sqrt{LN} : \sqrt{LB}$; and the uniform velocity for EB is equal to half that at the point B, therefore the time in Kk : time in EB :: $\frac{Kk}{\sqrt{LN}} : \frac{EB}{\frac{1}{2}\sqrt{LB}}$ :: (by ſim. tri.)

$\frac{Nn}{\sqrt{LN}} : \frac{LB}{\frac{1}{2}\sqrt{LB}}$ :: Nn or Mp : $2\sqrt{BL \cdot LN}$.

That is, the time in Kk : time in EB :: Mp : $2\sqrt{BL \cdot LN}$.
But it was, time in Gg : time in Kk :: $\sqrt{BL} : \sqrt{BN}$; theref. by comp. time in Gg : time in EB :: Mp : $2.\sqrt{BN.NL}$ or 2NM.
But, by ſim. tri. Mm : 2OM or BL :: Mp : 2NM.
Theref. time in Gg : time in EB :: Mm : BL.

Conſequently the ſum of all the times in all the Gg's is to the time in EB, or the time in DB, which is the ſame thing, as the ſum of all the Mm's is to LB;
that is, the time in Fg : time in DB :: Lm : LB,
and the time in FB : time in DB :: LMB : LB,
or the time in FBf : time in DB :: 2LMB : LB.

That is, the time of one whole vibration,
is to the time of falling through half CB,
as the circumference of any circle,
is to its diameter.

149. *Corol.* 1. Hence all the vibrations of a pendulum in a cycloid, whether great or ſmall, are performed in the ſame time; which time is to the time of falling through the axis, or half the length of the pendulum, as 3·1416 to 1, the ratio of the circumference to its diameter; and hence that time is eaſily found thus. Put $p = 3{\cdot}1416$, and l the length of the pendulum, alſo g the ſpace fallen by a heavy body in 1″ of time:

then $\sqrt{g} : \sqrt{\frac{1}{2}l} :: 1'' : \sqrt{\frac{l}{2g}}$ the time of falling through $\frac{1}{2}l$,

theref. $1 : p :: \sqrt{\frac{l}{2g}} : p\sqrt{\frac{l}{2g}}$, which therefore is the time of one vibration of the pendulum.

150. And if the pendulum vibrate in a ſmall arc of a circle; becauſe that ſmall arc nearly coincides with the ſmall cycloidal arc at the vertex B; therefore the time of vibration in the ſmall arc of a circle, is nearly equal to the time of vibration in the

the cycloidal arc; and consequently the time of vibration in a small circular arc, is equal to $p\sqrt{\frac{l}{2g}}$, where l is the radius of the circle.

151. So that, if one of these, g or l, be found by experiment, this theorem will give the other. Thus, if g, or the space fallen through by a heavy body in $1''$ of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way; this theorem will give g the descent of gravity in $1''$. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $p\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''$: and hence is found $g = \frac{1}{2}p^2l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in $1''$; which it has also been found to be very exactly, by many accurate experiments.

SCHOLIUM.

152. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

or $1 : \frac{1}{4} :: 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-seconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{7}{16}$ inches, the length of the quarter-seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here

$\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60''$ or $1' : 60\sqrt{\frac{39\frac{1}{8}}{80}} = 7\frac{1}{2}\sqrt{31{\cdot}3} =$ - - $41{\cdot}95987$, or almost 42 vibrations in a minute.

153. In these propositions, the thread is supposed to be very fine, or of no sensible weight, and the ball very small, or all the matter united in one point; also, the length of the pendulum, is the distance from the point of suspension, or centre of motion, to this point, or centre of the small ball.

ball. But if the ball be large, or the ſtring very thick, or the vibrating body be of any other figure; then the length of the pendulum is different, and is meaſured, from the centre of motion, not to the centre of magnitude of the body, but to ſuch a point, as that if all the matter of the pendulum were collected into it, it would then vibrate in the ſame time as the compound pendulum; and this point is called the Centre of Oſcillation, which will be treated of in what follows.

THE MECHANICAL POWERS, &c.

154. WEIGHT and Power, when oppoſed to each other, ſignify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

155. Equilibrium, is an equality of action or force, between two or more powers or weights, acting againſt each other, by which they deſtroy each other's effects, and remain at reſt.

156. Machine, or Engine, is any mechanical inſtrument contrived to move bodies. And it is compoſed of the mechanical powers.

157. Mechanical Powers, are certain ſimple machines, which are commonly employed for raiſing greater weights, or overcoming greater reſiſtances, than could be effected by the natural ſtrength without them.—Theſe are uſually accounted ſix in number, viz. the Lever and Balance, the Pulley, the Wheel and Axle, the Wedge, the Inclined Plane, and the Screw.

158. Mechanics, is the ſcience of forces, and the effects they produce, when applied to machines, in the motion of bodies.

159. Statics, is the ſcience of weights, eſpecially when conſidered in a ſtate of equilibrium.

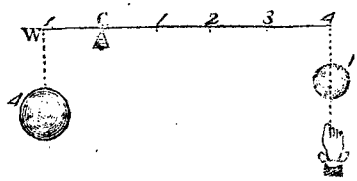
160. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

161. Centre of Gravity, is a certain point, upon which a body being freely ſuſpended, it will reſt in any poſition.

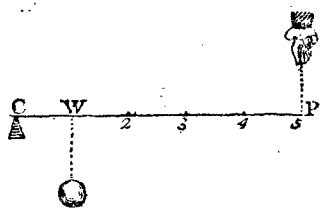
Of the LEVER.

162. A Lever is any inflexible rod, bar, or beam, which ſerves to raiſe weights, while it is ſupported at a point by a fulcrum or prop, which is the centre of motion. The lever is ſuppoſed to be void of gravity or weight, to render the demonſtrations eaſier and ſimpler. There are four kinds of levers.

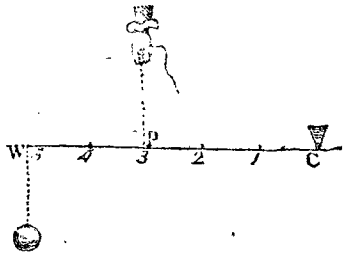
163. A Lever of the Firſt Kind has the prop C between the weight W and the power P. And of this kind are balances, ſcales, crows, handſpikes, ſciſſars, pinchers, &c.



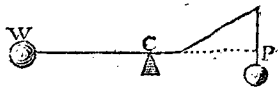
164. A Lever of the Second Kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.



165. A Lever of the Third Kind has the power between the weight and the prop. Such as tongs, the bones and muſcles of animals, a man rearing a ladder, &c.



166. A Fourth Kind is ſometimes added, called the Bended Lever. As a hammer drawing a nail.

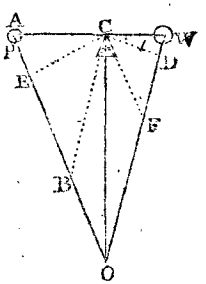


167. In all theſe machines, the power may be repreſented by a weight, which is its moſt natural meaſure, acting downwards; but having its direction changed, when neceſſary, by means of a fixed pulley.

PRO-

PROPOSITION XXXI.

168. *When the Weight and Power keep the Lever in Equilibrio, they are to each other Reciprocally as the Diſtances of their Lines of Direction from the Prop. That is,* P : W :: CD : CE; *where* CD *and* CE *are perp. to* WO *and* AO, *which are the Directions of the two Weights, or the Weight and Power* W *and* A.



For, draw CF parallel to AO, and CB parallel to WO: Alſo, join CO, which will be the direction of the preſſure on the prop C; for there cannot be an equilibrium unleſs the directions of the three forces all meet in, or tend to, the ſame point as O. Then, becauſe theſe three forces keep each other in equilibrio, they are proportional to the ſides of the triangle CBO or CFO, which are drawn in the direction of thoſe forces; therefore - - P : W :: CF : FO or CB.

But, becauſe of the parallels, the two triangles CDF, CEB are equiangular, therefore CD : CE :: CF : CB.

Hence, by equality, P : W :: CD : CE.

That is, each force is reciprocally proportional to the diſtance of its direction from the fulcrum.

And it will be found that this demonſtration will ſerve for all the other kinds of levers, by drawing the lines as directed.

169. *Corol.* 1. When the two forces act perpendicularly on the lever, as two weights, &c; then, in caſe of an equilibrium, D coincides with W, and E with P; conſequently then the above proportion becomes P : W :: CW : CP, or the diſtances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to thoſe forces.

170. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the ſine of the angle of direction CAE. For the perp. CE is as CA × s. ∠ A.

171. *Corol.* 3. Becauſe the product of the extremes is equal to the product of the means, therefore the product of the

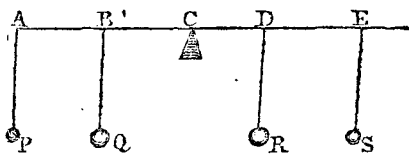
the power by the distance of its direction, is equal to the product of the weight by the distance of its direction.

That is, $P \times CE = W \times CD$.

172. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences, or spaces described, are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the momenta and radii; that is, as $P \times CE$ and $W \times CD$, which are equal by cor. 3.

173. *Corol.* 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.

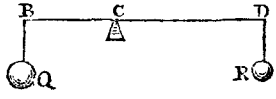
174. *Corol.* 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio, then the sum of the products on one side of the prop, will be equal to the sum on the other, made by multiplying each weight by its distance; namely,



$P \times AC + Q \times BC = R \times DC + S \times EC$.

For, the effect of each weight to turn the lever, is as the weight multiplied by its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal.

175. *Corol.* 7. Because, when two weights Q and R are in equilibrio, $Q : R :: CD : CB$;



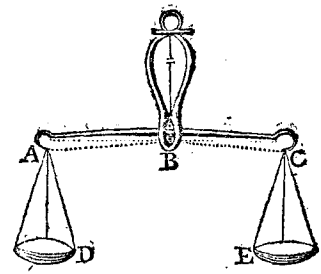
therefore, by composition, $Q + R : Q :: BD : CD$,
and $Q + R : R :: BD : CB$.

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHO-

SCHOLIUM.

176. Upon the foregoing principles depends the nature of ſcales and beams, for weighing all ſorts of commodities. For, if the weights be equal, then will the diſtances be equal alſo, which gives the conſtruction of the common ſcales, which ought to have theſe properties:



1*ſt*, The points of ſuſpenſion of the ſcales and the centre of motion of the beam, A, B, C, muſt be in a ſtraight line: 2*d*, The arms AB, BC muſt be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as poſſible at the centre B. A defect in any of theſe properties, makes the ſcales either imperfect or falſe. But it often happens that the one ſide of the beam is made ſhorter than the other, and the defect covered by making that ſcale the heavier, by which means the ſcales hang in equilibrio when empty; but when they are charged with any weights, ſo as to be ſtill in equilibrio, thoſe weights are not equal; but the deceit will be ſhewn by changing the weights to the contrary ſides, for then the equilibrium will be immediately deſtroyed.

177. To find the true weight of any body by ſuch a falſe balance:—Firſt weigh the body in one ſcale, and afterwards weigh it in the other; then the mean proportional between theſe two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the ſcale D, and only w pounds or ounces in the ſcale E; then we have theſe two equations, namely, $AB \cdot b = BC \cdot W$,

and $BC \cdot b = AB \cdot w$;

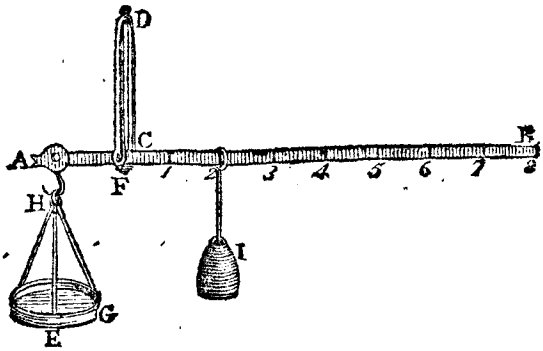
the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

hence then $b^2 = Ww$,

and $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

178. The Roman Statera, or Steelyard, is alſo a lever, but of unequal brachia or arms, ſo contrived that one weight only may ſerve to weigh a great many, by ſliding it backwards and forwards to different diſtances on the longer arm of the lever; and it is thus conſtructed:



Let AB be the ſteelyard, and C its centre of motion, from whence the diviſions muſt commence if the two arms juſt balance each other: if not, ſlide the conſtant moveable weight I along from B towards C, till it juſt balance the other end without a weight, and there make a notch in the beam, marking it with a cypher 0. Then hang on at A a weight W equal to I, and ſlide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and ſliding I back to balance it, and there mark it with 2. Do the ſame at 3, 4, 5, &c, by making W equal to 3, 4, 5, &c, times I; and the beam is finiſhed. Then, to find the weight of any body *b* by the ſteelyard; take off the weight W, and hang on the body *b* at A; then ſlide the weight I backwards and forwards till it juſt balance the body *b*, which ſuppoſe to be at the number 5; then is *b* equal to 5 times the weight of I. So, if I be one pound, then *b* is 5 pounds; but if I be 2 pounds, then *b* is 10 pounds; and ſo on.

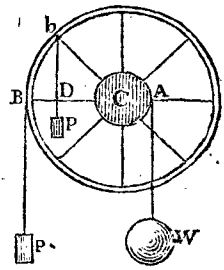
Of the WHEEL and AXLE.

PROPOSITION XXXII.

179. *In the Wheel-and-Axle; the Weight and Power will be in Equilibrio, when the Power* P *is to the Weight* W, *Reciprocally as the Radii of the Circles where they act; that is, as the Radius of the Axle* CA, *where the Weight hangs, to the Radius of the Wheel* CB, *where the Power acts. That is,* P : W :: CA : CB.

HERE the cord, by which the power P acts, goes about the

the circumference of the wheel, while that of the weight W goes round its axle, or another ſmaller wheel, attached to the larger, and having the ſame centre C. So that BA is a lever moveable about the point C, the power P acting always at the diſtance BC, and the weight W at the diſtance CA; therefore P : W :: CA : CB.

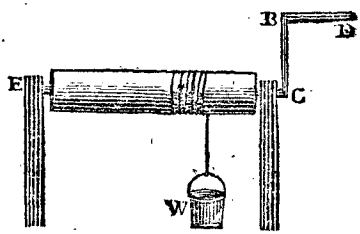


180. *Corol.* 1. If the wheel be put in motion; then, the ſpaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much ſlower, as it is heavier than the power; ſo that what is gained in power, is loſt in time. And this is the univerſal property of all machines and engines.

181. *Corol.* 2. If the power do not act at right angles to the radius Cb, but obliquely; draw CD perp. to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.

182. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or ſpindle CE, by the handle CBD; the power applied at B is to the weight W on the roller, as the radius of the roller is to the radius CB of the handle.



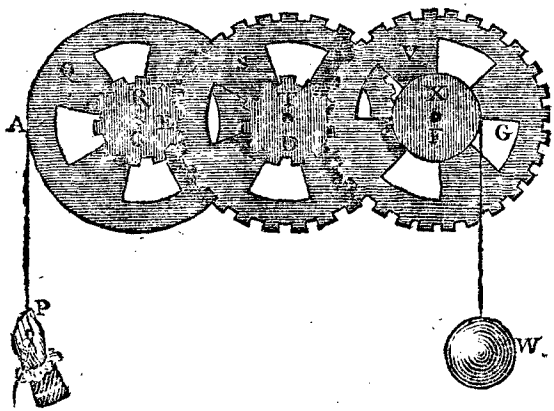
183. And the ſame for all cranes, capſtans, windlaſſes, and ſuch like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; ſo that they are always in the reciprocal ratio of their velocities. And to the ſame principle may be referred the gimblet and auger for boring holes.

184. But all this, however, is on ſuppoſition that the ropes or cords, ſuſtaining the weights, are of no ſenſible thickneſs. For, if the thickneſs be conſiderable, or if there be ſeveral folds of them, over one another, on the roller or barrel; then we muſt meaſure to the middle of the outermoſt rope, for

the radius of the roller; or, to the radius of the roller and half the thickneſs of the cord, when there is but one fold.

185. The wheel-and-axle has a great advantage over the ſimple lever, in point of convenience. For a weight can be raiſed only a little way by the lever. But, by the continual turning of the wheel and roller, the weight may be raiſed to any height, or from any depth.

186. By increaſing the number of wheels too, the power may be multiplied to any extent, making always the leſs wheels to turn greater ones, as far as we pleaſe: and this is commonly called Tooth and Pinion, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in caſe of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P



turn the wheel Q, and this turn the ſmall wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V, and this turn the axle X, which raiſes the weight W; then P : W :: CB . DE . FG : AC . BD . EF. And in the ſame proportion is the velocity of W ſlower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then P : W :: 1^3 : 10^3 or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faſter than the weight.

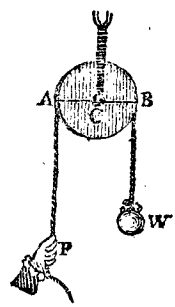
OF THE PULLEY.

187. A PULLEY is a ſmall wheel, commonly made of wood or braſs, which turns about an iron axis paſſing through the centre, and fixed in a block, by means of a cord paſſed round its circumference, which ſerves to draw up any weight. The pulley is either ſingle, or combined together, to increaſe the power. It is alſo either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

PROPOSITION XXXIII.

188. *If a Power ſuſtain a Weight by means of a Fixed Pulley; the Power and Weight are Equal.*

FOR through the centre C of the pulley draw the horizontal line AB: then will AB repreſent a lever of the firſt kind, its prop being the fixed centre C; from which the points A and B, where the power and weight act, being equally diſtant; the power P is therefore equal to the weight W.



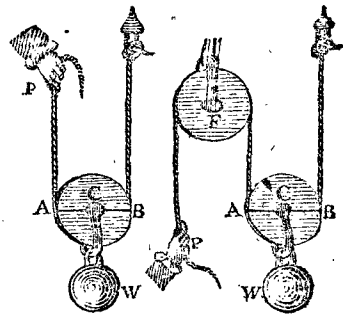
189. *Corol.* Hence, if the pulley be put in motion, the power P will deſcend as faſt as the weight W aſcends. So that the power is not increaſed by the uſe of the fixed pulley, even though the rope go over ſeveral of them. It is, however, of great ſervice in the raiſing of weights, both by changing the direction of the force, for the convenience of acting, and enabling a perſon to raiſe a weight to any height without moving from his place, and alſo by enabling a great many perſons at once to exert their force on the rope at P, which they could not do to the weight itſelf; as is evident in raiſing the hammer or weight of a pile-driver, as well as on many other occaſions.

PROPOSITION XXXIV.

190. *If a Power ſuſtain a Weight by means of a Moveable Pulley; the Power is but Half the Weight.*

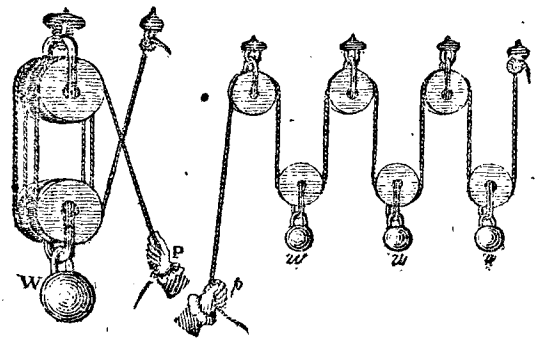
FOR, here AB may be conſidered as a lever of the ſecond

kind, the power acting at A, the weight at C, and the prop or fixed point at B; and becauſe P : W :: CB : AB, and CB = ½AB, therefore P = ½W, or W = 2P.



191. *Corol.* 1. Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P deſcends twice as faſt as the point C and weight W riſes. It is alſo evident, that the fixed pulley F makes no difference in the power P, but is only uſed to change the direction of it, from upwards to downwards.

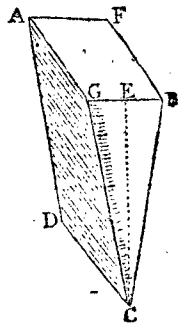
192. *Corol.* 2. Hence we may eſtimate the effect of a combination of any number of fixed and moveable pulleys; by which we ſhall find that every cord going over a moveable pulley, always doubles the power, ſince each end of the rope bears an equal ſhare of the weight; while each rope that is fixed to a pulley, only increaſes the power by unity.



Here $P = \frac{1}{6}W$.

Here $P = \frac{1}{2}w = \frac{w + w + w}{6}$.

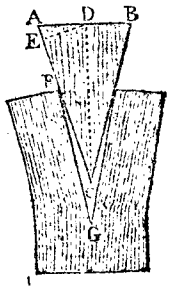
OF THE WEDGE.



193. THE WEDGE is a body of wood or metal, in form of a prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides; and its end GBC is composed of two equal inclined planes GCE, BCE.

PROPOSITION XXXV.

194. *When a Wedge is in Equilibrio; the Power acting against the Back, is to the Force acting Perpendicularly against either Side, as the Breadth of the Back* AB, *is to the Length of the Side* AC *or* BC.



FOR, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting upon them; therefore the three forces are as AB, AC, BC.

195. *Corol.* The force on the back, } AB,
Its effect in direct perp. to AC, } AC,
And its effect parallel to AB; } DC,
are as the three lines } which are perp. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

196. But it must be observed, that the resistance, or the forces above mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the lines AC and DC. In the

wedge, the friction againſt the ſides is very great, at leaſt equal to the force to be overcome, becauſe the wedge retains any poſition to which it is driven; and therefore the reſiſtance is doubled by the friction. But then the wedge has a great advantage over all the other powers, ariſing from the force of percuſſion or blow with which the back is ſtruck, which is a force incomparably greater than any dead weight or preſſure, ſuch as is employed in other machines. And accordingly we find it produces effects vaſtly ſuperior to thoſe of any other power; ſuch as the ſplitting and raiſing the largeſt and hardeſt rocks, the raiſing and lifting the largeſt ſhip, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the ſmall blow of a hammer, on the back of a wedge, is incomparably greater than any mere preſſure, and will overcome it.

Of the SCREW.

197. THE SCREW is one of the ſix mechanical powers, chiefly uſed in preſſing or ſqueezing bodies cloſe, though ſometimes alſo in raiſing weights.

The ſcrew is a ſpiral thread or groove cut round a cylinder, and every where making the ſame angle with the length of it. So that if the ſurface of the cylinder, with this ſpiral thread on it, were unfolded and ſtretched into a plane, the ſpiral thread would form a ſtraight inclined plane, whoſe length would be to its height, as the circumference of the cylinder, is to the diſtance between two threads of the ſcrew: as is evident by conſidering that, in making one round, the ſpiral riſes along the cylinder the diſtance between the two threads.

PROPOSITION XXXVI.

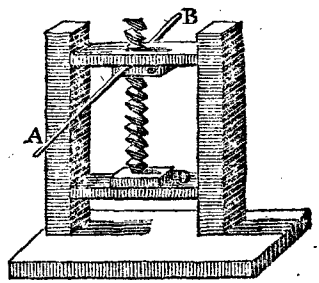
198. *The Force of a Power applied to turn a Screw round, is to the Force with which it preſſes upwards or downwards, ſetting aſide the Friction, as the Diſtance between two Threads, is to the Circumference where the Power is applied.*

THE ſcrew being an inclined plane, or half wedge, whoſe height is the diſtance between two threads, and its baſe the ſaid

ſaid circumference; and the force in the horizontal direction, being to that in the vertical one, as the lines perp. to them, namely, as the height of the plane, or diſtance of the two threads, is to the baſe of the plane, or circumference at the place where the power is applied; therefore the power is to the preſſure, as the diſtance of two threads, is to that circumference.

199. *Coral.* When the ſcrew is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

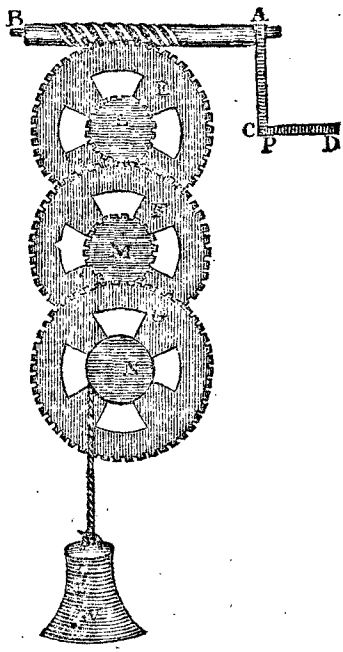
SCHOLIUM.



200. Hence we can eaſily compute the force of any machine turned by a ſcrew. Let the annexed figure repreſent a preſs driven by a ſcrew, whoſe threads are each a quarter of an inch aſunder; and that the ſcrew is turned by a handle of 4 feet long from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the ſcrew will preſs on the board at D, when the man turns the handle at A and B, with his whole force. The diameter AB of the power being 4 feet or 48 inches, its circumference is $48 \times 3{\cdot}1416$ or $150\frac{4}{5}$ nearly; and the diſtance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the preſſure, as 1 to $603\frac{1}{5}$: but the power is equal to 150lb; theref. as $1 : 603\frac{1}{5} :: 150 : 90480$; and conſequently the preſſure at D is equal to a weight of 90480 pounds; independent of friction.

201. Again, if the endleſs ſcrew AB be turned by a handle AC of 20 inches, the threads of the ſcrew being diſtant

diſtant half an inch each; and the ſcrew turn a toothed wheel E, whoſe pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raiſe, working at the handle C; ſuppoſing the diameters of the wheels to be 18 inches, and thoſe of the pinions and barrel 2 inches; the teeth and pinions being all of a ſize.



Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumf. of the power.

And 125·664 to $\frac{1}{2}$, or 251·328 to 1, is the force of the ſcrew alone.

Alſo, 18 to 2, or 9 to 1 being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1, or 729 to 1 is the power gained by the wheels.

Conſequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{9}$ to 1 nearly, is the ratio of the power to the weight, ariſing from the advantage of both the ſcrew and the wheels.

But the power is 150lb; and therefore $150 \times 183218\frac{1}{9}$, or 27482716 pounds, is the weight the man can ſuſtain; which is equal to 12269 tons weight.

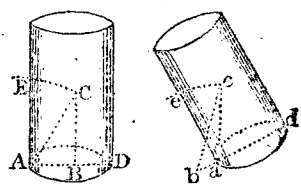
But the power has to overcome, not only the weight, but alſo the friction of the ſcrew, which is very great, in ſome caſes equal to the weight itſelf, ſince it is ſometimes ſufficient to ſuſtain the weight, when the power is taken off.

OF THE CENTRE OF GRAVITY.

202. THE CENTRE of GRAVITY of a body, is a certain point within it, upon which the body being freely ſuſpended, it will reſt in any poſition; and it will always deſcend to the loweſt place to which it can get, in other poſitions.

PROPOSITION XXXVII.

203. *If a Perpendicular to the Horizon, from the Centre of Gravity of any Body, fall Within the Baſe of the Body, it will reſt in that Poſition; but if the Perpendicular fall Without the Baſe, the Body will not reſt in that Poſition, but will tumble down.*



FOR, if CB be the perp. from the centre of gravity C, within the baſe: then the body cannot fall over towards A; becauſe, in turning on the point A, the centre of gravity C would deſcribe an arc which would riſe from C to E; contrary to the nature of that centre, which only reſts when in the loweſt place. For the ſame reaſon, the body will not fall towards D. And therefore it will ſtand in that poſition.

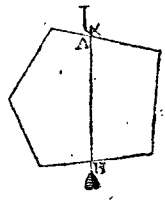
But if the perp. fall without the baſe, as Cb; then the body will tumble over on that ſide; becauſe, in turning on the point a, the centre c deſcends by deſcribing the deſcending arc ce.

204. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall juſt on the extremity of the baſe; the body may ſtand; but any the leaſt force will cauſe it to fall that way. And the nearer the perpendicular is to any ſide, or the narrower the baſe is, the eaſier it will be made to fall, or be puſhed over that way; becauſe the centre of gravity has the leſs height to riſe: which is the reaſon that a globe is made to roll on a ſmooth plane by any the leaſt force. But the nearer the perpendicular is to the middle of the baſe, or the broader the baſe is, the firmer the body ſtands.

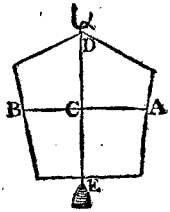
205. *Corol.* 2. Hence, if the centre of gravity of a body be ſupported, the whole body is ſupported. And the place of

of the centre of gravity muſt be accounted the place of the body; for into that point the whole matter of the body may be ſuppoſed to be collected, and therefore all the force with which it endeavours to deſcend.

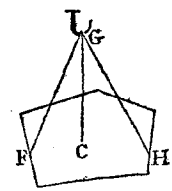
206. *Coral.* 3. From the property which the centre of gravity has, of always deſcending to the loweſt point, is derived an eaſy mechanical method of finding that centre.



For, if the body be hung up by any point A, and a plumb line AB be hung by the ſame point, it will paſs through the centre of gravity; becauſe that centre is not in the loweſt point till it fall in the plumb line. Mark the line AB upon it. Then hang the body up by any other point D, with a plumb line DE, which will alſo paſs through the centre of gravity, for the ſame reaſon as before; and therefore that centre muſt be at C where the two plumb lines croſs each other.



207. Or, if the body be ſuſpended by two or more cords GF, GH, &c, then a plumb line from the point G will cut the body in its centre of gravity C.



208. Likewiſe, becauſe a body reſts when its centre of gravity is ſupported, but not elſe; we hence derive another eaſy method of finding that centre mechanically. For, if the body be laid on the edge of a priſm, and moved backwards and forwards till it reſt, or balance itſelf; then is the centre of gravity juſt over the line of the edge. And if the body be then ſhifted into another poſition, and balanced on the edge again, this line will alſo paſs by the centre of gravity; and conſequently the interſection of the two will give the centre itſelf.

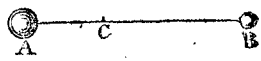
PROPOSITION XXXVIII.

209. *The Common Centre of Gravity* C *of any two Bodies* A, B, *divides the Line joining their Centres, into two Parts, which are Reciprocally as the Bodies.*

That is, AC : BC :: B : A.

FOR, if the centre of gravity C be ſupported, the two bodies

bodies A and B will be ſupported, and will reſt in equilibrio. But, by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their diſtances from that point; therefore A : B :: CB : CA.

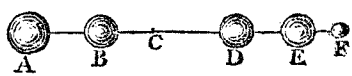


210. *Corol.* 1. Hence AB : AC :: A + B : B; or, the whole diſtance between the two bodies, is to the diſtance of either of them from the common centre, as the ſum of the bodies is to the other body.

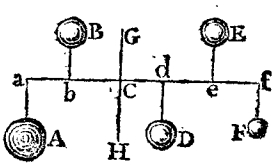
211. *Corol.* 2. Hence alſo, CA . A = CB . B; or the two products are equal, which are made by multiplying each body by its diſtance from the centre of gravity.

212. *Corol.* 3. As the centre C is preſſed with a force equal to both the weights A and B, while the points A and B are each preſſed with the reſpective weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be ſupported, each will ſtill bear, or be preſſed by the ſame weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, ſupported by two men at A and B, diſtant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight.

213. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and its diſtance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the ſtraight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the ſum of the products on one ſide, equal to the ſum of the products on the other, made by multiplying each body by its diſtance from that centre. And if ſeveral bodies be in equilibrium upon any ſtraight lever, then the prop is in the centre of gravity.



214. *Corol.* 5. And although the bodies be not ſituated in a ſtraight line, but ſcattered about in any promiſcuous manner, the ſame property as in the laſt corollary ſtill holds true, if perpendiculars to any line whatever af be drawn through the ſeveral bodies



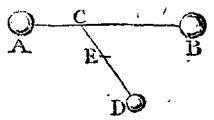
bodies and their common centre of gravity, namely, that $Ca \cdot A + Cb \cdot B = Cd \cdot D + Ce \cdot E + Cf \cdot F$. For the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

PROPOSITION XXXIX.

215. *If there be three or more Bodies, and, if a Line be drawn from any one Body* D *to the Centre of Gravity of the reſt* C; *then the Common Centre of Gravity* E *of all the Bodies, divides the line* CD *into two Parts in* E, *which are Reciprocally Proportional as the Body* D *to the Sum of all the other Bodies.*

That is, CE : ED :: D : A + B &c.

For, ſuppoſe the bodies A and B to be collected into their common centre of gravity C, and let their ſum be called S. Then, by the laſt prop. CE : ED :: D : S or A + B &c.



216. *Corol.* Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by firſt finding the centre of any two of them, then the centre of that centre and a third, and ſo on for a fourth, or fifth, &c.

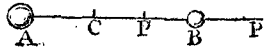
PROPOSITION XL.

217. *If there be taken any Point* P, *in the Line paſſing through the Centres of two Bodies; then the Sum of the two Products, of each Body multiplied by its Diſtance from that Point, is equal to the Product of the Sum of the Bodies multiplied by the Diſtance of their Common Centre of Gravity* C *from the ſame Point* P.

That is, $PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}$.

For, by the 38th, $CA \cdot A = CB \cdot B$,

that is, $\overline{PA - PC} \cdot A = \overline{PC - PB} \cdot B$;



theref. by adding,

$PA \cdot A + P\beta \cdot B = PC \cdot \overline{A + B}$.

218. *Corol.* 1. Hence, the two bodies A and B have the ſame force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or,

Or, if the line, with the bodies, move about the point P; the ſum of the momenta of A and B, is equal to the momentum of the ſum S or A + B placed at the centre C.

219. *Corol.* 2. The ſame is alſo true of any number of bodies whatever, as will appear by cor. 4, prop. 38, namely, $PA \cdot A + PB \cdot B + PD \cdot D$ &c. $= PC \cdot \overline{A + B + D}$ &c, where P is any point whatever in the line AC.

And, by cor. 5, prop. 38, the ſame thing is true when the bodies are not placed in that line, but any where in the perpendiculars paſſing through the points A, B, D, &c; namely, $Pa \cdot A + Pb \cdot B + Pd \cdot D$ &c. $= PC \cdot \overline{A + B + D}$ &c.

220. *Corol.* 3. And if a plane paſs through the point P perpendicular to the line CP; then the diſtance of the common centre of gravity from that plane, is

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c}}{A + B + D \text{ \&c}},$$

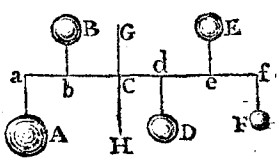
that is, equal to the ſum of all the forces divided by the ſum of all the bodies. Or, if A, B, D, &c, be the ſeveral particles of one maſs or compound body; then the diſtance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole maſs or body, that is, equal to all the Pa . A, Pb . B, Pd . D, &c, divided by the body or ſum of the particles A, B, D, &c.

PROPOSITION XLI.

221. *To find the Centre of Gravity of any Body, or of any Syſtem of Bodies.*

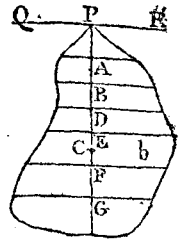
THROUGH any point P draw a plane, and let Pa, Pb, Pd, &c, be the diſtances of the bodies A, B, D, &c, from the plane; then, by the laſt cor. the diſtance of the common centre of gravity from the plane, will be

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c}}{A + B + D \text{ \&c}}.$$



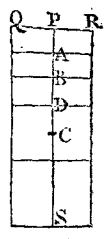
222. Or, if *b* be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw

draw innumerable ſections of the body b parallel to the plane QR. Let s denote any of theſe ſections, and $d = PA$, or PB, &c, its diſtance from the plane QR. Then will the diſtance of the centre of gravity of the body from the plane be $PC = \frac{\text{ſum of all the } ds}{b}$. And if the diſtance be thus found for two interſecting planes, they will give the point in which the centre is placed.



223. But the diſtance from one plane is ſufficient for any regular body, becauſe it is evident that, in ſuch a figure, the centre of gravity is in the axis, or line paſſing through the centres of all the parallel ſections.

Thus, if the figure be a parallelogram, or a cylinder, or any priſm whatever; then the axis or line, or plane PS, which biſects all the ſections parallel to QR, will paſs through the centre of gravity of all thoſe ſections, and conſequently through that of the whole figure C. Then, all the ſections s being equal, and the body $b = PS \cdot s$, the diſtance of the centre will be $PC = \frac{PA \cdot s + PB \cdot s + \&c}{b} = \frac{PA + PB + PD \&c}{b} \times s = \frac{PA + PB + \&c}{PS}$.



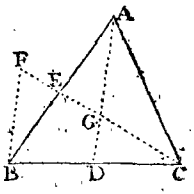
But $PA + PB + \&c$, is the ſum of an arithmetical progreſſion, beginning at o, and increaſing to the greateſt term PS, the number of the terms being alſo equal to PS; therefore the ſum $PA + BP + \&c = \frac{1}{2}PS \cdot PS$; and conſequently $PC = \frac{\frac{1}{2}PS \cdot PS}{PS} = \frac{1}{2}PS$; that is, the centre of gravity is in the middle of the axis of any figure whoſe parallel ſections are equal.

224. In other figures, whoſe parallel ſections are not equal, but varying according to ſome general law, it will not be eaſy to find the ſum of all the $PA \cdot s$, $PB \cdot s'$, $PD \cdot s''$, &c, except by the general method of Fluxions; which caſe, therefore, will be beſt reſerved till we come to treat of that doctrine. It will be proper, however, to add here ſome examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

PRO-

PROPOSITION XLII.

225. *To find the Centre of Gravity of a Triangle.*



FROM any two of the angles draw lines AD, CE, to bisect the opposite sides; so will their intersection G be the centre of gravity of the triangle.

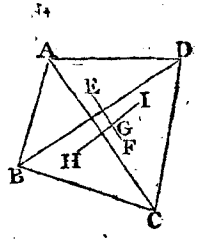
For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure; therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it also lies in the line CE. And consequently it is in their common point of intersection G.

226. *Corol.* The distance of the point G, is $AG = \frac{2}{3}AD$, and $CG = \frac{2}{3}CE$; or $AG = 2GD$, and $CG = 2GE$.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are similar, and also equal, because AE = BE; consequently AG = BF. But the triangles CDG, CBF are also equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is also = AG; conseq. AG = 2GD or $\frac{2}{3}AD$. In like manner, CG = 2GE or $\frac{2}{3}CE$.

PROPOSITION XLIII.

227. *To find the Centre of Gravity of a Trapezium.*



DIVIDE the trapezium ABCD into two triangles, by the diagonal BD, and find E, F the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

228. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by

the other diagonal AC, and the centres of gravity H and I of theſe two triangles be alſo found; then the centre of gravity of the trapezium will alſo lie in the line HI.

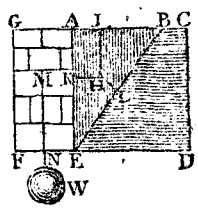
So that, lying in both the lines, EF, HI, it muſt neceſſarily lie in their interſection G.

229. And thus we are to proceed for a figure of any greater number of ſides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of theſe, till they be all reduced into one only.

Of the uſe of the place of the centre of gravity, and the nature of forces, the following practical problem is added, to find the force of a bank of earth preſſing againſt a wall, and the force of the wall to ſupport it.

PROPOSITION XLIV.

230. *To determine the Force with which a Bank of Earth, or ſuch like, preſſes againſt a Wall, and the Dimenſions of the Wall neceſſary to ſupport it.*



LET ACDE be a vertical ſection of a bank of earth, and ſuppoſe that, if it were not ſupported, a triangular part of it, as ABE, would ſlide down, leaving it at what is called the natural ſlope BE; but that, by means of a wall AEFG, it is ſupported, and kept in its place.—It is required to find the force of ABE, and the dimenſions of the wall AEFG.

Let H be the centre of gravity of the triangle ABE, from which draw HI perpendicular to the horizontal line AB, HK to the upright face of the wall AE, and HL to the ſlope face of the earth BE. Now the centre of gravity H may be accounted the place of the triangle ABE, or the point into which it is all collected; and it is ſuſtained by three forces, namely, its weight acting in the direction IH, the reſiſtance of the plane BE in the direction LH, and the reſiſtance of the plane AE in the direction KH; and theſe three forces, ſuſtaining the body in equilibrio, are as the three lines perpendicular to their directions, namely, as the three lines AB, BE, AE; therefore the weight of the body ABE, is to its preſſure againſt K, as AB is to AE. But $\frac{1}{2}$AE. AB is the area of the triangle ABE; and if m be the ſpecific gravity of the earth, then $\frac{1}{2}$AE. AB. m is as its weight.

Therefore,

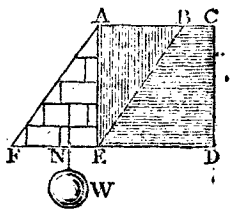
Therefore, as AB : AE :: $\frac{1}{2}$AE . AB . m : $\frac{1}{2}$AE^2 . m, the force or preſſure againſt K: which therefore is proportional to the ſquare of the altitude AE, whatever be the breadth AB, or the angle of the ſlope AEB. And the effect of this preſſure to overturn the wall, is alſo as the length of the lever KE or $\frac{2}{3}$AE: conſequently its effect is $\frac{1}{2}$AE^2 . m . $\frac{2}{3}$AE, or $\frac{1}{3}$AE^3 . m. Which muſt be balanced by the counter reſiſtance of the wall, in order that it may at leaſt be ſupported.

Now, if M be the centre of gravity of the wall, into which its whole matter may be ſuppoſed to be collected, and acting in the direction MNW, its effect will be the ſame as if a weight W were ſuſpended from the point N of the lever FN. Hence, if A be put for the area of the wall AEFG, and n its ſpecific gravity; then A . n will be equal to the weight W, and A . n . FN its effect on the lever to prevent it from turning about the point F. And as this effort muſt be equal to that of the triangle of earth, that it may juſt ſupport it, which was before found equal to $\frac{1}{3}$AE^3 . m; therefore A . n . FN = $\frac{1}{3}$AE^3 . m, in caſe of an equilibrium.

231. But now, both the breadth of the wall FE, and the lever FN, or place of the centre of gravity M, will depend on the figure of the wall. If the wall be rectangular, or as broad at top as bottom; then FN = $\frac{1}{2}$FE, and the area A = AE . FE; and conſequently the effort of the wall A . n . FN is = $\frac{1}{2}$FE^2 . AE . n; which muſt be = $\frac{1}{3}$AE^3 . m, the effort of the earth. And the reſolution of this equation gives the breadth of the wall FE = AE$\sqrt{\frac{2m}{3n}}$. So that the breadth of the wall is always proportional to its height, and is always the ſame at the ſame height, whatever the ſlope may be. But the breadth muſt be made a little more than the above value of it, that it may be more than a bare balance to the earth.

232. If the wall be of brick; its ſpecific gravity is about 2000, and that of earth about 1984; namely, m to n as 1984 to 2000: then $\sqrt{\frac{2m}{3n}}$ = ·813 = $\frac{13}{16}$ very nearly; and hence FE = $\frac{13}{16}$ AE. That is, whenever a brick rectangular wall is made to ſupport earth, its thickneſs muſt be at leaſt $\frac{13}{16}$ of its height. But if the wall be of ſtone, whoſe ſpecific gravity is about 2520; then $\sqrt{\frac{2m}{3n}}$ = ·7246 = $\frac{8}{11}$ nearly: that

is, when the rectangular wall is of ſtone, the breadth muſt be at leaſt $\frac{8}{11}$ of its height.

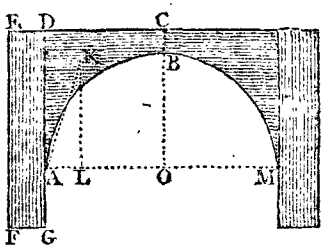


233. But if the figure of the wall be a triangle, the outer ſide tapering to a point at top. Then the lever $FN = \frac{2}{3}FE$, and the area $A = \frac{1}{2}FE \cdot AE$; conſequently its effort $A \cdot n \cdot FN$ is $= \frac{1}{3}FE^2 \cdot AE \cdot n$; which being put $= \frac{1}{3}AE^3 \cdot m$, the equation gives $FE = AE\sqrt{\frac{m}{n}}$ for the breadth of the wall at the bottom, for an equilibrium in this caſe alſo. Where again FE is as AE, and indeed equal to it when the two ſpecific gravities are equal; which is nearly the caſe when this wall is of brick. But when it is of ſtone; then $\sqrt{\frac{m}{n}} = \cdot 8873 = \frac{8}{9}$ nearly: that is, the triangular ſtone wall muſt have its thickneſs at bottom equal to $\frac{8}{9}$ of its height.

And in like manner, for other figures of the wall.

PROPOSITION XLV.

234. *To determine the Thickneſs of a Pier neceſſary to ſupport a Given Arch.*



LET ABCD be half the arch, and DEFG the pier. From the centre of gravity K of the arch draw KL perpendicular to the horizon. Then the weight of the arch in direction KL will be to the horizontal puſh at A in direction LA, as KL to LA.

For the weight of the arch in direction KL, the horizontal puſh or lateral preſſure in direction LA, and the puſh in direction KA, will be as the three ſides KL, LA, KA. So that if A denote the weight or area of the arch; then $\frac{LA}{KL} \cdot A$ will be its force at A in the direction LA; and $\frac{LA}{KL} \cdot GA \cdot A$ its effect on the lever GA to overſet the pier, or to turn it about the point F.

Again,

Again, the weight or area of the pier, is as EF . FG ; and therefore EF . FG . $\frac{1}{2}$FG, or $\frac{1}{2}$EF . FG^2, is its effect on the lever $\frac{1}{2}$FG, to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and arch be in equilibrio, these two effects must be equal. Therefore we have $\frac{1}{2}$EF . FG^2 = $\frac{LA}{KL}$. GA . A, and consequently the thickness of the pier is

$$FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL}} \times A.$$

Example 1. Suppose the arc ABN to be a semicircle; and that DC or AO = 45, BC = 6, and GA = 18 feet. Then KL will be found = 40, AL = 15 nearly, and EF = 69; also, the area ABCD or A = 704$\frac{1}{2}$. Therefore FG = $\sqrt{\frac{2GA \cdot AL}{EF \cdot KL}}$. A is = $\sqrt{\frac{36 \cdot 15}{69 \cdot 40}}$. 704$\frac{1}{2}$ = 11$\frac{4}{5}$ nearly, which is the thickness of the pier.

Example 2. Suppose, in the segment ABN, AN = 100, OB = 41$\frac{1}{2}$, BC = 6$\frac{1}{2}$, and AG = 10. Then EF = 58, KL = 35, AL = 15 nearly, and ABCD or A = 842. Therefore FG = $\sqrt{\frac{2GA \cdot AL}{EF \cdot KL}}$. A = $\sqrt{\frac{20 \cdot 15}{58 \cdot 35}}$. 842 = 11$\frac{2}{13}$ nearly, the thickness of the pier in this case.

OF THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

235. THE CENTRE of PERCUSSION, of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest as it were in equilibrio, without acting on the centre of suspension.

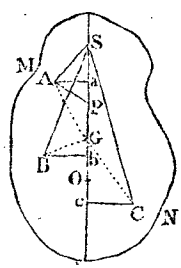
236. The Centre of Oscillation is that point, in a vibrating body, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

237. The Centre of Gyration, is that point, in which if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

238. The angular motion of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different, unconnected bodies, each revolving about a centre, the angular velocity is as the absolute velocity directly, and distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

PROPOSITION XLVI.

239. *To find the Centre of Percussion of a Body, or System of Bodies.*



LET the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion, as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as A . SA; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as A . Sa, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as $A . Sa . PO = A . Sa . \overline{SO - SP} = A . Sa . SO - A . Sa . SP = A . Sa . SO - A . SA^2$. In like manner, the forces of B and C, to turn the system about O, are as

$B . Sb . SO - B . SB^2$, and
$C . Sc . SO - C . SC^2$, &c.

But,

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities, must be equal to the sum of the negative parts,

that is, $A \cdot Sa \cdot SO + B \cdot Sb \cdot SO + C \cdot Sc \cdot SO$ &c $=$ - -
$A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$ &c;

and hence $SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{A \cdot Sa + B \cdot Sb + C \cdot Sc \text{ \&c}}$, which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points a, b, &c, fall on the contrary side of S, the corresponding product A . Sa, or B . Sb, &c, must be made negative.

240. *Corol.* 1. Since, by cor. 3, pr. 40, A + B + C &c, or the body $b \times$ the distance of the centre of gravity, SG, is $= A \cdot Sa + B \cdot Sb + C \cdot Sc$ &c, which is the denominator of the value of SO; therefore the distance of the centre of percussion is $SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{SG \times \text{body } b}$.

241. *Corol.* 2. Since, by Geometry, theor. 36, 37,
it is $SA^2 = SG^2 + GA^2 - 2SG \cdot Ga$,
and $SB^2 = SG^2 + GB^2 + 2SG \cdot Gb$,
and $SC^2 = SG^2 + GC^2 + 2SG \cdot Gc$, &c;
and, by cor. 5, pr. 39, the sum of the last terms is nothing, namely, $-2SG \cdot Ga + 2SG \cdot Gb + 2SG \cdot Gc$ &c $= 0$; therefore the sum of the others, or $A \cdot SA^2 + B \cdot SB^2$ &c
$= \overline{A + B \text{ \&c}} \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2$ &c,
or $= b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2$ &c;
which being substituted in the numerator of the foregoing value of SO, gives

$$SO = \frac{b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + \text{\&c}}{b \cdot SG},$$

$$\text{or } SO = SG + \frac{A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \text{ \&c}}{b \cdot SG}.$$

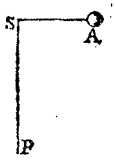
242. *Corol.* 3. Hence the distance of the centre of percussion, always exceeds the distance of the centre of gravity, and the excess is always $GO = \frac{A \cdot GA^2 + B \cdot GB^2 \text{ \&c}}{b \cdot SG}$.

243. And hence also, $SG \cdot GO = \frac{A \cdot GA^2 + B \cdot GB^2 \text{ \&c}}{\text{the body } b}$;
that is, SG . GO is always the same constant quantity, wherever the point of suspension S is placed; since the point G and the bodies A, B, &c, are constant. Or GO is always

reciprocally as SG, that is, GO is leſs, as SG is greater; and conſequently the point riſes upwards and approaches towards the point G, as the point S is removed to the greater diſtance; and they coincide when SG is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite diſtance.

PROPOSITION XLVII.

244. *If a Body* A, *at the Diſtance* SA *from an Axis paſſing through* S, *be made to revolve about that Axis by any Force* f *acting at* P *in the Line* SP, *Perpendicular to the Axis of Motion: It is required to determine the Quantity or Matter of another Body* Q, *which being placed at* P, *the Point where the Force acts, it ſhall be accelerated in the Same Manner, as when* A *revolved at the Diſtance* SA; *and conſequently, that the Angular Velocity of* A *and* Q *about* S, *may be the Same in Both Caſes.*

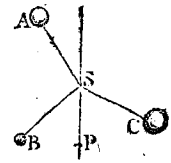


By the nature of the lever, $SA : SP :: f : \frac{SP}{SA} \cdot f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the ſame effect on the body A, as the force $\frac{SP}{SA} f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the abſolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circles deſcribed by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire velocities which are as SP and SA. But the motive forces of bodies are as their maſs and velocity; therefore $\frac{SP}{SA} f : f :: A \cdot SA : Q \cdot SP$, and $SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2} A$, which therefore is the maſs of matter which, being placed at P, would receive the ſame angular motion from the action of any force at P, as the body A receives. So that the reſiſtance of any body A, to a force acting at any point P, is directly as the ſquare of its diſtance SA from the axis of motion, and reciprocally as the ſquare of the diſtance SP of the point where the force acts.

245. *Corol.* 1,

245. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f \cdot SP^2}{A \cdot SA^2}$ to 1, or as $f \cdot SP^2$ to $A \cdot SA^2$.

246. *Corol.* 2. If any number of bodies, A, B, C, be put in motion, about a fixed axis paſſing through S, by a force acting at P; the point P will be accelerated in the ſame manner, and conſequently the whole ſyſtem will have the ſame angular velocity, if, inſtead of the bodies A, B, C, placed at the diſtances SA, SB, SC, there be ſubſtituted the bodies $\frac{SA^2}{SP^2}$ A, $\frac{SB^2}{SP^2}$ B, $\frac{SC^2}{SP^2}$ C; theſe being collected into the point P. And hence, the moving force being f, and the matter moved being - - - - - $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$; therefore the accelerating force is $\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$; which is to the accelerating force of gravity, as $f \cdot SP^2$ to $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$.



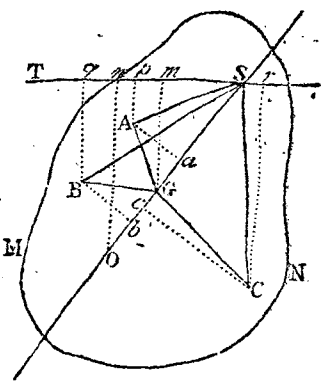
247. *Corol.* 3. The angular velocity of the whole ſyſtem of bodies, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. For the abſolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inverſely as the maſs $\frac{A \cdot SA^2 \&c}{SP^2}$: but the angular velocity is as the abſolute velocity directly, and the radius SP inverſely; and therefore the angular velocity of P, or of the whole ſyſtem, which is the ſame thing, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$.

PROPOSITION XLVIII.

248. *To determine the Centre of Oſcillation of any Compound Maſs or Body* MN, *or of any Syſtem of Bodies* A, B, C, *&c.*

LET MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle,

particle, to this plane. Let G be the centre of gravity, and O the centre of oſcillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to theſe two lines; and join SA, SB, SC; alſo, draw Gm, On perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are A . Sp, B . Sq, $-$C . Sr; and therefore, by cor. 3, prop. 47, the angular motion generated by all theſe forces is $\frac{A.Sp+B.Sq-C.Sr}{A.SA^2+B.SB^2+C.SC^2}$.



Alſo, the angular veloc. any particle p, placed in O, generates in the ſyſtem, by its weight, is $\frac{p.Sn}{p.SO^2}$ or $\frac{Sn}{SO^2}$, or $\frac{Sm.}{SG.SO}$, becauſe of the ſimilar triangles SGm, SOn. But, by the problem, the vibrations are performed alike in both caſes, and therefore theſe two expreſſions muſt be equal to each other, that is, $\frac{Sm}{SG.SO}=\frac{A.Sp+B.Sq-C.Sr}{A.SA^2+B.SB^2+C.SC^2}$; and hence $SO=\frac{Sm}{SG}\times\frac{A.SA^2+B.SB^2+C.SC^2}{A.Sp+B.Sq-C.Sr}$.

But, by cor. 2, pr. 40, the ſum A . Sp + B . Sq $-$ C . Sr = (A + B + C) . Sm; therefore the diſtance SO = $\frac{A.SA^2+B.SB^2+C.SC^2}{SG.(A+B+C)}=\frac{A.SA^2+B.SB^2+C.SC^2}{A.Sa+B.Sb+C.Sc}$ by prop. 41, the diſtance of the centre of oſcillation O, below the axis of ſuſpenſion; where any of the products A . Sa, B . Sb, muſt be negative, when a, b, &c, lie on the other ſide of S; which is the ſame expreſſion as that for the diſtance of the centre of percuſſion, found in prop. 46.

Hence it appears, that the centres of percuſſion and of oſcillation, are in the very ſame point. And therefore the properties in all the corollaries there found for the former, are to be here underſtood of the latter.

219. *Coral.* 1. If p be any particle of a body b, and d its diſtance

diſtance from the axis of motion S; alſo G, O the centres of gravity and oſcillation. Then the diſtance of the centre of oſcillation of the body, from the axis of motion, is - -

$$SO = \frac{\text{ſum of all the } pd^2}{SG \times \text{the body } b}.$$

250. *Corol.* 2. If b denote the matter in any compound body, whoſe centres of gravity and oſcillation are G and O; the body P, which being placed at P, where the force acts as in the laſt propoſition, and which receives the ſame motion from that force as the compound body b, is $P = \frac{SG.SO}{SP^2}.b$.

For, by corol. 2, prop. 46, this body $P =$ - - - - $\frac{A.SA^2 + B.SB^2 + C.SC^2}{SP^2}$. But, by corol. 1, prop. 45, $SG.SO.b = A.SA^2 + B.SB^2 + C.SC^2$; therefore $P = \frac{SG.SO}{SP^2}.b$.

SCHOLIUM.

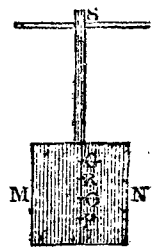
251. By the method of Fluxions, the centre of oſcillation, for a regular body, will be found from cor. 1. But for an irregular one; ſuſpend it at the given point; and hang up alſo a ſimple pendulum of ſuch a length, that, making them both vibrate, they may keep time together. Then the length of the ſimple pendulum, is equal to the diſtance of the centre of oſcillation of the body, below the point of ſuſpenſion.

252. Or it will be ſtill better found thus: Suſpend the body very freely by the given point, and make it vibrate in ſmall arcs, counting the number of vibrations it makes in any time, as a minute, by a good ſtop watch; and let that number of vibrations made in a minute be called n: Then ſhall the diſtance of the centre of oſcillation, be $SO = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating ſeconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the ſquare of the number of vibrations made in the ſame time; therefore - - - - $n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$, the length of the pendulum which vibrates n times in a minute, or the diſtance of the centre of oſcillation below the axis of motion.

253. The foregoing determination of the point, into which all the matter of a body being collected, it ſhall oſcillate in the

PROPOSITION L.

257. *To determine the Velocity with which a Ball moves, which, being ſhot againſt a Balliſtic Pendulum, cauſes it to vibrate through a Given Angle.*



THE Balliſtic Pendulum, is a heavy block of wood MN, ſuſpended vertically by a ſtrong horizontally iron axis at S, to which it is connected by a firm iron ſtem. This problem is the application of the laſt propoſition, or of prop. 46, and was invented by the very ingenious Mr. Robins, to determine the initial velocities of military projectiles; a circumſtance very uſeful in that ſcience; and it is the only method yet known for determining them with any degree of accuracy.

Let G, R, O be the centres of gravity, gyration, and oſcillation, as determined by the foregoing propoſitions; and let P be the point where the ball ſtrikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expreſſed by the force f, acting at P, in the foregoing propoſitions.

Put p = the whole weight of the pendulum,
b = the weight of the ball,
g = SG the diſt. of the centre of gravity,
o = SO the diſt. of the centre of oſcillation,
$r = \text{SR} = \sqrt{go}$ the diſt. of the centre of gyration,
i = SP the diſtance of the point of impact,
v = the velocity of the ball,
u = the velocity of the point of impact P,
c = chord of the arc deſcribed by the point O.

By prop. 48, if the maſs p be placed all at R, the pendulum will receive the ſame motion from the blow in the point P: and as $\text{SP}^2 : \text{SR}^2 :: p : \frac{\text{SR}^2}{\text{SP}^2} \cdot p$ or $\frac{r^2}{i^2} p$ or $\frac{go}{ii} p$, (prop. 46), the maſs which being placed at P, the pendulum will ſtill receive the ſame motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii} p$, the former moving with the velocity v, and ſtriking the latter at reſt; to determine

determine their common velocity u, with which they will jointly proceed forward together after the ſtroke. In which caſe, by the law of the impact of non-elaſtic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimenſions and weights of the bodies.

But now to determine the value of u, we muſt have recourſe to the angle through which the pendulum vibrates; for when the pendulum deſcends down again to the vertical poſition, it will have acquired the ſame velocity with which it began to aſcend, and, by the laws of falling bodies, the velocity of the centre of oſcillation is ſuch, as a heavy body would acquire by freely falling through the verſed ſine of the arc deſcribed by the ſame centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the verſed ſine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the verſed ſine of the arc deſcribed by O. Then, by the laws of falling bodies, $\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point O in deſcending through the arc whoſe chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by ſubſtituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball, is directly as the chord of the arc deſcribed by the pendulum in its vibration.

SCHOLIUM.

258. In the foregoing ſolution, the change in the centre of oſcillation is omitted, which is cauſed by the ball lodging in the point P. But the allowance for that ſmall change, and that of ſome other ſmall quantities, may be ſeen in my Tracts, where all the circumſtances of this method are treated at full length.

259. For an example in numbers of this method, ſuppoſe the weights and dimenſions to be as follow: namely,

$p =$

$p = 57$ clb,
$b = 18$ oz $1\frac{1}{2}$ dr $= 1{\cdot}131$ lb,
$g = 78\frac{1}{2}$ inc.
$o = 84\frac{7}{9}$ inc. $= 7{\cdot}065$ feet
$i = 94\frac{3}{10}$ inc.
$c = 18{\cdot}73$ inc.

Then

$$\frac{bii + gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}} \times \frac{18{\cdot}73}{12} = 656{\cdot}56.$$

And $\sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\frac{193}{42{\cdot}39}} = 2{\cdot}1337.$

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per second, with which the ball moved when it ſtruck the pendulum.

OF HYDROSTATICS.

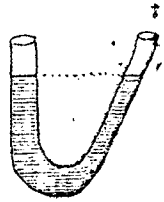
260. HYDROSTATICS is the ſcience which treats of the preſſure, or weight, and equilibrium of water and other fluids, eſpecially thoſe that are non-elaſtic.

261. A fluid is elaſtic, when it can be reduced into a leſs bulk by compreſſion, and which reſtores itſelf to its former bulk again when the preſſure is removed; as air. And it is non-elaſtic, when it is not compreſſible or expanſible, as water, &c.

PROPOSITION LI.

262. *If any Part of a Fluid be raiſed higher than the reſt, by any Force, and then left to itſelf; the higher Parts will deſcend to the lower Places, and the Fluid will not reſt, till its Surface be quite even and level.*

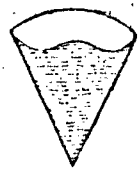
FOR, the parts of a fluid being eaſily moveable every way, the higher parts will deſcend by their ſuperior gravity, and raiſe the lower parts, till the whole come to reſt in a level or horizontal plane.



263. *Corol.* 1. Hence, water which communicates with other water, by means of a cloſe canal or pipe, will ſtand at the ſame height in both places. Like as water in the two legs of a ſyphon.

264. *Corol.* 2.

264. *Corol.* 2. For the ſame reaſon, if a fluid gravitate towards a centre; it will diſpoſe itſelf into a ſpherical figure, the centre of which is the centre of force. Like as the ſea in reſpect of the earth.



PROPOSITION LII.

265. *When a Fluid is at reſt in a Veſſel, the Baſe of which is Parallel to the Horizon; Equal Parts of the Baſe are Equally Preſſed by the Fluid.*

For, upon every equal part of the baſe there is an equal column of the fluid ſupported by it. And as all the columns are of equal height, by the laſt propoſition, they are of equal weight, and therefore they preſs the baſe equally; that is, equal parts of the baſe ſuſtain an equal preſſure.

266. *Corol.* 1. All parts of the fluid preſs equally at the ſame depth.

For, if a plane parallel to the horizon be conceived to be drawn at that depth; then, the preſſure being the ſame in any part of that plane, by the propoſition, therefore the parts of the fluid, inſtead of the plane, ſuſtain the ſame preſſure at the ſame depth.

267. *Corol.* 2. The preſſure of the fluid at any depth, is as the depth of the fluid.

For the preſſure is as the weight, and the weight is as the height of the fluid.

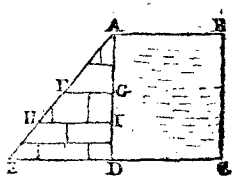
PROPOSITION LIII.

268. *When a Fluid is preſſed by its own Weight, or by any other Force; at any Point it preſſes Equally, in all Directions whatever.*

This ariſes from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will preſs againſt other parts of the fluid in the direction of that force. And the preſſure in all directions will be the ſame. For if it were leſs in any part, the fluid would move that way, till the preſſure be equal every way.

269. *Corol.* 1. In a veſſel containing a fluid; the preſſure is the ſame againſt the bottom, as againſt the ſides, or even upwards, at the ſame depth.

270. *Corol.* 2.



270. *Corol.* 2. Hence, and from the laſt propoſition, if ABCD be a veſſel of water, and there be taken, in the baſe produced, DE to repreſent the preſſure at the bottom; joining AE, and drawing any parallels to the baſe, as FG, HI; then ſhall FG repreſent the preſſure at the depth AG, and HI the preſſure at the depth AI, and ſo on; becauſe the parallels - FG, HI, ED, by ſim. triangles, are as the depths AG, AI, AD; which are as the preſſures by the propoſition.

And hence the ſum of all the FG, HI, &c, or area of the triangle ADE, is as the preſſure againſt all the points G, I, &c, that is, againſt the line AD. But as every point in the line CD is preſſed with a force as DE, and that thence the preſſure on the whole line CD is as the rectangle ED . DC, while that againſt the ſide is as the triangle ADE or $\frac{1}{2}$AD.DE; therefore the preſſure on the horizontal line DC, is to the preſſure againſt the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the veſſel be an upright rectangular one, the preſſure on the bottom, or whole weight of the fluid, is to the preſſure againſt one ſide, as the baſe is to half that ſide. And therefore the weight of the fluid is to the preſſure againſt all the four upright ſides, as the baſe is to half the upright ſurface. And the ſame holds true alſo in any upright veſſel, whatever the ſides be, or in a cylindrical veſſel. Or, in the cylinder, the weight of the fluid, is to the preſſure againſt the upright ſurface, as the radius of the baſe is to double the altitude.

Moreover, when the rectangular priſm becomes a cube, it appears that the weight of the fluid on the baſe, is double the preſſure againſt one of the upright ſides, or half the preſſure againſt the whole upright ſurface.

271. *Corol.* 3. The preſſure of a fluid againſt any upright ſurface, as the gate of a ſluice or canal, is equal to half the weight of a column of the fluid whoſe baſe is the ſurface preſſed, and its altitude the ſame as the altitude of that ſurface.

For the preſſure on a horizontal baſe equal to the upright ſurface, is equal to that column; and the preſſure on the upright ſurface is but half that on the baſe, of the ſame area.

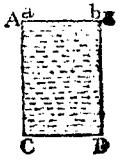
So that, if *b* be the breadth, and *d* the depth of ſuch a gate or upright ſurface; then the preſſure againſt it, is equal

to the weight of the fluid whoſe magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}AB \cdot AD^2$.

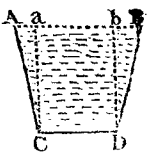
If the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}AB \cdot AD^2$, is 1440 feet; and the preſſure is 1440000 ounces, or 90000 pounds, or $40\frac{1}{5}$ tons weight nearly.

PROPOSITION LIV.

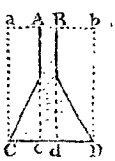
272. *The Preſſure of a Fluid, on the Baſe of the Veſſel in which it is contained, is as the Baſe and Perpendicular Altitude; whatever be the Figure of the Veſſel that contains it.*



If the ſides of the baſe be upright, ſo that it be a priſm of an uniform width throughout; then the caſe is evident; for then the baſe ſupports the whole fluid, and the preſſure is juſt equal to the weight of the fluid.



But if the veſſel be wider at top than bottom; then the bottom ſuſtains, or is preſſed by, only the part contained within the upright lines aC, bD; becauſe the parts ACa, BDb are ſupported by the ſides AC, BD; and thoſe parts have no other effect on the part abDC than keeping it in its poſition, by the lateral preſſure againſt aC and bD, which does not alter its perpendicular preſſure downwards. And thus the preſſure on the bottom is leſs than the weight of the contained fluid.



And if the veſſel be wideſt at bottom; then the bottom is ſtill preſſed with a weight which is equal to that of the whole upright column ABDC. For, as the parts of the fluid are in equilibrio, all the parts have an equal preſſure at the ſame depth; ſo that the parts within Cc and dD preſs equally as thoſe in cd, and therefore equally the ſame as if the ſides of the veſſel had gone upright to A and B, the defect of fluid in the parts ACa and BD*b* being exactly compenſated by the downward preſſure or reſiſtance of the ſides aC and bD againſt the contiguous fluid. And thus the preſſure on the baſe may be made to exceed the weight of the contained fluid, in any proportion whatever.

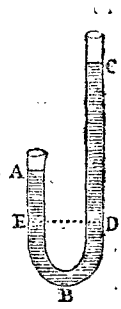
So

So that, in general, be the veſſels of any figure whatever, regular or irregular, upright or ſloping, or variouſly wide and narrow, in different parts, if the baſes and perpendicular altitudes be but equal, the baſes always ſuſtain the ſame preſſure. And as that preſſure, in the regular upright veſſel, is the whole column of the fluid, which is as the baſe and altitude; therefore the preſſure in all figures is in the ſame ratio.

273. *Corol.* 1. Hence, when the heights are equal, the preſſures are as the baſes. And when the baſes are equal, the preſſure is as the heights. But when both the heights and baſes are equal, the preſſures are equal in all, though their contents be ever ſo different.

274. *Corol.* 2. The preſſure on the baſe of any veſſel, is the ſame as on that of a cylinder, of an equal baſe and height.

275. *Corol.* 3. If there be an inverted ſyphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or reſt in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their denſities.



For, if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the ſurface at D is the ſame, where they act againſt each other, therefore AE : CD :: denſity of CD : denſity of AE.

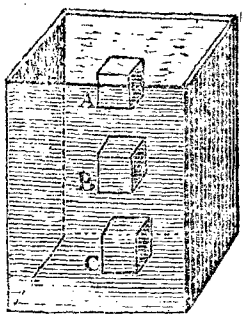
So, if CD be water, and AE quickſilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and ſo on.

PROPOSITION LV.

276. *If a Body be immerſed in a Fluid of the Same Denſity or Specific Gravity; it will reſt in any Place where it is put. But a Body of Greater Denſity will Sink; and one of a Leſs Denſity will Aſcend to the Top, and Float.*

The body, being of the ſame denſity, or of the ſame weight with the like bulk of the fluid, will preſs the fluid

under it, juſt as much as if its ſpace was filled with the fluid itſelf. The preſſure then all around it will be the ſame as if the fluid were in its place; conſequently there is no force, neither upwards nor downwards, to put the body out of its place. And therefore it will remain wherever it is put.



But if the body be lighter; its preſſure downwards will be leſs than before, and leſs than the water upwards at the ſame depth; therefore the greater force will overcome the leſs, and puſh the body upwards to A.

And if the body be heavier, the preſſure downwards will be greater than the fluid at the ſame depth; and therefore the greater force will prevail, and carry the body to the bottom at C.

277. *Corol.* 1. A body immerſed in a fluid, loſes as much weight, as an equal bulk of the fluid weighs. And the fluid gains the ſame weight.

Thus, if the body be of equal denſity with the fluid, it loſes all its weight, and ſo requires no force but the fluid to ſuſtain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the ſame bulk of water; and it requires a force to ſuſtain it juſt equal to that difference. But if it be lighter, it requires a force equal to the ſame difference of weights to keep it from riſing up in the fluid.

278. *Corol.* 2. The weights loſt, by immerging the ſame body in different fluids, are as the ſpecific gravities of the fluids. And bodies of equal weight, but different bulk, loſe, in the ſame fluid, weights which are reciprocally as the ſpecific gravities of the bodies, or directly as their bulks.

279. *Corol.* 3. The whole weight of a body, which will float in a fluid, is equal to as much of the fluid, as the immerſed part of the body takes up, when it floats.

For the preſſure under the floating body, is juſt the ſame as ſo much of the fluid as is equal to the immerſed part; and therefore the weights are the ſame.

280. *Corol.* 4. Hence the magnitude of the whole body, is to the magnitude of the part immerſed, as the ſpecific gravity of the fluid, is to that of the body.

For,

For, in bodies of equal weight, the denſities, or ſpecific gravities, are reciprocally as their magnitudes.

281. *Corol.* 5. And becauſe, when the weight of a body taken in a fluid, is ſubtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the ſpecific gravity of the fluid, is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the ſpecific gravity of the body, and
s the ſpecific gravity of the fluid;
then $W - w : W :: s : S$, which proportion will give either of thoſe ſpecific gravities, the one from the other.

Thus $S = \frac{W}{W - w} s$, the ſpecific gravity of the body;

and $s = \frac{W - w}{W} S$, the ſpecific gravity of the fluid.

So that the ſpecific gravities of bodies, are as their weights in the air directly, and their loſs in the ſame fluid inverſely.

282. *Corol.* 6. And hence, for two bodies connected together, or mixed together into one compound, of different ſpecific gravities, we have the following equations, denoting their weights and ſpecific gravities, as below, viz.

H = weight of the heavier body in air, } S its ſpec. gravity;
h = weight of the ſame in water, }

L = weight of the lighter body in air, } s its ſpec. gravity;
l = weight of the ſame in water, }

C = weight of the compound in air, } f its ſpec. gravity;
c = weight of the ſame in water, }

w = the ſpecific gravity of water. Then

1ſt, $(H - h) S = Hw$,
2d, $(L - l) s = Lw$,
3d, $(C - c) f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}$.

From which equations, may be found any of the above quantities, in terms of the reſt.

Thus, from one of the firſt three equations, is found the ſpecific gravity of any body, as $s = \frac{Lw}{L - l}$, by dividing the abſolute weight of the body by its loſs in water, and multiplying by the ſpecific gravity of water.

But if the body L be lighter than water; then l will be negative, and we muſt divide by $L + l$ inſtead of $L - l$, and to find l we muſt have recourſe to the compound maſs C; and

becauſe, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - b}$, therefore $s = \frac{Lw}{(C - c) - (H - b)}$; that is, divide the abſolute weight of the light body, by the difference between the loſſes in water, of the compound and heavier body, and multiply by the ſpecific gravity of water. Or thus, $s = \frac{SfL}{CS - Hf}$, as found from the laſt equation.

Alſo, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s)S}{(S - s)f}C, \text{ and } L = \frac{(S - f)s}{(S - s)f}C,$$

the quantities of the two ingredients H and L, in the compound C. And ſo for any other demand.

PROPOSITION LVI.

To find the Specific Gravity of a Body.

283. CASE I.—*When the body is heavier than water;* weigh it both in water and out of water, and take the difference, which will be the weight loſt in water. Then, by corol. 6, prop. 54, $s = \frac{Bw}{B - b}$, where B is the weight of the body out of water, b its weight in water, s its ſpecific gravity, and w the ſpecific gravity of water. That is,

As the weight loſt in water,
Is to the whole or abſolute weight,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE. If a piece of ſtone weigh 10lb, but in water only $6\frac{3}{4}$lb, required its ſpecific gravity, that of water being 1000? Anſ. 3077.

284. CASE II.—*When the body is lighter than water, ſo that it will not ſink;* annex to it a piece of another body, heavier than water, ſo that the maſs compounded of the two may ſink together. Weigh the denſer body and the compound maſs, ſeparately, both in water, and out of it; then find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then ſay,

As

As the laſt remainder,
Is to the weight of the light body in air,
So is the ſpecific gravity in water,
To the ſpecific gravity of the body.

That is, the ſpec. grav. $s = \frac{Lw}{(C-c)-(H-b)}$ by cor. 6, prop. 54.

EXAMPLE. Suppoſe a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the ſpecific gravity of the elm?

Anſ. 600.

285. CASE III.—*For a fluid of any ſort.* Take a piece of a body of known ſpecific gravity; weigh it both in and out of the fluid, finding the loſs of weight by taking the difference of the two; then ſay,

As the whole or abſolute weight,
Is to the loſs of weight,
So is the ſpecific gravity of the ſolid,
To the ſpecific gravity of the fluid.

That is, the ſpec. grav. $w = \frac{B-b}{B}\, s$, by cor. 6, pr. 55.

EXAMPLE. A piece of caſt iron weighed $34\frac{61}{100}$ ounces in a fluid, and 40 ounces out of it; of what ſpecific gravity is that fluid? Anſ. 1000.

PROPOSITION LVII.

286. *To find the Quantities of Two Ingredients in a Given Compound.*

TAKE the three differences of every pair of the three ſpecific gravities, namely, the ſpecific gravities of the compound and each ingredient; and multiply each ſpecific gravity by the difference of the other two. Then ſay,

As the greateſt product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, the one $H = \frac{(f-s)S}{(S-s)f}C$; and the other $L = \frac{(S-f)s}{(S-s)f}C$, by cor. 6, prop. 55.

EXAMPLE. A compoſition of 112lb being made of tin and copper, whoſe ſpecific gravity is found to be 8784; required the quantity of each ingredient, the ſpecific gravity of tin being 7320, and that of copper 9000?

Anſwer, there is 100lb of copper, and conſequently 12lb of tin } in the compoſition.

SCHOLIUM.

287. The ſpecific gravities of ſeveral ſorts of matter, as found from experiments, are expreſſed by the numbers annexed to their names in the following Table:

A Table of the Specific Gravities of Bodies.

Platina (pure)	23400	Brick	2000
Fine gold	19640	Common earth	1984
Standard gold	18888	Nitre	1900
Quickſilver	13600	Ivory	1825
Lead	11325	Brimſtone	1810
Fine ſilver	11091	Solid gunpowder	1745
Standard ſilver	10535	Sand	1520
Copper	9000	Coal	1250
Copper halfpence	8915	Box-wood	1030
Gun metal	8784	Sea water	1030
Caſt braſs	8000	Common water	1000
Steel	7850	Oak	925
Iron	7645	Gunpowder, cloſe ſhaken	937
Caſt iron	7425	Ditto, in a looſe heap	836
Tin	7320	Aſh	800
Clear cryſtal glaſs	3150	Maple	755
Marble and hard ſtone	2700	Elm	600
Common green glaſs	2600	Fir	559
Flint	2570	Charcoal	
Common ſtone	2520	Cork	240
Clay	2160	Air at a mean ſtate	$1\frac{2}{9}$

288. *Note.* The ſeveral ſorts of wood are ſuppoſed to be dry. Alſo, as a cubic foot of water weighs juſt 1000 ounces averdupois, the numbers in this table expreſs, not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in averdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propoſitions.

PRO-

PROPOSITION LVIII.

289. *To find the Magnitude of any Body, from its Weight.*

As the tabular ſpecific gravity of the body,
Is to its weight in averdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, reſpectively.

Example 1. Required the content of an irregular block of common ſtone, which weighs 1 cwt, or 112lb?
Anſ. $1228\frac{20}{25}\frac{16}{20}$ cubic inches.

Example 2. How many cubic inches of gunpowder are there in 1lb weight? Anſ. 30 cubic inches nearly.

Example 3. How many cubic feet are there in a ton weight of dry oak? Anſ. $38\frac{1}{1}\frac{38}{85}$ cubic feet.

PROPOSITION LIX.

290. *To find the Weight of a Body, from its Magnitude.*

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular ſpecific gravity,
To the weight of the body.

Example 1. Required the weight of a block of marble, whoſe length is 63 feet, and breadth and thickneſs each 12 feet; being the dimenſions of one of the ſtones in the walls of Balbeck?
Anſ. $683\frac{4}{10}$ ton, which is nearly equal to the burthen of an Eaſt-India ſhip.

Example 2. What is the weight of 1 pint, ale meaſure, of gunpowder? Anſ. 19 oz. nearly.

Example 3. What is the weight of a block of dry oak, which meaſures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Anſ. $4335\frac{15}{16}$lb.

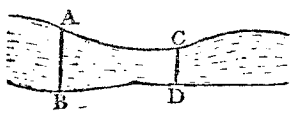
OF HYDRAULICS.

291. HYDRAULICS is the ſcience which treats of the motion of fluids, and the forces with which they act upon bodies.

PROPOSITION LX.

292. *If a Fluid run through a Canal or River, or Pipe of various Widths, always filling it; the Velocity of the Fluid in different Parts of it,* AB, CD, *will be Reciprocally as the Tranſverſe Sections in thoſe Parts.*

THAT is, veloc. at A : veloc. at C :: $\frac{1}{AB} : \frac{1}{CD}$; or :: CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or ſections, there.



For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the ſame time; that is, the column through AB is equal to the column through CD, in the ſame time; or AB × length of its column = CD × length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the ſpace run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

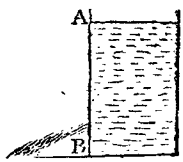
293. *Corol.* Hence, by obſerving the velocity at any place AB, the quantity of water diſcharged in a ſecond, or any other time, will be found, namely, by multiplying the ſection AB by the velocity there.

But if the channel be not a cloſe pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the ſection will not be the ſame, becauſe the velocity towards the bottom and ſides will be diminiſhed by the friction againſt the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the ſides, - 50
3) 210 ſum;
dividing their ſum by 3 gives 70 for the mean velocity, which

which is to be multiplied by the ſection, to give the quantity diſcharged in a minute.

PROPOSITION LXI.

294. *The Velocity with which a Fluid runs out by a Hole in the Bottom or Side of a Veſſel, kept always full, is equal to that which is generated by Gravity through the Height of the Water above the Hole; that is, the Velocity of a Heavy Body acquired by falling freely through the Height* AB.



DIVIDE the altitude AB into a great number of very ſmall parts, each being 1, their number a, or $a =$ the altitude AB.

Now, by prop. 53, the preſſure of the fluid againſt the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whoſe height is AB or a, and baſe the area of the hole B. Therefore the preſſure on the hole, or ſmall part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 28, the velocities generated in the ſame body in any time, are as thoſe forces; and becauſe gravity generates the velocity 2 in deſcending through the ſmall ſpace 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the preſſure of the column of fluid in the ſame time. But $2a$ is alſo, by corol. 1, prop. 6, the velocity generated by gravity in deſcending through a or AB. That is, the velocity of the iſſuing water, is equal to that which is acquired by a body in falling through the height AB.

295. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the ſquare roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

296. *Corol.* 2. The water ſpouts out with the ſame velocity, whether it be downwards, or upwards, or ſideways; becauſe the preſſure of fluids is the ſame in all directions, at the ſame depth. And therefore, if the adjutage be turned upwards, the jet will aſcend to the height of the ſurface of the water in the veſſel. And this is confirmed by experience, by which it is found that jets really aſcend nearly to the height of the reſervoir, abating a ſmall quantity only, for the friction againſt the ſides, and ſome reſiſtance from the oblique motion of the water in the hole.

297. *Corol.* 3. The quantity run out in any time, is equal to

to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if h denote the height of the fluid,

a the area of the orifice, and

$g = 16\frac{1}{12}$ feet, or 193 inches;

then $2a\sqrt{gh}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{10}a\sqrt{h}$ cubic feet, when a and h are taken in feet.

So, for example, if the height h be 25 inches, and the orifice $a = 1$ square inch; then $2a\sqrt{gh} = 2\sqrt{25} \times 193 = 139$ cubic inches, which is the quantity that would be discharged per second.

SCHOLIUM.

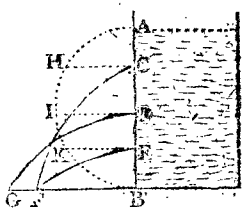
298. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

299. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from whence it happens, that the real velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through apertures, shew that the velocity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity equal to that of a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. Or else,

, that

that the orifice is not quite full of particles that ſpout out with the whole velocity, aſſigned in the propoſition.

300. Experiments have alſo been made on the extent to which the ſpout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile diſcharged with the velocity acquired by deſcending through the height of the fluid. For, when the aperture is in the ſide of the veſſel, the fluid ſpouts out horizontally with an uniform velocity, which, combined with the perpendicular velocity from the action of gravity, cauſes the jet to form the curve of a parabola. Then the diſtances to which the jet will ſpout on the horizontal plane BG, will be as the roots of the rectangles of the ſegments AC . CB, AD . DB, AE . EB. For the ſpaces BF, BG, are as the times and horizontal velocities; but the velocities are as $\sqrt{AC}$, and the time of the fall, which is the ſame as the time of moving, or as $\sqrt{CB}$; therefore the diſtance BF is as $\sqrt{AC \cdot CB}$; and the diſtance BG as $\sqrt{AD \cdot DB}$. And hence, if two holes are made equidiſtant from the top and bottom, they will project the water to the ſame diſtance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB; which makes EF the ſame for both. Or, if on the diameter AB a ſemicircle be deſcribed; then, becauſe the ſquares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c; therefore the diſtances BF, BG are as the ordinates CH, DI. And hence alſo it follows, that the projection from the middle point D will be fartheſt, for DI is the greateſt ordinate.



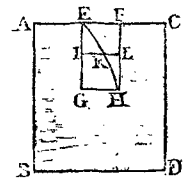
Theſe are the *proportions* of the diſtances: but for the abſolute diſtances, it will be thus. The velocity through any hole C, is ſuch as will carry the water horizontally through a ſpace equal to 2AC in the time of falling through AC: but, after quitting the hole, it deſcribes a parabola, and comes to F in the time a body will fall through CB; and to find this diſtance, ſince the times are as the roots of the ſpaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC \cdot CB} = 2CH = BF$, the ſpace ranged on the horizontal plane. And the greateſt range BG = 2DI, or 2AD, or equal to AB.

And as theſe ranges anſwer very exactly to the experiments, this confirms the theory as to the velocity aſſigned.

PRO-

PROPOSITION LXII.

301. *If a Notch or Slit* EH, *in form of a Parallelogram, be cut in the Side of a Veſſel, full of Water*, AD; *the Quantity of Water flowing through it, will be* $\frac{2}{3}$ *of the Quantity flowing through an equal Orifice, placed at the whole Depth* EG, *or at the Baſe* GH, *in the Same Time; it being ſuppoſed that the Veſſel is always kept full.*



FOR the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$, that is, as GH or IL to IK, the ordinate of a parabola EKH, whoſe axis is EG. Therefore the ſum of the velocities at all the points I, is to as many times the velocity at G, as the ſum of all the ordinates IK, to the ſum of all the IL's, namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumſcribing parallelogram.

302. *Corol.* 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

303. *Corol.* 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic fruſtum IGHK, is to the rectangle IGHL. As appears from the demonſtration.

OF PNEUMATICS.

304. PNEUMATICS is the ſcience which treats of the properties of air, or elaſtic fluids.

PROPOSITION LXIII.

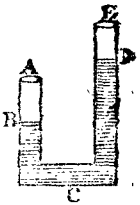
305. *Air is a Heavy Fluid Body; and it Surrounds, and Gravitates on, all Parts of the Surface of the Earth.*

THESE properties of air are proved by experience.—That it is a fluid, is evident from its eaſily yielding to any the leaſt force impreſſed on it, without making a ſenſible reſiſtance.

But

But when it is moved briſkly, by any means, as by a fan or a pair of bellows; or when any body is moved very briſkly through it; in theſe caſes we become ſenſible of it as a body, by the reſiſtance it makes in ſuch motions, and likewiſe by its impelling or blowing away any light ſubſtances. So that, being capable of reſiſting, or moving other bodies by its impulſe, it muſt itſelf be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will preſs on all bodies that are placed under it.

Alſo, as it is a fluid, it will ſpread itſelf all over on the earth; and, like other fluids, it will gravitate and preſs every where on the earth's ſurface.



306. The gravity and preſſure of the air is alſo evident from many experiments. Thus, for inſtance, if water, or quickſilver, be poured into the tube ACE, and the air be ſuffered to preſs on it, in both ends of the tube, the fluid will reſt at the ſame height in both legs of the tube: but if the air be drawn out of one end as E, by any means; then the air preſſing on the other end A, will preſs down the fluid in this leg at B, and raiſe it up in the other to D, as much higher than at B, as the preſſure of the air is equal to. By which it appears, not only that the air does really preſs, but alſo what the quantity of that preſſure is equal to. And this is the principle of the barometer.

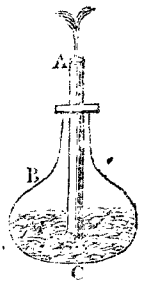
PROPOSITION LXIV.

307. *The Air is alſo an Elaſtic Fluid, being Condenſible and Expanſible. And the Law it obſerves is this, that its Denſity is proportional to the Force which compreſſes it.*

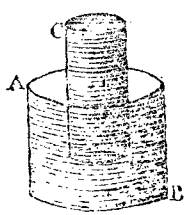
THIS property of the air is proved by many experiments. Thus, if the handle of a ſyringe be puſhed inwards, it will condenſe the incloſed air into leſs ſpace, thereby ſhewing its condenſibility. But the included air, thus condenſed, will be felt to act ſtrongly againſt the hand, reſiſting the force compreſſing it more and more; and, on withdrawing the hand, the handle is puſhed back again to where it was at firſt. Which ſhews that the air is elaſtic.

308. Again, fill a ſtrong bottle half full of water, and then inſert a pipe into it, putting its lower end down near to the

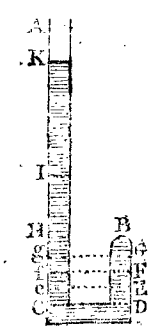
the bottom, and cementing it very close round the mouth of the bottle. Then, if air be ſtrongly injected through the pipe, as by blowing with the mouth or otherwiſe, it will paſs through the water from the lower end, aſcending into the parts before occupied with air at B, and the whole maſs of air become there condenſed, becauſe the water is not compreſſible into a leſs ſpace. But, on removing the force which injected the air at A, the water will begin to riſe from thence in a jet, being puſhed up the pipe by the increaſed elaſticity of the air B, by which it preſſes on the ſurface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the ſame denſity as at firſt, and, the balance being reſtored, the jet will ceaſe.



309. Likewiſe, if into a jar of water AB, be inverted an empty glaſs tumbler CD, or ſuch-like, the mouth downwards; the water will enter it, and partly fill it, but not near ſo high as the water in the jar, compreſſing and condenſing the air into a leſs ſpace in the upper parts C, and cauſing the glaſs to make a ſenſible reſiſtance to the hand in puſhing it down. Then, on removing the hand, the elaſticity of the internal condenſed air throws the glaſs up again. All theſe ſhewing that the air is condenſible and elaſtic.



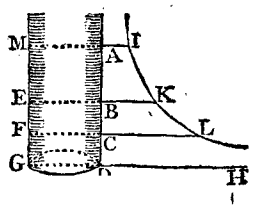
310. Again, to ſhew the rate or proportion of the elaſticity to the condenſation; take a long crooked glaſs tube, equally wide throughout, or at leaſt in the part BD, open at A, but cloſe at the other end B. Pour in a little quickſilver at A, juſt to cover the bottom to the bend at CD, and to ſtop the communication between the external air and the air in BD. Then pour in more quickſilver, and mark the correſponding heights at which it ſtands in the two legs: ſo, when it riſes to H in the open leg AC, let it riſe to E in the cloſe one, reducing its included air from the natural bulk BD to the contracted ſpace BE, by the preſſure of the column He; and when the quickſilver ſtands at I and K, in the open leg, let it riſe to F and G in the



the other, reducing the air to the reſpective ſpaces BF, BG, by the weights of the columns If, Kg. Then it is always found that the condenſations and elaſticities are as the compreſſing weights, or columns, of the quickſilver, and the atmoſphere together. So, if the natural bulk of the air ED be compreſſed into the ſpaces BE, BF, BG, or reduced by the ſpaces DE, DF, DG, which are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ of DB, or as the numbers 1, 2, 3; then the atmoſphere, together with the correſponding columns He, If, Kg, will alſo be found to be in the ſame proportion, or as the numbers 1, 2, 3. And then $He = \frac{1}{3}A$, $If = A$, and $Kg = 3A$; where A is the weight of the atmoſphere. Which ſhews, that the condenſations are directly as the compreſſing forces. And the elaſticities are in the ſame ratio, ſince the columns in AC are ſuſtained by the elaſticities in BD.

From the foregoing principles may be deduced many uſeful remarks, as in the following corollaries, viz.

311. *Corol.* 1. The ſpace which any quantity of air is confined in, is reciprocally as the force that compreſſes it. So, the forces which confine a quantity of air in the cylindrical ſpaces AG, BG, CG, are reciprocally as the ſame, or reciprocally as the heights AD, BD, CD. And therefore, if to the two perpendicular lines DA, DH, as aſymptotes, the hyperbola IKL be deſcribed, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the ſpaces AG, BG, CG, will be directly as the correſponding ordinates AI, BK, CL, ſince theſe are reciprocally as the abſciſſes AD, BD, CD, by the nature of the hyperbola.



312. *Corol.* 2. All the air near the earth is in a ſtate of compreſſion, by the weight of the incumbent atmoſphere.

313. *Corol.* 3. The air is denſer near the earth, than in high places; or denſer at the foot of a mountain than at the top of it. And the higher above the earth, the leſs denſe it is.

314. *Corol.* 4. The ſpring or elaſticity of the air, is equal to the weight of the atmoſphere above it; and they will produce the ſame effects; ſince they always ſuſtain and balance each other.

315. *Corol.* 5. If the denſity of the air be increaſed, preſerving the ſame heat or temperature; its ſpring or elaſticity will likewiſe be increaſed, and in the ſame proportion.

316. *Corol.* 6. By the gravity and preſſure of the atmoſphere, on the ſurface of fluids, the fluids are made to riſe in any pipes or veſſels, when the ſpring or preſſure within is decreaſed or taken off.

PROPOSITION LXV.

317. *Heat Increaſes the Elaſticity of the Air, and Cold Diminiſhes it. Or, Heat Expands, and Cold Condenſes the Air.*

This property is alſo proved by experience.

318. Thus, tie a bladder very cloſe with ſome air in it; and lay it before the fire: then as it warms, it will more and more diſtend the bladder, and at laſt burſt it, if the heat be continued, and increaſed high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was upon this principle, that the firſt air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air diſtends them to a ſize which occupies a ſpace in the atmoſphere, whoſe weight of common air exceeds that of the balloon.

319. Alſo, if a cup or glaſs, with a little air in it, be inverted into a veſſel of water; and the whole be heated over the fire, or otherwiſe: the air in the top will expand till it fill the glaſs, and expel the water out of it; and part of the air itſelf will follow, by continuing or increaſing the heat.

Many other experiments, to the ſame effect, might be adduced, all proving the properties mentioned in the propoſition.

SCHOLIUM.

320. So that, when the force of the elaſticity of air is conſidered, regard muſt be had to its heat or temperature; the ſame quantity of air being more or leſs elaſtic, as its heat is more or leſs. And it has been found, by experiment, that the elaſticity is increaſed by the 435th part, by each degree of heat, of which there are 180 between the freezing and boiling heat of water.

321. *N E.*

321. *N. B.* Water expands about the $\frac{3}{10000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philoſ. Tranſ. 1777, p. 560, &c.)

Alſo, the

Spec. grav. of	air	1	when the barom. is at 29·27, and the thermom. at 53°.
	water	836	
	mercury	11365	
Or thus,	air	1	when the barom. is 29·27, and the thermom. at 55°.
	water	832	
	mercury	11315	
Or thus,	air	1	when the barom. is 29·5, and the thermom. is 55°, which are their mean heights in this country.
	water	826	
	mercury	11227	
Or thus,	air	1·201 or $1\frac{1}{5}$	in the laſt circumſtances.
	water	1000	
	mercury	13592	
Or thus,	air	1·222 or $1\frac{2}{9}$	nearly, when the barom. is 30, and thermometer 55.
	water	1000	
	merc.	13600	

PROPOSITION LXVI.

322. *The Weight or Preſſure of the Atmoſphere, on any Baſe at the Earth's Surface, is equal to the Weight of a Column of Quickſilver, of the Same Baſe, and the Height of which is between* 28 *and* 31 *inches.*

This is proved by the barometer, an inſtrument which meaſures the preſſure of the air, and which is deſcribed below. For, at ſome ſeaſons, and in ſome places, the air ſuſtains and balances a column of mercury, of about 28 inches; but at other times it balances a column of 29, or 30, or near 31 inches high; ſeldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the ſurface of the earth, and partly on the commotions and changes in the atmoſphere, from winds and other cauſes, by which it is accumulated in ſome places, and depreſſed in others, being thereby rendered denſer and heavier, or rarer and lighter; which changes in its ſtate are almoſt continually happening in any one place. But the medium ſtate is commonly about $29\frac{1}{2}$ or 30 inches.

323. *Corol.* 1. Hence the preſſure of the atmoſphere on every ſquare inch at the earth's ſurface, at a medium, is very near 15 pounds averdupois.

For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almoſt 8 ounces, or near half a pound, which is the weight of the atmoſphere for every inch of the barometer on a baſe of a ſquare inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

324. *Corol.* 2. Hence alſo, the weight or preſſure of the atmoſphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high.

For, water and quickſilver are in weight nearly as 1 to 13·6; ſo that the atmoſphere will balance a column of water 13·6 times as high as one of quickſilver; conſequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

And hence a common ſucking pump will not raiſe water higher than about 34 feet. And a ſyphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.

325. *Corol.* 3. If the air were of the ſame uniform denſity at every height up to the top of the atmoſphere, as at the ſurface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium.

For, the weights of the ſame bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And ſo high the atmoſphere would be, if it were all of uniform denſity, like water. But, inſtead of that, from its expanſive and elaſtic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as alſo the method of meaſuring heights by the barometer, which depends on it.

326. *Corol.* 4. From this propoſition and the laſt it follows, that the height is always the ſame, of an uniform atmoſphere above any place, which ſhall be all of the uniform denſity with the air there, and of equal weight or preſſure with the real height of the atmoſphere above that place, whether it be at the ſame place at different times, or at any different places or heights above the earth; and that height

is

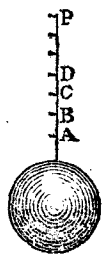
is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the denſity varies in exact proportion to the weight of the column, therefore it requires a column of the ſame height in all caſes, to make the reſpective weights or preſſures. Thus, if W and w be the weights of atmoſphere above any places, D and d their denſities, and H and h the heights of the uniform columns, of the ſame denſities and weights; Then $H \times D = W$,

and $h \times d = w$;

therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h. The temperature being the ſame.

PROPOSITION LXVII.

327. *The Denſity of the Atmoſphere, at different Heights above the Earth, Decreaſes in ſuch Sort, that when the Heights increaſe in Arithmetical Progreſſion, the Denſities decreaſe in Geometrical Progreſſion.*



LET the perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very ſmall parts A, B, C, D, &c, forming ſo many thin ſtrata of air in the atmoſphere, all of different denſity, gradually decreaſing from the greateſt at A: then the denſity of the ſeveral ſtrata A, B, C, D, &c, will be in geometrical progreſſion decreaſing.

For, as the ſtrata A, B, C, &c, are all of equal thickneſs, the quantity of matter in each of them, is as the denſity there; but the denſity in any one, being as the compreſſing force, is as the weight or quantity of matter from that place upwards to the top of the atmoſphere; therefore the quantity of matter in each ſtratum, is alſo as the whole quantity from that place upwards. Now, if from the whole weight at any place as B, the weight or quantity in the ſtratum B be ſubtracted, the remainder is the weight at the next ſtratum C; that is, from each weight ſubtracting a part which is proportional to itſelf, leaves the next weight; or, which is the ſame thing, from each denſity ſubtracting a part which is always proportional to itſelf, leaves the next denſity. But when any quantities are continually diminiſhed by parts which are proportional to themſelves, the remainders form a ſeries of continued proportionals; conſequently theſe denſities are in geometrical progreſſion.

Thus, if the firſt denſity be D, and from each be taken its nth part; then there will remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the ſeries of denſities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the ſeries being $\frac{m}{n}$.

SCHOLIUM.

328. Becauſe the terms of an arithmetical ſeries, are proportional to the logarithms of the terms of a geometrical ſeries; therefore different altitudes above the earth's ſurface, are as the logarithms of the denſities, or weights of air, at thoſe altitudes.

So that, if D denote the denſity at the altitude A,
and d - the denſity at the altitude a;
then A being as the log. of D, and a as the log. of d, the dif. of alt. $A-a$ will be as the log. D − log. d or log. $\frac{D}{d}$.

And if $A = 0$, or D the denſity at the ſurface of the earth; then any alt. above the ſurface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the ſurface of the earth, or any where elſe.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which is an inſtrument that meaſures the preſſure or denſity of the air at any place. For, by taking, with this inſtrument, the preſſure or denſity, at the foot of a hill for inſtance, and again at the top of it, the difference of the logarithms of theſe two preſſures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; ſuppoſing the temperatures of the air to be the ſame at both places, and the gravity of air not altered by the different diſtances from the earth's centre.

329. But as this formula expreſſes only the relations between different altitudes, with reſpect to their denſities, recourſe muſt be had to ſome experiment, to obtain the real altitude

altitude which corresponds to any given denſity, or the denſity which corresponds to a given altitude. And there are various experiments by which this may be done. The firſt, and moſt natural, is that which reſults from the known ſpecific gravity of air, with reſpect to the whole preſſure of the atmoſphere on the ſurface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; aſſume b ſo that $a = b \times$ log. $\frac{D}{d}$, where b will be of one conſtant value for all altitudes; and to determine that value, let a caſe be taken in which we know the altitude a corresponding to a known denſity d; as for inſtance, take $a = 1$ foot, or 1 inch, or ſome ſuch ſmall altitude; then, becauſe the denſity D may be meaſured by the preſſure of the atmoſphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the denſity D at the lower place, and 27599 the leſs denſity d at 1 foot above it; conſequently $1 = b \times$ log. $\frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= b \times \frac{\cdot 43429448}{27600}$ $= \frac{b}{63551}$ nearly; and hence $b = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times$ log. $\frac{D}{d}$, or $= 63551 \times$ log. $\frac{M}{m}$ feet, or $10592 \times$ log. $\frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the preſſure or weight of the atmoſphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any meaſure, either feet, or inches, &c.

330. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which muſt be added when that medium exceeds 55°, otherwiſe ſubtracted.

331. Note alſo, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

332. But the formula may be rendered much more convenient for uſe, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corre-

ſponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the reſult is to be increaſed by ſo many times its 435th part.

333. *Exam.* 1. To find the height of a hill when the preſſure of the atmoſphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Anſ. 4378 feet, or 730 fathoms.

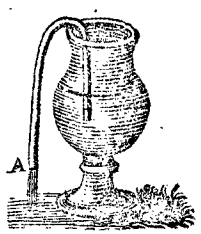
334. *Exam.* 2. To find the height of a hill when the atmoſphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°?
Anſ. 2385 feet, or 397½ fathoms.

335. *Exam.* 3. At what altitude is the denſity of the atmoſphere only the 4th part of what it is at the earth's ſurface? Anſ. 6020 fathoms.

By the weight and preſſure of the atmoſphere, the effect and operations of pneumatic engines may be accounted for, and explained; ſuch as ſyphons, pumps, barometers, &c; of which it may not be improper here to give a brief deſcription.

OF THE SIPHON.

336. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.



If it be filled with water, and then inverted, with the two open ends downward, and held level in that poſition; the water will remain ſuſpended in it, if the two legs be equal. For the atmoſphere will preſs equally on the ſurface of the water in each end, and ſupport them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoiſe by their equal weights; ſo that the one has no power to move more than the other; and they are both ſupported by the atmoſphere.

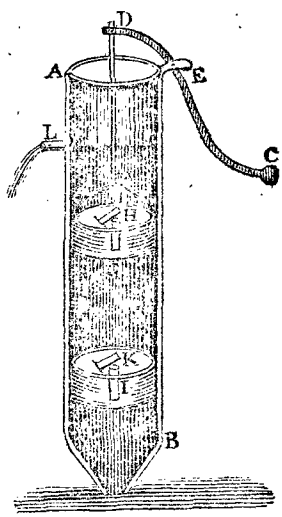
But

But if now the ſyphon be a little inclined to one ſide, ſo that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the ſame thing; then the equilibrium is deſtroyed, and the water will all deſcend out by the lower end, and riſe up in the higher. For, the air preſſing equally, but the two ends weighing unequally, a motion muſt commence where the power is greateſt, and ſo continue till all the water has run out by the lower end. And if the ſhorter leg be immerſed into a veſſel of water, and the ſyphon be ſet a running as above, it will continue to run till all the water be exhauſted out of the veſſel, or at leaſt as low as that end of the ſyphon. Or, it may be ſet a running without filling the ſyphon as above, by only inverting it, with its ſhorter leg into the veſſel of water; then, with the mouth applied to the lower orifice A, ſuck the air out; and the water will preſently follow, being forced up into the ſyphon by the preſſure of the air on the water in the veſſel.

OF THE PUMP.

337. THERE are three ſorts of PUMPS; the Sucking, the Lifting, and the Forcing Pump. By the former, water can be raiſed only to about 34 feet, viz. by the preſſure of the atmoſphere; but by the others, to any height; but then they require more apparatus and power.

The annexed figure repreſents a common ſucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and ſmooth within, or of wood for very common purpoſes. CD is the handle, moveable about the pin D, by moving the end C up and down. DF an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piſton, bucket, or ſucker, FG, by which this is moved up and down within the barrel, which it muſt fit very tight and cloſe, that no air or water may paſs between the piſton and the ſides of the barrel;



barrel; and for this purpoſe it is commonly armed with leather. The piſton is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, alſo perforated, and covered by a valve K opening upwards.

338. When the pump is firſt to be worked, and the water is below the plug I; raiſe the end C of the handle, and the piſton deſcending, compreſſes the air in HI, which by its ſpring ſhuts faſt the valve K, and puſhes up the valve H, and ſo enters into the barrel above the piſton. Then putting the end C of the handle down again, raiſes the piſton or ſucker, which lifts up with it the column of air above it, the external atmoſphere by its preſſure keeping the valve H ſhut: the air in the barrel being thus exhauſted, or rarefied, is no longer a counterpoiſe to that which preſſes on the ſurface of the water in the well, this is forced up the pipe, and through the valve K, into the barrel of the pump. Then puſhing the piſton down again into this water, now in the barrel, its weight ſhuts the lower valve K, and its reſiſtance forces up the valve of the piſton, and enters the upper part of the barrel, above the piſton. Then, the bucket being raiſed, lifts up with it the water which had paſſed above its valve, and it runs out by the cock L; and taking off the weight below it, the preſſure of the external atmoſphere on the water in the well again forces it up through the pipe and lower valve cloſe to the piſton, all the way as it aſcends, thus keeping the barrel always full of water. And thus, by repeating the ſtrokes of the piſton, a continued diſcharge is made at the cock L.

OF THE AIR-PUMP.

339. NEARLY on the ſame principles as the water-pump, is the invention of the Air-pump, by which the air is drawn out of any veſſel, like as water is drawn out by the former. A braſs barrel is bored and poliſhed truly cylindrical, and exactly fitted with a turned piſton, ſo that no air can paſs by the ſides of it, and furniſhed with a proper valve opening upwards. Then, by lifting up the piſton, the air in the cloſe veſſel below it follows the piſton, and fills the barrel; and being thus diffuſed through a larger ſpace than before, when it

it occupied the veſſel or receiver only, but not the barrel, it is made rarer than it was before; in proportion as the capacity of the barrel and receiver together, exceeds the receiver alone. Another ſtroke of the piſton exhauſts another barrel of this now rarer air, which again rarifies it in the ſame proportion as before. And ſo on, for any number of ſtrokes of the piſton, ſtill exhauſting in the ſame geometrical progreſſion, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhauſted to any propoſed degree, or as far as the nature of the machine is capable of performing; which happens when the elaſticity of the included air is ſo far diminiſhed, by rarefying, that it is too feeble to puſh up the valve of the piſton, and eſcape.

340. From the nature of this exhauſting, in geometrical progreſſion, we may eaſily find how much the air in the receiver is rarefied by any number of ſtrokes of the piſton; or what number of ſuch ſtrokes is neceſſary, to exhauſt the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural denſity of the air at firſt; then

$c : r :: 1 : \frac{r}{c}$, the denſity after 1 ſtroke of the piſton,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the denſity after 2 ſtrokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the denſity after 3 ſtrokes,

&c, and $\frac{r^n}{c^n}$, the denſity after n ſtrokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r :: 5 : 4$; and $\frac{4^n}{5^n} = 0{\cdot}8^n$ is $= d$ the denſity after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the denſity of the included air after 20 ſtrokes of the piſton; which being the $86\frac{7}{10}$ part of 1, or the firſt denſity, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 ſtrokes.

341. Or, if it were required to find the number of ſtrokes neceſſary to rarefy the air any number of times; becauſe $\frac{r^n}{c^n}$ is $=$ the propoſed denſity d; therefore taking the logarithms $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{\text{l. } r - \text{l. } c}$, the number

number of ſtrokes required. So, if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times; then $d = \frac{1}{100}$ or $\cdot 01$; and hence $n = \frac{\log. 100}{l. 5 - l. 4} = 20\frac{3}{5}$ nearly. So that in $20\frac{3}{5}$ ſtrokes the air will be rarefied 100 times.

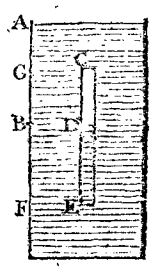
OF THE DIVING BELL, AND CONDENSING MACHINE.

342. ON the ſame principles too, depend the operations and effect of the Condenſing Engine, by which air may be condenſed to any degree, inſtead of rarefied as in the air-pump. And, like as the air-pump rarifies the air, by extracting always one barrel of air after another; ſo, by this other machine, the air is condenſed by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piſton and barrel, that is, making them to open the contrary way, and working the piſton in the ſame manner: ſo that, as they both open upwards, or outwards, in the air-pump, or rarefier, they will both open downwards, or inwards, in the condenſer.

343. And on the ſame principles, namely of the compreſſion and elaſticity of the air, depends the uſe of the Diving Bell, which is a large veſſel, in which a perſon deſcends to the bottom of the ſea, the open end of the veſſel being downwards; only, in this caſe, the air is not condenſed by forcing more of it into the ſame ſpace, as in the condenſing engine; but by compreſſing the ſame quantity of air into a leſs ſpace in the bell, by increaſing always the force which compreſſes it.

344. If a veſſel of any ſort be inverted into water, and puſhed or let down to any depth in it; then by the preſſure of the water ſome of it will aſcend into the veſſel, but not ſo high as the water without, and will compreſs the air into leſs ſpace, according to the difference between the heights of the internal and external water; and the denſity and elaſtic force of the air will be increaſed in the ſame proportion, as its ſpace in the veſſel is diminiſhed.

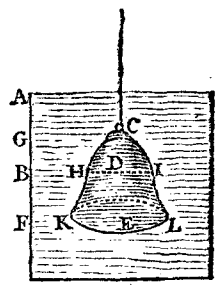
So, if the tube CE be inverted, and puſhed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the ſpace



CD; then that air is preſſed both by a column of water of the height AB, and by the whole atmoſphere which preſſes on the upper ſurface of the water; conſequently the ſpace CD is to the whole ſpace CE, as the weight of the atmoſphere, is to the weights both of the atmoſphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmoſphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and ſo on. And hence, by knowing the depth AF, to which the veſſel is ſunk, we can eaſily find the point D, to which the water will riſe within it at any time. For, let the weight of the atmoſphere at that time be equal to that of 34 feet of water; alſo, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water DE $= x$,

it is - $34 + AB : 34 :: CE : CD$,
that is, $34 + AF - DE : 34 :: CE : CE - DE$,
or - $54 - x : 34 :: 4 : 4 - x$;

hence, multiplying extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will riſe within the tube.

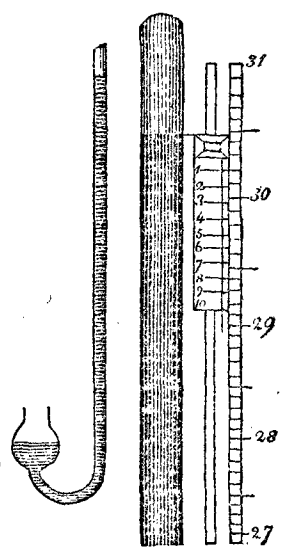


345. But if the veſſel be not equally wide throughout, but of any other ſhape, as of a bell-like form, ſuch as is uſed in diving; then the altitudes will not obſerve the proportion above, but the ſpaces or bulks only, will reſpect that proportion, namely, $34 + AB : 34 ::$ capacity CKL : capacity CHI, if it be common or freſh water; and $33 + AB : 33 ::$ capacity CKL : capacity CHI, if it be ſea-water. From which proportion, the height DE may be found, when the nature or ſhape of the veſſel or bell CKL is known.

Of the BAROMETER.

346. THE BAROMETER is an inſtrument for meaſuring the preſſure of the atmoſphere, and elaſticity of the air, at any time. It is commonly made of a glaſs tube, of near 3 feet

3 feet long, close at one end, and filled with mercury. When the tube is full, by ſtopping the open end with the finger, then inverting the tube, and immerſing that end with the finger into a baſon of quickſilver, on removing the finger from the orifice, the quickſilver in the tube will deſcend into the baſon, till what remains in the tube be of the ſame weight with a column of the atmoſphere; which is commonly between 28 and 31 inches of quickſilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no preſſure downwards but from the column of quickſilver, and therefore that will be an exact balance to the counter preſſure of the whole column of atmoſphere, acting on the orifice of the tube by the quickſilver in the baſon. The upper three inches of the tube, namely, from 28 to 31 inches, have a ſcale attached to them, divided into inches, tenths, and hundredths, for meaſuring the length of the column at all times, by obſerving which diviſion of the ſcale the top of the quickſilver is oppoſite to; as it aſcends and deſcends within theſe limits, according to the ſtate of the atmoſphere.



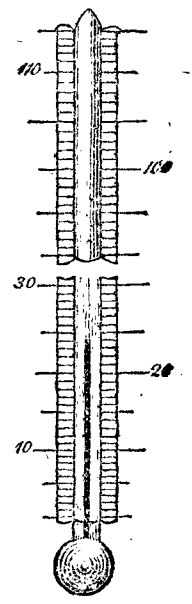
So that the weight of the quickſilver in the tube, above that in the baſon, is at all times equal to the weight or preſſure of the column of atmoſphere above it, and of the ſame baſe with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound averdupois for every inch of quickſilver in the tube, on every ſquare inch of baſe; or more exactly, it is $\frac{5}{1}\frac{9}{2}\frac{}{0}$ of a pound on the ſquare inch, for every inch in the altitude of the quickſilver: for the cubic inch of quickſilver weighs juſt $\frac{5}{1}\frac{9}{2}\frac{}{0}$ lb, or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55° of heat. And conſequently, when the barometer ſtands at 30 inches, or $2\frac{1}{2}$ feet high, which is the medium or ſtandard height, the whole preſſure of the atmoſphere is equal to $14\frac{3}{4}$ pounds, on every ſquare inch of the baſe. And ſo in proportion for other heights.

On

OF THE THERMOMETER.

347. THE THERMOMETER is an inſtrument for meaſuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and hence the degrees of expanſion become the meaſure of the degrees of heat. Fluids are more convenient for this purpoſe, than ſolids: and quickſilver is now moſt commonly uſed for it.



A very fine glaſs tube, having a pretty large hollow ball at the bottom, is filled about half way up with quickſilver: the whole being then heated very hot till the quickſilver riſe quite to the top, the top is then hermetically ſealed, ſo as perfectly to exclude all communication with the outward air. Then, in cooling, the quickſilver contracts, and conſequently its ſurface deſcends in the tube, till it come to a certain point, correſpondent to the temperature or heat of the air. And when the weather becomes warmer, the quickſilver expands, and its ſurface riſes in the tube; and again contracts and deſcends when the weather becomes cooler. So that, by placing a ſcale of any diviſions againſt the ſide of the tube, it will ſhow the degrees of heat, by the expanſion and contraction of the quickſilver in the tube; obſerving at what diviſion of the ſcale the top of the quickſilver ſtands. And the method of preparing the ſcale, as uſed in England, is thus:—Bring the thermometer into a temperature of juſt freezing, by immerſing the ball in water juſt freezing, or in ice juſt thawing, and mark the ſcale where the mercury then ſtands, for the point of freezing. Next, immerge it in boiling water; and the quickſilver will riſe to a certain height in the tube; which mark alſo on the ſcale, for the boiling point, or the heat of boiling water. Then the diſtance between theſe two points is divided into 180 equal diviſions, or degrees; and the like equal degrees are alſo continued to any extent below the freezing point, and above the boiling point. Theſe diviſions are then numbered as follows,

follows, namely, at the freezing point is ſet the number 32, and conſequently 212 at the boiling point; and all the other numbers in their order.

This diviſion of the ſcale, is commonly called Fahrenheit's. According to this diviſion, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmoſphere which ſuſtains a column of 30 inches of quickſilver in the barometer, that all meaſures and ſpecific gravities are taken, unleſs when otherwiſe mentioned; and in this temperature and preſſure, the relative weights, or ſpecific gravities, of air, water, and quickſilver, are as

$1\frac{2}{9}$ for air,
1000 for water,
13600 for mercury;

and theſe alſo are the weights of a cubic foot of each, in averdupois ounces, in that ſtate of the barometer and thermometer. For other ſtates of the thermometer, each of theſe bodies expands or contracts according to the following rate, with each degree of heat; viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9666}$ part of its bulk.

OF THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

348. FROM the principles laid down in the Scholium to prop. 66, concerning the meaſuring of altitudes by the barometer, and the foregoing deſcriptions of the barometer and thermometer, we may now collect together the precepts for the practice of ſuch meaſurements, which are as follow:

Firſt, Obſerve the height of the barometer at the bottom of any height, or depth, intended to be meaſured; with the temperature of the quickſilver by means of a thermometer attached to the barometer, and alſo the temperature of the air in the ſhade by a detached thermometer.

Second, Let the ſame thing be done alſo at the top of the ſaid height or depth, and at the ſame time, or as near the ſame time as may be. And let thoſe altitudes of barometer be reduced to the ſame temperature, if it be thought neceſſary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminiſh that in the warmer, by its $\frac{1}{9666}$ part for every degree of difference of the two.

Third,

Third, Take the difference of the common logarithms of the two heights of the barometer, corrected as above if neceſſary, cutting off three figures next the right hand for decimals, the reſt being fathoms in whole numbers.

Fourth, Correct the number laſt found for the difference of temperature of the air, as follows: Take half the ſum of the two temperatures, for the mean one; and for every degree which this differs from the temperature 31°, take ſo many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but ſubtract them if the mean temperature be below 31°; and the ſum or difference will be the true altitude in fathoms; or, being multiplied by 6, it will be the altitude in feet.

349. *Example* 1. Let the ſtate of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.	Anſ. the alt. is
Lower 29·68	57	57	$727\frac{3}{10}$ fath.
Upper 25·28	43	42	

350. *Exam.* 2. To find the altitude, when the ſtate of the barometers and thermometers are as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	Anſ. the alt. is
Lower 29·45	38	31	$408\frac{1}{2}$ fath.
Upper 26·82	41	35	

OF THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTION ON BODIES.

PROPOSITION LXVIII.

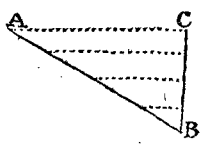
351. *If any Body move through a Fluid at Reſt, or the Fluid move againſt the Body at Reſt; the Force or Reſiſtance of the Fluid againſt the Body, will be as the Square of the Velocity and the Denſity of the Fluid. That is,* $R \propto dv^2$.

FOR, the force or reſiſtance is as the quantity of matter or particles ſtruck, and the velocity with which they are ſtruck. But the quantity or number of particles ſtruck, in any time,

are as the velocity and the denſity of the fluid. Therefore the reſiſtance, or force of the fluid, is as the denſity and ſquare of the velocity.

352. *Corol.* 1. The reſiſtance to any plane, is alſo more or leſs, as the plane is greater or leſs; and therefore the reſiſtance on any plane, is as the area of the plane a, the denſity of the medium, and the ſquare of the velocity. That is, $R \propto adv^2$.

353. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the reſiſtance, in the direction of motion, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination of the plane to the direction of motion, or as the cube of radius to the cube of the ſine of that angle. So that $R \propto adv^2 s^3$, putting 1 = radius, and s = ſine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminiſhed as AB to BC, or as 1 to s. But the force of each particle, ſtriking the plane obliquely in the direction CA, is alſo diminiſhed as AB to BC, or as 1 to s; therefore the reſiſtance, which is perpendicular to the face of the plane by art. 52, is as 1^2 to s^2. But again, this reſiſtance in the direction perpendicular to the face of the plane, is to that in the direction AC, by art. 51, as AB to BC, or as 1 to s. Conſequently, on all theſe accounts, the reſiſtance to the plane, when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the reſiſtance in the direction of the motion, is diminiſhed as 1 to s^3, or in the triplicate ratio of radius to the ſine of inclination.

PROPOSITION LXIX.

354. *The Real Reſiſtance to a Plane, by a Fluid acting in a Direction Perpendicular to its Face, is equal to the Weight of a Column of the Fluid, whoſe Baſe is the Plane, and Altitude equal to that which is due to the Velocity of the Motion, or through which a Heavy Body muſt fall to acquire that Velocity.*

THE reſiſtance to the plane moving through a fluid, is the

ſame as the force of the fluid in motion with the ſame velocity, on the plane at reſt. But the force of the fluid in motion, is equal to the weight or preſſure which generates that motion; and this is equal to the weight or preſſure of a column of the fluid, whoſe baſe is the area of the plane, and its altitude that which is due to the velocity.

355. *Corol.* 1. If a denote the area of the plane, v the velocity, n the denſity or ſpecific gravity of the fluid, and $g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole reſiſtance, or motive force R, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$.

356. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in an angle whoſe ſine is s. Then the reſiſtance to the plane will be $\frac{anv^2s^3}{4g}$.

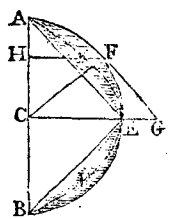
357. *Corol.* 3. Alſo, if w denote the weight of the body, whoſe plane face a is reſiſted by the abſolute force R; then the retarding force f, or $\frac{R}{w}$, will be $\frac{anv^2s^3}{4gw}$.

358. *Corol.* 4. And if the body be a cylinder, whoſe face or end is a, and radius r, moving in the direction of its axis: becauſe then $s = 1$, and $a = pr^2$, where $p = 3{\cdot}1416$; the reſiſting force R will be $\frac{pnv^2r^2}{4g}$, and the retarding force $f = \frac{pnv^2r^2}{4gw}$.

359. *Corol.* 5. This is the value of the reſiſtance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic ſection, or a conical ſurface, or any other figure every where equally inclined to the axis, or direction of motion, the ſine of inclination being s: then, the number of particles of the fluid ſtriking the face being ſtill the ſame, but the force of each, oppoſed to the direction of motion, diminiſhed in the duplicate ratio of radius to the ſine of inclination, the reſiſting force R will be $\frac{pnr^2v^2s^2}{4g}$.

PROPOSITION LXX.

360. *The Reſiſtance to a Sphere moving through a Fluid, is but Half the Reſiſtance to its Great Circle, or to the End of a Cylinder of the Same Diameter, moving with the ſame Velocity.*



LET AFEB be half the ſphere, moving in the direction CEG. Deſcribe the paraboloid AIEKB on the ſame baſe. Let any particle of the medium meet the ſemicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the ſurface at F, is to its force on the baſe at H, as the ſquare of the ſine of the angle G, or its equal the angle FCH, to the ſquare of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole ſurface, is to its force on the circle of the baſe, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC.CB$, and $HF^2 = AH.HB$ by the nature of the circle; alſo, $AH.HB : AC.CB :: HI : CE$ by the nature of the parabola; conſequently the force on the ſpherical ſurface, is to the force on its circular baſe, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumſcribed cylinder, namely, as 1 to 2.

361. *Corol.* Hence, the reſiſtance to the ſphere is $R = \frac{p n v^2 r^2}{8g}$, being the half of that of a cylinder of the ſame diameter. For example, a 9lb iron ball, whoſe diameter is 4 inches, when moving through the air with a velocity of 1600 feet per ſecond, would meet a reſiſtance which is equal to a weight of $132\frac{2}{3}$lb, independent of the preſſure of the atmoſphere, for want of the counterpoize behind the ball.

PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimenſions; and what do all three come to at 45s. per ſquare, or 100 ſquare feet?

Anſ. dif. 280 ſq feet, Amount 18 guineas.

QUEST. 2.

QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have juſt a ſquare yard ſlit off it; at what diſtance from the edge muſt the line be ſtruck? Anſ. $7\frac{1}{19}$ inches.

QUEST. 3. A ceiling contains 114 yards 6 feet of plaſtering, and the room 28 feet broad; what is the length of it?
Anſ. $36\frac{6}{7}$ feet.

QUEST. 4. A common joiſt is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a ſcantling juſt as big again, that ſhall be 3 inches thick; what will the other dimenſion be?
Anſ. $11\frac{2}{3}$ inches.

QUEST. 5. A wooden trough coſt me 3s. 2d. painting within, at 6d. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?
Anſ. $27\frac{1}{4}$ inches.

QUEST. 6. If my court-yard be 47 feet 9 inches ſquare, and I have laid a foot-path with Purbeck ſtone, of 4 feet wide, along one ſide of it; what will paving the reſt with flints come to, at 6d. per ſquare yard? Anſ. 5l. 16s. $0\frac{1}{2}$d.

QUEST. 7. A ladder, 40 feet long, may be ſo planted, that it ſhall reach a window 33 feet from the ground on one ſide of the ſtreet; and, by only turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high on the other ſide: what is the breadth of the ſtreet? Anſ. 56 feet $7\frac{3}{4}$ inches.

QUEST. 8. The paving of a triangular court, at 18d. per foot, came to 100l; the longeſt of the three ſides was 88 feet; required the ſum of the other two equal ſides?
Anſ. 106·85 feet.

QUEST. 9. There are two columns in the ruins of Perſepolis left ſtanding upright; the one is 64 feet above the plain, and the other 50: in a ſtraight line between theſe ſtands an ancient ſmall ſtatue, the head of which is 97 feet from the ſummit of the higher, and 86 feet from the top of the lower column, the baſe of which meaſures juſt 76 feet to the centre of the figure's baſe. Required the diſtance between the tops of the two columns? Anſ. 157 feet nearly.

QUEST. 10. The perambulator, or ſurveying wheel, is ſo contrived, as to turn juſt twice in the length of a pole, or $16\frac{1}{2}$ feet; required the diameter? Anſ. 2·626 feet.

QUEST. 11. In turning a one-horſe chaiſe within a ring of a certain diameter, it was obſerved, that the outer wheel made two turns, while the inner made but one: the wheels

were both 4 feet high; and, ſuppoſing them fixed at the ſtatutable diſtance of 5 feet aſunder on the axletree, what was the circumference of the track deſcribed by the outer wheel? Anſ. 62·832 feet.

QUEST. 12. What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8d. a foot, as the palliſading the three ſides did at a guinea a yard?

Anſ. 72·746 feet.

QUEST. 13. In the trapezium ABCD are given AB = 13, BC = $31\frac{1}{5}$, CD = 24, and DA = 18, alſo B a right angle; required the area? Anſ. 410·122.

QUEST. 14. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per ſquare foot; what will it come to at 18s. per cwt. Anſ. 22l. 19s. $10\frac{1}{4}$d.

QUEST. 15. Having a rectangular marble ſlab, 58 inches by 27, I would have a ſquare foot cut off parallel to the ſhorter edge; I would then have the like quantity divided from the remainder parallel to the longer ſide; and this alternately repeated, till there ſhall not be the quantity of a foot left: what will be the dimenſions of the remaining piece? Anſ. 20·7 inches by 6·086.

QUEST. 16. Given two ſides of an obtuſe-angled triangle, which are 20 and 40 poles; required the third ſide, that the triangle may contain juſt an acre of land?

Anſ. 58·876 or 23·099.

QUEST. 17. The end wall of a houſe is 24 feet 6 inches in breadth, and 40 feet to the eaves; $\frac{1}{3}$ of which is 2 bricks thick, $\frac{1}{3}$ more is $1\frac{1}{2}$ brick thick, and the reſt 1 brick thick. Now the triangular gable riſes 38 courſes of bricks, 4 of which uſually make a foot in depth, and this is but $4\frac{1}{2}$ inches, or half a brick thick: what will this piece of work come to at 5l. 10s. per ſtatute rod? Anſ. 20l. 11s. $7\frac{1}{2}$d.

QUEST. 18. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courſes to the foot in height? Anſ. 72000.

QUEST. 19. How many bricks will build a ſquare pyramid of 100 feet on each ſide at the baſe, and alſo 100 feet perpendicular height: the dimenſions of a brick being ſuppoſed 10 inches long, 5 inches broad, and 3 inches thick?

Anſ. 3840000.

QUEST. 20. If, from a right-angled triangle, whoſe baſe is 12, and perpendicular 16 feet, a line be drawn parallel to the

the perpendicular, cutting off a triangle whoſe area is 24 ſquare feet; required the ſides of this triangle?

Anſ. 6, 8, and 10.

QUEST. 21. The ellipſe in Groſvenor-ſquare meaſures 840 links acroſs the longeſt way, and 612 the ſhorteſt, within the rails: now the walls being 14 inches thick, what ground do they incloſe, and what do they ſtand upon?

Anſ. { incloſe 4 ac. 0r. 6 p.
{ ſtand on 1760½ ſq. feet.

QUEST. 22. If a round pillar, 7 inches over, have 4 feet of ſtone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Anſ. 22·136 inches.

QUEST. 23. A circular fiſh-pond is to be made in a garden, that ſhall take up juſt half an acre; what muſt be the length of the chord that ſtrikes the circle? Anſ. 27¾ yards.

QUEST. 24. When a roof is of a true pitch, the rafters are ¾ of the breadth of the building: now ſuppoſing the eaves-boards to project 10 inches on a ſide, what will the new ripping a houſe coſt, that meaſures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15s. per ſquare?

Anſ. 8l. 15s. 9½d.

QUEST. 25. A cable, which is 3 feet long, and 9 inches in compaſs, weighs 22lb; what will a fathom of that cable weigh, which meaſures a foot about? Anſ. 78⅔lb.

QUEST. 26. My plumber has put 28lb. per ſquare foot into a ciſtern, 74 inches and twice the thickneſs of the lead long, 26 inches broad, and 40 deep; he has alſo put three ſtays acroſs it within, 16 inches deep, of the ſame ſtrength, and reckons 22s. per cwt, for work and materials. I, being a maſon, have paved him a workſhop, 22 feet 10 inches broad, with Purbeck ſtone, at 7d. per foot; and upon the balance I find there is 3s. 6d. due to him; what was the length of the workſhop? Anſ. 32 feet, 0¾ inch.

QUEST. 27. The diſtance of the centres of two circles, whoſe diameters are each 50, being given, equal to 30; what is the area of the ſpace incloſed by their circumferences? Anſ. 559·119.

QUEST. 28. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter ſquare; what will 50 feet come to at 3½d. per lb, the bars being but ⅞ of an inch ſquare? Anſ. 20l. 0s. 2d.

QUEST. 29. The area of an equilateral triangle, whoſe baſe falls on the diameter, and its vertex in the middle of the arc of a ſemicircle, is equal to 100: what is the diameter of the ſemicircle? Anſ. 26·32148.

QUEST. 30. It is required to find the thickneſs of the lead in a pipe, of an inch and quarter bore, which weighs 14 lb per yard in length; the cubic foot of lead weighing 11325 ounces? Anſ. ·20737 inches.

QUEST. 31. Suppoſing the expence of paving a ſemicircular plot, at 2s. 4d. per foot, come to 10l.; what is the diameter of it? Anſ. 14·7737 feet.

QUEST. 32. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whoſe diameter is 289? Anſ. 278·6716.

QUEST. 33. My plumber has ſet me up a ciſtern, and, his ſhop-book being burnt, he has no means of bringing in the charge, and I do not chooſe to take it down to have it weighed; but by meaſure he finds it contains $64\frac{3}{10}$ ſquare feet, and that it is preciſely $\frac{1}{8}$ of an inch in thickneſs. Lead was then wrought at 2l. per fother of $19\frac{1}{2}$ cwt. It is required from theſe items to make out the bill, allowing $6\frac{5}{9}$ oz. for the weight of a cubic inch of lead? Anſ. 4l. 11s. 2d.

QUEST. 34. What will the diameter of a globe be, when the ſolidity and ſuperficial content are expreſſed by the ſame number? Anſ. 6.

QUEST 35. A ſack, that would hold 3 buſhels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another ſack contain, which, being of the ſame length, has twice its breadth or circumference? Anſ. 12 buſhels.

QUEST. 36. A carpenter is to put an oaken curb to a round well, at 8d. per foot ſquare; the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $3\frac{1}{2}$ feet: what will be the expence? Anſ. 5s. $2\frac{1}{4}$d.

QUEST. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what muſt the breadth of the walk be, to take up juſt half the ground? Anſ. 25·968 feet.

QUEST. 38. The top of a may-pole, being broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the foot of the pole; what was the height of the whole may-pole, ſuppoſing the length of the broken piece to be 39 feet? Anſ. 75 feet.

QUEST. 39.

QUEST. 39. Seven men bought a grinding ſtone, of 60 inches diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter muſt each grind down for his ſhare?

Anſ. the 1ſt 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778 inches.

QUEST. 40. A maltſter has a kiln, that is 16 feet 6 inches ſquare: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what muſt be the length of its ſide? Anſ. 28 feet, 7 inches.

QUEST 41. How many 3 inch cubes may be cut out of a 12 inch cube? Anſ. 64.

QUEST. 42. How long muſt the tether of a horſe be, that will allow him to graze, quite around, juſt an acre of ground? Anſ. $39\frac{1}{4}$ yards.

QUEST. 43. What will the painting of a conical ſpire come to, at 8d. per yard; ſuppoſing the height to be 118 feet, and the circumference of the baſe 64 feet? Anſ. 14l. 0s. $8\frac{3}{4}$d.

QUEST. 44. The diameter of a ſtandard corn buſhel is $18\frac{1}{2}$ inches, and its depth 8 inches; then what muſt the diameter of that buſhel be, whoſe depth is $7\frac{1}{2}$ inches?

Anſ. 19·1067 inches.

QUEST. 45. Suppoſe the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it coſt at $3\frac{1}{2}$d per ſquare inch? Anſ. 237l. 10s. 1d.

QUEST. 46. What will a fruſtum of a marble cone come to, at 12s. per ſolid foot; the diameter of the greater end being 4 feet, that of the leſs end $1\frac{1}{2}$, and the length of the ſlant ſide 8 feet? Anſ. 30l. 1s. $10\frac{1}{4}$d.

QUEST. 47. To divide a cone into three equal parts by ſections parallel to the baſe, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Anſ. the upper part 13·867
the middle part 3·605
the lower part 2·528

QUEST. 48. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raiſe 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth muſt the ditch be dug, ſuppoſing its breadth to be every where 8 feet? Anſ. $7\frac{23}{86}$ feet.

QUEST. 49. How high above the earth muſt a perſon be raiſed, that he may ſee $\frac{1}{3}$ of its ſurface?

Anſ. to the height of the earth's diameter.

QUEST. 50.

QUEST. 50. A cubic foot of brafs is to be drawn into wire, of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no lofs in the metal?

Anf. 97784·797 yards, or 55 miles 984·797 yards.

QUEST. 51. Of what diameter muft the bore of a cannon be, which is caft for a ball of 24lb weight, fo that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Anf. 5·647 inches.

QUEST. 52. Suppofing the diameter of an iron 9lb ball to be 4 inches, as it is very nearly; it is required to find the diameters of the feveral balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb, and the caliber of their guns, allowing $\frac{1}{30}$ of the caliber, or $\frac{2}{40}$ of the ball's diameter, for windage. Anfwer.

Wt ball.	Diameter ball.	Caliber gun.
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·5469	5·6601
32	6·1051	6·2297
36	6·3496	6·4792
42	6·6844	6·8208

QUEST. 53. Suppofing the windage of all mortars to be $\frac{2}{40}$ of the caliber, and the diameter of the hollow part of the fhell to be $\frac{7}{10}$ of the caliber of the mortar: It is required to determine the diameter and weight of the fhell, and the quantity or weight of powder requifite to fill it, for each of the feveral forts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar. Anfwer.

Calib. mort.	Diameter fhell.	Wt fhell empty.	Wt of powder.	Wt fhell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

QUEST. 54.

QUEST. 54. If a heavy ſphere, whoſe diameter is 4 inches, be let fall into a conical glaſs, full of water, whoſe diameter is 5, and altitude 6 inches; it is required to determine how much water will run over?

Anſ. 26·272 cubic inches, or near $\frac{3}{4}\frac{5}{7}$ parts of a pint.

QUEST. 55. The dimenſions of the ſphere and cone being the ſame as in the laſt queſtion, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the ſphere is immerſed in the water? Anſ. ·546 parts of an inch.

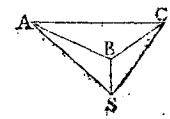
QUEST. 56. The cone being ſtill the ſame, and $\frac{1}{5}$ full of water; required the diameter of a ſphere which ſhall be juſt all covered by the water? Anſ. ·2·445996 inches.

QUEST. 57. If a perſon, with an air balloon, aſcend vertically from London, to ſuch height that he can juſt ſee Oxford appear in the horizon; it is required to determine his height above the earth, ſuppoſing its circumference to be 25000 miles, and the diſtance between London and Oxford 49·5933 miles? Anſ. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

QUEST. 58. In a garriſon there are three remarkable objects, A, B, C, the diſtances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am deſirous of knowing my poſition and diſtance at a place or ſtation S, from whence I obſerved the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Anſwer.

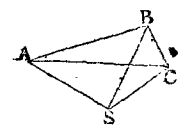
AS	605·7122;
BS	429·6814;
CS	524·2365.



QUEST. 59. Required the ſame as in the laſt queſtion, when the point B is on the other ſide of AC, ſuppoſing AB 9, AC 12, and BC 6 furlongs; alſo the angle ASB 33° 45′, and the angle BSC 22° 30′.

Anſwer.

AS 10·64, BS 15·64, CS 14·01.



QUEST. 60. It is required to determine the magnitude of a cube of gold, of the ſtandard fineneſs, which ſhall be equal

equal to a ſum of 240 million of pounds ſterling; ſuppoſing a guinea to weigh 5 dwts $9\frac{1}{2}$ grains. Anſ. 14·835 feet.

QUEST. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Anſ. 1158127 gallons nearly.

PRACTICAL EXERCISES CONCERNING SPECIFIC GRAVITY.

THE Specific Gravities of Bodies, are their relative weights contained under the ſame given magnitude; as a cubic foot, or a cubic inch, &c.

The ſpecific gravities of ſeveral ſorts of matter, are expreſſed by the numbers annexed to their names in the Table of Specific Gravities, at page 220; from whence the numbers are to be taken, when wanted.

Note. The ſeveral ſorts of wood are ſuppoſed to be dry. Alſo, as a cubic foot of water weighs juſt 1000 ounces averdupois, the numbers in the table expreſs, not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in averdupois ounces; and thence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems:

PROBLEM I.

To find the Magnitude of any Body, from its Weight.

As the tabular ſpecific gravity of the body,
Is to its weight in averdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, reſpectively.

EXAMPLES.

EXAM. 1. Required the content of an irregular block of common ſtone, which weighs 1 cwt, or 112lb.

Anſ. $1228\frac{4}{5}$ cubic inches.

EXAM. 2.

EXAM. 2. How many cubic inches of gunpowder are there in 1lb weight? Anſ. 30 cubic inches nearly.

EXAM. 3. How many cubic feet are there in a ton weight of dry oak? Anſ. $38\frac{1}{1}\frac{3}{8}\frac{8}{5}$ cubic feet.

PROBLEM II.

To find the Weight of a Body, from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular ſpecific gravity,
To the weight of the body.

EXAMPLES.

EXAM. 1. Required the weight of a block of marble, whoſe length is 63 feet, and breadth and thickneſs each 12 feet; being the dimenſions of one of the ſtones in the walls of Balbeck?

Anſ. $683\frac{7}{16}$ ton, which is nearly equal to the burthen of an Eaſt-India ſhip.

EXAM. 2. What is the weight of 1 pint, ale meaſure, of gunpowder? Anſ. 19 oz. nearly.

EXAM. 3. What is the weight of a block of dry oak, which meaſures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Anſ. $4335\frac{1}{1}\frac{5}{6}$lb.

PROBLEM III.

To find the Specific Gravity of a Body.

CASE 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight loſt in water. Then ſay,

As the weight loſt in water,
Is to the whole weight,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

A piece of ſtone weighed 10lb, but in water only $6\frac{1}{8}$lb, required its ſpecific gravity? Anſ. 2609.

CASE 2. When the body is lighter than water, ſo that it will not quite ſink; affix to it a piece of another body, heavier than

than water, ſo that the maſs compounded of the two may ſink together. Weigh the denſer body and the compound maſs, ſeparately, both in water, and out of it; then find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then ſay,

As the laſt remainder,
Is to the weight of the light body in air,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

Suppoſe a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the ſpecific gravity of the elm? Anſ. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients in a Given Compound.

TAKE the three differences of every pair of the three ſpecific gravities, namely, the ſpecific gravities of the compound and each ingredient; and multiply the difference of every two ſpecific gravities by the third. Then, as the greateſt product, is to the whole weight of the compound, ſo is each of the other products, to the two weights of the ingredients.

EXAMPLE.

A compoſition of 112lb being made of tin and copper, whoſe ſpecific gravity is found to be 8784; required the quantity of each ingredient, the ſpecific gravity of tin being 7320, and of copper 9000?

Anſwer, there is 100lb of copper, and conſequently 12lb of tin } in the compoſition.

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimenſions of Balls and Shells might be found from the problems laſt given, concerning ſpecific gravity. But they may be found ſtill eaſier by means of the experimented weight of a ball of a given ſize, from the known proportion of ſimilar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb, and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9, ſo is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

EXAM. 1. The diameter of an iron ſhot being 6·7, required its weight? Anſ. 42·294 lb.

EXAM. 2. What is the weight of an iron ball, whoſe diameter is 5·54 inches? Anſ. 24 lb.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of $4\frac{1}{4}$ inches diameter weighs 17 lb; therefore, as the cube of $4\frac{1}{4}$ is to 17, or nearly as 9 is to 2, ſo is the cube of the diameter of a leaden ball, to its weight.

Or, take $\frac{2}{9}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Anſ. 63·888 lb.

EXAM. 2. What is the weight of a leaden ball of 5·24 inches diameter? Anſ. 32 lb nearly.

PRO-

PROBLEM III.

To find the Diameter of an Iron Ball.

MULTIPLY the weight by $7\frac{1}{5}$, and the cube root of the product will be the diameter.

EXAMPLES.

EXAM. 1. Required the diameter of a 42 lb iron ball?
Anſ. 6·685 inches.

EXAM. 2. What is the diameter of a 24 lb iron ball?
Anſ. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

MULTIPLY the weight by 9, and divide the product by 2; then the cube root of the quotient will be the diameter.

EXAMPLES.

EXAM. 1. Required the diameter of a 64lb leaden ball?
Anſ. 6·605 inches.

EXAM. 2. What is the diameter of an 8lb leaden ball?
Anſ. 3·303 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

TAKE $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the ſhell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

EXAM. 1. The outſide diameter of an iron ſhell being 12·8, and the inſide diameter 9·1 inches; required its weight?
Anſ. 188·941lb.

EXAM. 2. What is the weight of an iron ſhell, whoſe external and internal diameters are 9·8 and 7 inches?
Anſ. $84\frac{1}{4}$lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

DIVIDE the cube of the internal diameter, in inches, by 57·3, for the lbs of powder.

EXAMPLES.

EXAM. 1. How much powder will fill the ſhell whoſe internal diameter is 9·1 inches? Anſ. $13\frac{2}{15}$ lb nearly.

EXAM. 2. How much powder will fill the ſhell whoſe internal diameter is 7 inches? Anſ. 6 lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

FIND the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

EXAM. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Anſ. 60lb.

EXAM. 2. How much powder will fill a cubical box whoſe ſide is 12 inches? Anſ. $57\frac{3}{5}$ lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

MULTIPLY the ſquare of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAMPLES.

EXAM. 1. How much powder will the cylinder hold, whoſe diameter is 10 inches, and length 20 inches?
Anſ. $52\frac{1}{3}$ lb nearly.

EXAM. 2. How much powder can be contained in the cylinder, whoſe diameter is 4 inches, and length 12 inches?
Anſ. $5\frac{5}{191}$ lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

MULTIPLY the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

EXAM. 1. What is the diameter of a ſhell that will hold $13\frac{1}{6}$ lb of powder? Anſ. 9·1 inches.

EXAM. 2. What is the diameter of a ſhell to contain 6lb of powder? Anſ. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

MULTIPLY the weight in pounds by 30, and the cube root of the product will be the ſide of the box in inches.

EXAMPLES.

EXAM. 1. Required the ſide of a cubical box, to hold 50lb of gun-powder? Anſ. 11·44 inches.

EXAM. 2. Required the ſide of a cubical box, to hold 400 lb of gunpowder? Anſ. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

MULTIPLY the weight in pounds by 38·2, and divide the product by the ſquare of the diameter in inches, for the length.

EXAMPLES.

EXAM. 1. What length of a 36 pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12 lb of gunpowder?
Anſ. 10·314 inches.

EXAM. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20 lb of powder? Anſ. $11\frac{15}{16}$ inches.

OF THE PILING OF BALLS AND SHELLS.

IRON Balls and Shells are commonly piled, by horizontal courſes, either in a pyramidical or wedge-like form; the baſe being either an equilateral triangle, a ſquare, or a rectangle. In the triangle and ſquare, the pile will finiſh in a ſingle ball; but in the rectangle, it will finiſh in a ſingle row of balls, like an edge.

In triangular and ſquare piles, the number of horizontal rows, or courſes, is always equal to the number of balls in one ſide of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Alſo, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

MULTIPLY continually together, the number in one ſide of the bottom row, that number increaſed by 1, and the ſame number increaſed by 2; and $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n \cdot n+1 \cdot n+2}{6}$ is the number or ſum; where n is the number in the bottom row.

EXAMPLES.

EXAM. 1. Required the number of balls in a triangular pile, each ſide of the baſe containing 30 balls? Anſ. 4960.

EXAM. 2. How many balls are in the triangular pile, each ſide of the baſe containing 20? Anſ. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

MULTIPLY continually together, the number in one ſide of the bottom courſe, that number increaſed by 1, and double the ſame number increaſed by 1; then $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n \cdot n+1 \cdot 2n+1}{6}$ is the number.

EXAMPLES.

EXAM. 1. How many balls are in a ſquare pile of 30 rows?

Anſ. 9455.

EXAM. 2. How many balls are in a ſquare pile of 28 rows? Anſ. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

FROM 3 times the number in the length of the baſe row, ſubtract one leſs than the breadth of the ſame, multiply the remainder by the ſaid breadth, and the product by one more than the ſame; and divide by 6 for the anſwer.

That is, $\frac{b \cdot b + 1 \cdot 3l - b + 1}{6}$ is the number; where l is the length, and b the breadth, of the loweſt courſe.

Note. In all the piles, the breadth of the bottom is equal to the number of courſes. And, in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the baſe row being 46 and 15? Anſ. 4960.

EXAM. 2. How many ſhot are in a rectangular complete pile, the length of the bottom courſe being 59, and its breadth 20? Anſ. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

FROM the number in the whole pile, conſidered as complete, ſubtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the fruſtum, or incomplete pile.

EXAMPLES.

EXAM. 1. To find the number of ſhot in the incomplete triangular pile, one ſide of the bottom courſe being 40, and the top courſe 20? Anſ. 10150.

EXAM. 2. How many ſhot are in the incomplete triangular pile, the ſide of the baſe being 24, and of the top 8? Anſ. 2516.

EXAM. 3. How many balls are in the incomplete ſquare pile, the ſide of the baſe being 24, and of the top 8? Anſ. 4760.

EXAM. 4. How many ſhot are in the incomplete rectangular pile, of 12 courſes, the length and breadth of the baſe being 40 and 20? Anſ. 6146.

OF

OF DISTANCES BY THE VELOCITY OF SOUND.

BY various experiments it has been found, that ſound flies, through the air, uniformly at the rate of about 1142 feet in 1 ſecond of time, or a mile in $4\frac{2}{3}$ ſeconds. And therefore, by proportion, any diſtance may be found correſponding to any given time; namely, multiplying the given time, in ſeconds, by 1142, for the correſponding diſtance in feet; or taking $\frac{3}{14}$ of the given time for the diſtance in miles.

Note. The time for the paſſage of ſound in the interval between ſeeing the flaſh of a gun, or lightning, and hearing the report, may be obſerved by a watch, or a ſmall pendulum. Or, it may be obſerved by the beats of the pulſe in the wriſt, counting, on an average, about 70 to a minute for perſons in moderate health, or $5\frac{1}{2}$ pulſations to a mile; and more or leſs according to circumſtances.

EXAMPLES.

EXAM. 1. After obſerving a flaſh of lightning, it was 12 ſeconds before the thunder was heard; required the diſtance of the cloud from whence it came? Anſ. $2\frac{4}{7}$ miles.

EXAM. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, ſuppoſing the diſtance to be 8 miles in a ſtraight line? Anſ. $37\frac{1}{3}$ ſeconds.

EXAM. 3. After obſerving the firing of a large cannon at a diſtance, it was 7 ſeconds before the report was heard; what was its diſtance? Anſ. $1\frac{1}{2}$ mile.

EXAM. 4. Perceiving a man at a diſtance hewing down a tree with an axe, I remarked that 6 of my pulſations paſſed between ſeeing him ſtrike and hearing the report of the blow; what was the diſtance between us, allowing 70 pulſes to a minute? Anſ. 1 mile and 198 yards.

EXAM. 5. How far off was the cloud from which thunder iſſued, whoſe report was 5 pulſations after the flaſh of lightning; counting 75 to a minute? Anſ. 1523 yards.

EXAM. 6. If I ſee the flaſh of a cannon, fired by a ſhip in diſtreſs at ſea, and hear the report 33 ſeconds after, how far is ſhe off? Anſ. $7\frac{1}{14}$ miles.

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, and other Branches of Natural Philoſophy.

QUESTION 1. REQUIRED the weight of a caſt iron ball of 3 inches diameter, ſuppoſing the weight of a cubic inch of the metal to be 0·258 lb averdupois? Anſ. 5·64739 lb.

QUEST. 2. To determine the weight of a hollow ſpherical iron ſhell, 5 inches in diameter, the thickneſs of the metal being one inch? Anſ. 13·2387 lb.

QUEST. 3. Being one day ordered to obſerve how far a battery of cannon was from me, I counted, by my watch, 17 ſeconds between the time of ſeeing the flaſh and hearing the report; what was the diſtance? Anſ. $3\frac{2}{3}$ miles.

QUEST. 4. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles; required the proportion of their ſurfaces, and alſo of their ſolidities; ſuppoſing them both to be globular, as they are very nearly?

Anſ. the ſurfaces are as $13\frac{1}{2}$ to 1 nearly.
and the ſolidities as $49\frac{1}{2}$ to 1 nearly.

QUEST. 5. It is propoſed to determine the proportional quantities of matter in the earth and moon; the denſity of the former being to that of the latter, as 10 to 7, and their diameters as ſpecified in the preceding problem.

Anſ. as 71 to 1 nearly.

QUEST. 6. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of braſs of the ſame dimenſions?

Anſ. 496 lb 14 oz.

QUEST. 7. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three ſtones laid end to end, now in ſight, that meaſure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, ſo as to prepare it for moving?

Anſ. $683\frac{7}{8}$ tons, the burthen of an Eaſt-India ſhip.

QUEST. 8. The battering-ram of Veſpaſian weighed, ſuppoſe 100000 pounds; and was moved, let us admit, with ſuch a velocity, by ſtrength of hands, as to paſs through 20 feet in one ſecond of time; and this was found ſufficient

to

to demoliſh the walls of Jeruſalem. The queſtion is, with what velocity a 32 lb ball muſt move, to do the ſame execution? Anſ. 62500 feet.

QUEST. 9. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the leſs moves with 1000 times the velocity of the greater: in what proportion then are the momenta, or forces, with which they are moved?

Anſ. the leſs moves with a force 40 times greater.

QUEST. 10. A body, weighing 20 lb, is impelled by ſuch a force, as to ſend it through 100 feet in a ſecond; with what velocity then would a body of 8 lb weight move, if it were impelled by the ſame force? Anſ. 250 feet per ſecond.

QUEST. 11. There are two bodies, the one of which weighs 100 lb, the other 60; but the leſs body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which theſe bodies move, is required?

Anſ. the velocity of the greater to that of the leſs, as 3 to 40.

QUEST. 12. There are two bodies, the greater contains 8 times the quantity of matter in the leſs, and is moved with a force 48 times greater; the ratio of the velocities of theſe two bodies is required?

Anſ. the greater to the leſs, as 6 to 1.

QUEST. 13. There are two bodies, one of which moves 40 times ſwifter than the other; but the ſwifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the ſpaces deſcribed by theſe two bodies is required?

Anſ. the ſwifter to the ſlower, as 1 to 3.

QUEST. 14. Suppoſing one body to move 30 times ſwifter than another, as alſo the ſwifter to move 12 minutes, the other only 1: what difference will there be between the ſpaces deſcribed by them, ſuppoſing the laſt has moved 60 inches? Anſ. 1795 feet.

QUEST. 15. There are two bodies, the one of which has paſſed over 50 miles, the other only 5; and the firſt had moved with 5 times the celerity of the ſecond: what is the ratio of the times they have been in deſcribing thoſe ſpaces?

Anſ. as 2 to 1.

QUEST. 16. If a lever, 40 effective inches long, will, by a certain power thrown ſucceſſively upon it, in 13 hours, raiſe a weight 104 feet; in what time will two other levers,

each 18 effective inches long, raiſe an equal weight 73 feet?
Anſ. 10 hours $8\frac{1}{3}$ minutes.

QUEST. 17. What weight will a man be able to raiſe, who preſſes with the force of a hundred and a half, on the end of an equipoiſed handſpike, 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine? Anſ. 2072 lb.

QUEST. 18. A weight of $1\frac{1}{2}$ lb, laid on the ſhoulder of a man, is no greater burthen to him than its abſolute weight, or 24 ounces: what difference will he feel, between the ſaid weight applied near his elbow, at 12 inches from the ſhoulder, and in the palm of his hand, 28 inches from the ſame; and how much more muſt his muſcles then draw, to ſupport it at right angles, that is, having his arm ſtretched right out?
Anſ. 24 lb averdupois.

QUEST. 19. What weight hung on at 70 inches from the centre of motion of a ſteel-yard, will balance a ſmall gun of $9\frac{1}{2}$ cwt, freely ſuſpended at 2 inches diſtance from the ſaid centre on the contrary ſide? Anſ. $30\frac{2}{5}$ lb.

QUEST. 20. It is propoſed to divide the beam of a ſteel-yard, or to find the points of diviſion where the weights of 1, 2, 3, 4, &c lb on the one ſide, will juſt balance a conſtant weight of 95 lb at the diſtance of 2 inches on the other ſide of the fulcrum; the weight of the beam being 10 lb, and its whole length 36 inches.
Anſ. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

QUEST. 21. Two men carrying a burthen of 200lb weight between them, hung on a pole, the ends of which reſt on their ſhoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet? Anſ. 125 lb and 75 lb.

QUEST. 22. If, in a pair of ſcales, a body weigh 90 lb in one ſcale, and only 40 lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each ſide of the point of ſuſpenſion?
Anſ. the weight 60 lb, and the proportion 3 to 2.

QUEST. 23. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For inſtance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itſelf when the prop is 13 feet from the leſs end; but removing the prop a foot nearer to the ſaid end,

end, it takes a man's weight of 210 lb, ſtanding on the leſs end, to hold it in equilibrium. Required the weight of the tree? Anſ. 2520 lb.

QUEST. 24. If AB be a cane or walking-ſtick, 40 inches long, ſuſpended by a ſtring SD faſtened to the middle point D: now a body being hung on at E, 6 inches diſtant from D, is balanced by a weight of 2 lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2 lb weight on the other ſide is moved to G, within 8 inches of D, before the cane will reſt in equilibrio. Required the weight of the body? Anſ. 24 lb.

QUEST. 25. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio on the planes, in all poſitions of them; and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?

Anſ. BD = 24; and P to Q as AB to BC, or as 3 to 4.

QUEST. 26. A lever, of 6 feet long, is fixed at right angles in a ſcrew, whoſe threads are one inch aſunder, ſo that the lever turns juſt once round in raiſing or depreſſing the ſcrew one inch. If then this lever be urged by a weight or force of 50 lb, with what force will the ſcrew preſs?

Anſ. 22619$\frac{1}{2}$ lb.

QUEST. 27 If a man can draw a weight of 150 lb up the ſide of a perpendicular wall, of 20 feet high; what weight will he be able to raiſe along a ſmooth plank of 30 feet long, laid aſlope from the top of the wall? Anſ. 225 lb.

QUEST. 28. If a force of 150 lb be applied on the head of a rectangular wedge, its thickneſs being 2 inches, and the length of its ſide 12 inches; what weight will it riſe or balance perpendicular to its ſide? Anſ. 900 lb.

QUEST. 29. If a round pillar, of 30 feet diameter, be raiſed on a plane, inclined to the horizon in an angle of 75°, or the ſhaft inclining 15 degrees out of the perpendicular; what length will it bear before it overſet?

Anſ. 30 $(2 + \sqrt{3})$ or 111·9615 feet.

QUEST. 30. If the greateſt angle at which a bank of natural earth will ſtand, be 45°; it is propoſed to determine what thickneſs an upright wall of ſtone muſt be made throughout, juſt to ſupport a bank of 12 feet high; the ſpecific gravity of the ſtone being to that of earth, as 5 to 4.

Anſ. $12\sqrt{\frac{8}{15}}$, or 8·76356 feet.

QUEST. 31.

QUEST. 31. If the ſtone wall be made like a wedge, or having its upright ſection a triangle, tapering to a point at top, but its ſide next the bank of earth perpendicular to the horizon; what is its thickneſs at the bottom, ſo as to ſupport the ſame bank? Anſ. $12\sqrt{\frac{4}{3}}$, or 10·733126 feet.

QUEST. 32. But if the earth will only ſtand at an angle of 30 degrees to the horizontal line; it is required to determine the thickneſs of wall in both the preceding caſes?

Anſ. the breadths are the ſame as before, becauſe the area of the triangular bank of earth is increaſed in the ſame proportion as its horizontal puſh is decreaſed.

QUEST. 33. To find the thickneſs of an upright rectangular wall, neceſſary to ſupport a body of water; the water being 10 feet deep, and the wall 12 feet high; alſo the ſpecific gravity of the wall to that of the water, as 11 to 7.

Anſ. 4·204374 feet.

QUEST. 34. To determine the thickneſs of the wall at the bottom, when the ſection of it is triangular, and the altitudes as before. Anſ. 5·1492866 feet.

QUEST. 35. Suppoſing the diſtance of the earth from the ſun to be 95 millions of miles; I would know at what diſtance from him another body muſt be placed, ſo as to receive light and heat quadruple to that of the earth.

Anſ. at half the diſtance, or $47\frac{1}{2}$ millions.

QUEST. 36. If the mean diſtance of the ſun from us be 106 of his diameters; how much hotter is it at the ſurface of the ſun, than under our equator?

Anſ. 11236 times hotter.

QUEST. 37. The diſtance between the earth and ſun being accounted 95 millions of miles, and between Jupiter and the ſun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Anſ. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 38. A certain body on the ſurface of the earth weighs a cwt, or 112 lb; the queſtion is, whither this body muſt be carried, that it may weigh only 10 lb?

Anſ. either at 3·3466 ſemi-diameters, or $\frac{5}{56}$ of a ſemi-diameter, from the centre.

QUEST. 39. If a body weigh 1 pound, or 16 ounces, on the ſurface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Anſ. 15 oz. $9\frac{5}{8}$ dr. nearly.

QUEST. 40.

QUEST. 40. Whereabouts, in the line between the earth and moon, is their common centre of gravity; ſuppoſing the earth's diameter to be 7930 miles, and the moon's 2160; alſo the denſity of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean diſtance 30 of the earth's diameters?

Anſ. at $\frac{10}{21}\frac{5}{11}$ parts of a diameter from the earth's centre, or $\frac{41}{50}\frac{1}{2}$ parts of a diameter, or 648 miles below the ſurface.

QUEST. 41. Whereabouts, between the earth and moon, are their attractions equal to each other? Or, where muſt another body be placed, ſo as to remain ſuſpended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimenſions being as in the laſt queſtion.

Anſ. From the earth's centre $26\frac{9}{11}$ } of the earth's di-
From the moon's centre $3\frac{2}{11}$ } ameters.

QUEST. 42. Suppoſe a ſtone, dropt into an abyſs, ſhould be ſtopped at the end of the 11th ſecond after its delivery; what ſpace would it have gone through? Anſ. $1946\frac{1}{2}$ feet.

QUEST. 43. What is the difference between the depths of two wells, into each of which ſhould a ſtone be dropped at the ſame inſtant, the one will ſtrike the bottom at 6 ſeconds, the other at 10? Anſ. $1029\frac{1}{3}$ feet.

QUEST. 44. If a ſtone be $19\frac{1}{2}$ ſeconds in deſcending from the top of a precipice to the bottom, what is its height?

Anſ. $6115\frac{11}{16}$ feet.

QUEST. 45. In what time will a muſket ball, dropped from the top of Saliſbury ſteeple, ſaid to be 400 feet high, reach the bottom? Anſ. 5 ſeconds nearly.

QUEST. 46. If a heavy body be obſerved to fall through 100 feet in the laſt ſecond of time, from what height did it fall, and how long was it in motion?

Anſ. time $3\frac{2}{3}\frac{3}{8}\frac{5}{6}$ ſec. and height $209\frac{4}{9}\frac{2}{2}\frac{7}{6}\frac{3}{4}$ feet.

QUEST. 47. A ſtone being let fall into a well, it was obſerved that, after being dropped, it was 10 ſeconds before the ſound of the fall at the bottom reached the ear. What is the depth of the well? Anſ. 1270 feet nearly.

QUEST. 48. It is propoſed to determine the length of a pendulum vibrating ſeconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the firſt ſecond of time? Anſ. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

QUEST. 49.

QUEST. 49. What is the length of a pendulum vibrating in 2 ſeconds; alſo in half a ſecond, and in a quarter ſecond?

Anſ. the 2 ſecond pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ ſecond pendulum $9\frac{2}{3}\frac{5}{2}$
the $\frac{1}{4}$ ſecond pendulum $2\frac{5}{1}\frac{7}{2}\frac{7}{8}$ inches.

QUEST. 50. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Anſ. $2692\frac{1}{2}$.

QUEST. 51. Obſerved that while a ſtone was deſcending, to meaſure the depth of a well, a ſtring and plummet, that from the point of ſuſpenſion, or the place where it was held, to the centre of oſcillation, meaſured juſt 18 inches, had made 8 vibrations, when the ſound from the bottom returned. What was the depth of the well? Anſ. 412·61 feet.

QUEST. 52. If a ball vibrate in the arch of a circle, 10 degrees on each ſide of the perpendicular; or a ball roll down the loweſt 10 degrees of the arch; required the velocity at the loweſt point? the radius of the circle, or length of the pendulum, being 20 feet. Anſ. 4·4213 feet per ſecond.

QUEST. 53. If a ball deſcend down a ſmooth inclined plane, whoſe length is 100 feet, and altitude 10 feet; how long will it be in deſcending, and what will be the laſt velocity?

Anſ. the veloc. 25·364 feet per ſec. and time 7·8852 ſec.

QUEST. 54. If a cannon ball, of 1 lb weight, be fired againſt a pendulous block of wood, and, ſtriking the centre of oſcillation, cauſe it to vibrate an arc whoſe chord is 30 inches; the radius of that arc, or diſtance from the axis to the loweſt point of the pendulum, being 118 inches, and the pendulum vibrating in ſmall arcs 40 oſcillations per minute. Required the velocity of the ball, and the velocity of the centre of oſcillation of the pendulum, at the loweſt point of the arc; the whole weight of the pendulum being 500lb?

Anſ. veloc. ball 1956·6054 feet per ſec.
and veloc. cent. oſcil. 3·9054 feet per ſec.

QUEST. 55. How deep will a cube of oak ſink in common water; each ſide of the cube being 1 foot?

Anſ. $11\frac{1}{10}$ inches.

QUEST. 56. How deep will a globe of oak ſink in water; the diameter being 1 foot? Anſ. 9·9867 inches.

QUEST. 57. If a cube of wood, floating in common water, have 3 inches of its height dry above the water, and $4\frac{8}{10}\frac{3}{3}$ inches

inches dry when in ſea-water; it is propoſed to determine the magnitude of the cube, and what ſort of wood it is made of? Anſ. the wood is oak, and each ſide 40 inches.

QUEST. 58. An irregular piece of lead ore weighs, in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative denſities, or ſpecific gravities?

Anſ. as 145 to 132.

QUEST. 59. An irregular fragment of glaſs, in the ſcale, weighs 171 grains, and another of magnet 102 grains; but in water the firſt fetches up no more than 120 grains, and the other 79: what then will their ſpecific gravities turn out to be? Anſ. glaſs to magnet as 3933 to 5202, or nearly as 10 to 13.

QUEST. 60. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold: The workmen thought that, ſubſtituting part ſilver was only a proper perquiſite; which taking air, Archimedes was appointed to examine it; who, on putting it into a veſſel of water, found it raiſed the fluid 8·2245 cubic inches: and having diſcovered that the inch of gold more critically weighed 10·36 ounces, and that of ſilver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are deſired to repeat the proceſs. Anſ. 28·8 ounces.

QUEST. 61. Suppoſing the cubic inch of common glaſs weigh 1·4921 ounces troy, the ſame of ſea-water ·59542, and of brandy ·5368; then a ſeaman having a gallon of this liquor in a glaſs bottle, which weighs 3·84lb out of water, and, to conceal it from the officers of the cuſtoms, throws it overboard. It is propoſed to determine, if it will ſink, how much force will juſt buoy it up? Anſ. 14·1496 ounces.

QUEST. 62. Another perſon has half an anker of brandy, of the ſame ſpecific gravity as in the laſt queſtion; the wood of the caſk ſuppoſe meaſures $\frac{1}{8}$ of a cubic foot; it is propoſed to aſſign what quantity of lead is juſt requiſite to keep the caſk and liquor under water? Anſ. 89·743 ounces.

QUEST. 63. Suppoſe, by meaſurement, it be found that a man of war, with its ordnance, rigging, and appointments, ſinks ſo deep as to diſplace 50000 cubic feet of freſh water; what is the whole weight of the veſſel? Anſ. $1395\frac{1}{10}$ tons.

QUEST. 64. It is required to determine what would be the height of the atmoſphere, if it were every where of the ſame

ſame denſity as at the ſurface of the earth, when the quick-ſilver in the barometer ſtands at 30 inches; and alſo, what would be the height of a water barometer at the ſame time?

Anſ. height of the air 29.66⅔ feet or 5·5240 miles,
height of water 35 feet.

QUEST. 65. With what velocity would each of thoſe three fluids, viz. quickſilver, water, and air, iſſue through a ſmall orifice in the bottom of veſſels, of the reſpective heights of 30 inches, 35 feet, and 5·5240 miles; eſtimating the preſſure by the half altitudes, and the air ruſhing into a vacuum?

Anſ. the veloc. of quickſilver 8·967 feet.
the veloc. of water - 33·55
the veloc. of air - - 968·6

But, eſtimating by the whole altitude, the veloc.
of air is - - - - - - - 1369·8
And the mean between theſe two is - - 1169·2
which is nearly the velocity of ſound, and alſo nearly equal to the velocity of a ball through the air when it ſuffers a reſiſtance equal to the preſſure of the atmoſphere.

QUEST. 66. A very large veſſel of 10 feet high (no matter what ſhape) being kept conſtantly full of water, by a large ſupplying cock at the top; if 9 ſmall circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular ſide at every foot of the depth; it is required to determine the ſeveral diſtances to which they will ſpout on the horizontal plane of the baſe, and the quantity of water diſcharged by all of them in 10 minutes?

Anſ. the diſtances are

$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000

and the quantity diſcharged in 10 min. 123·8849 gallons.

Note. In this ſolution, the velocity of the water is ſuppoſed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice.

QUEST. 67.

QUEST. 67. If the inner axis of a hollow globe of copper, exhauſted of air, be 100 feet; what thickneſs muſt it be of, that it may juſt float in air?

Anſ. ·02688 of an inch thick.

QUEST. 68. If a ſpherical balloon of copper of $\frac{1}{1000}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air; what weight will juſt balance it, and prevent it from riſing up into the atmoſphere? Anſ. 27981 lb.

QUEST. 69. If a glaſs tube, 36 inches long, cloſe at top, be ſunk perpendicularly into water, till its lower or open end be 30 inches below the ſurface of the water; how high will the water riſe within the tube, the quickſilver in the common barometer at the ſame time ſtanding at $29\frac{1}{2}$ inches?

Anſ. 2·26545 inches.

QUEST. 70. If a diving bell, of the form of a parabolic conoid, be let down into the ſea to the ſeveral depths of 5, 10, 15, and 20 fathoms; it is required to aſſign the reſpective heights to which the water will riſe within it: its axis and the diameter of its baſe being each 8 feet, and the quickſilver in the barometer ſtanding at 30·9 inches?

Anſ. at 5 fathoms deep the water riſes 2·03546 feet,
at 10 - - - - 3·06393
at 15 - - - - 3·70267
at 20 - - - - 4·14658.

THE

THE DOCTRINE OF FLUXIONS.

DEFINITIONS AND PRINCIPLES.

Art. 1. IN the Doctrine of Fluxions, magnitudes or quantities of all kinds, are conſidered, not as made up of a number of ſmall parts, but as generated by continued motion, by means of which they increaſe or decreaſe. As, a line by the motion of a point; a ſurface by the motion of a line; and a ſolid by the motion of a ſurface. So likewiſe, time may be conſidered as repreſented by a line, increaſing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increaſe and decreaſe, may, in like manner, be repreſented by lines, ſurfaces, or ſolids, conceived to be generated by motion.

2. Any quantity, thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increaſes, at any poſition or inſtant, is the Fluxion of the ſaid quantity, at that poſition or inſtant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increaſed, in a given time, with the generating celerity uniformly continued during that time.

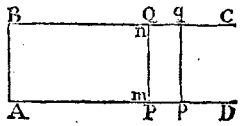
3. The ſmall quantities that are actually generated, produced, or deſcribed, in any ſmall given time, and by any continued motion, either uniform or variable, are called Increments.

4. Hence, if the motion of increaſe be uniform, by which increments are generated, the increments will in that caſe be proportional, or equal, to the meaſures of the fluxions: but if the motion of increaſe be accelerated, the increment ſo generated, in a given finite time, will exceed the fluxion: and if it be a decreaſing motion, the increment, ſo generated, will be leſs than the fluxion. But if the time be indefinitely ſmall, ſo that the motion be conſidered as uniform for that inſtant: then theſe naſcent increments will always be proportional, or equal, to the fluxions, and may be ſubſtituted for them, in any calculation.

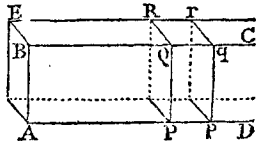
5. To

5. To illuſtrate theſe definitions: Suppoſe a point *m* be conceived to move from the poſition A, and to generate a line AP, by a motion any how regulated; and ſuppoſe the celerity of the point *m*, at any poſition P, to be ſuch, as would, if from thence it ſhould become or continue uniform, be ſufficient to cauſe the point to deſcribe, or paſs uniformly over, the diſtance P*p*, in the given time allowed for the fluxion: then will the ſaid line P*p* repreſent the fluxion of the fluent, or flowing line, AP, at that poſition.

6. Again, ſuppoſe the right line *mn* to move, from the poſition AB, continually parallel to itſelf, with any continued motion, ſo as to generate the fluent or flowing rectangle ABQP, whilſt the point *m* deſcribes the line AP; alſo, let the diſtance P*p* be taken, as above, to expreſs the fluxion of the line or baſe AP; and complete the rectangle PQ*qp*. Then, like as P*p* is the fluxion of the line AP, ſo is P*q* the fluxion of the flowing parallelogram AQ; both theſe fluxions, or increments, being uniformly deſcribed in the ſame time.

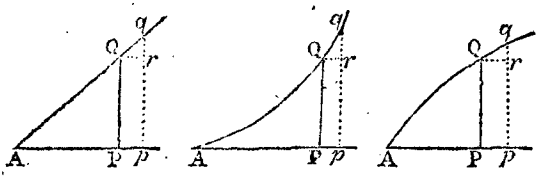


7. In like manner, if the ſolid AERP be conceived to be generated by the plane PQR moving, from the poſition ABE, always parallel to itſelf, along the line AD; and if P*p* denote the fluxion of the line AP: Then, like as the rectangle PQ*qp*, or PQ × P*p*, expreſſes the fluxion of the flowing rectangle ABQP, ſo alſo ſhall the fluxion of the variable ſolid, or priſm ABERQP, be expreſſed by the priſm PQR*rqp*, or the plane PR × P*p*. And, in both theſe laſt two caſes, it appears, that the fluxion of the generated rectangle, or priſm, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.



8. Hitherto the generating line, or plane, has been conſidered as of a conſtant or invariable magnitude; in which caſe the fluent, or quantity generated, is a rectangle, or a priſm, the former being deſcribed by the motion of a line, and the latter by the motion of a plane. So, in like manner, are other figures, whether plane or ſolid, conceived to be

deſcribed, by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried with a parallel motion along AP, or whilſt a point P is carried along, and deſcribes the line AP, ſuppoſe another



point Q to be carried by a motion perpendicular to the former, and to deſcribe the line PQ: let *pq* be another poſition of PQ, indefinitely near to the former; and draw Q*r* parallel to AP. Now, in this caſe, there are ſeveral fluents, or flowing quantities, with their reſpective fluxions: namely, the line or fluent, AP, the fluxion of which is P*p* or Q*r*; the line or fluent PQ, the fluxion of which is *rq*; the curve or oblique line AQ, deſcribed by the oblique motion of the point Q, the fluxion of which is Q*q*; and laſtly, the ſurface APQ, deſcribed by the variable line PQ, the fluxion of which is the rectangle PQ*rp*, or PQ × P*p*. In the ſame manner may any ſolid be conceived to be deſcribed, by the motion of a variable plane parallel to itſelf, ſubſtituting the variable plane for the variable line; in which caſe, the fluxion of the ſolid, at any poſition, is repreſented by the variable plane, at that poſition, drawn into the fluxion of the line along which it is carried.

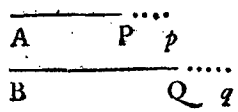
9. Hence then it follows in general, that the fluxion of any figure, whether plane or ſolid, at any poſition, is equal to the ſection of it, at that poſition, drawn into the fluxion of the axis, or line along which the variable ſection is ſuppoſed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ × P*p*, when that figure is a ſolid, or to the ordinate PQ × P*p*, when the figure is a ſurface.

10. It alſo follows, from the ſame premiſes, that, in any curve, or oblique line, AQ, whoſe abſciſs is AP, and ordinate is PQ, the fluxions of theſe three form a ſmall right-angled plane triangle Q*qr*; for Q*r* = P*p* is the fluxion of the abſciſs AP, *qr* the fluxion of the ordinate PQ, and Q*q* the fluxion of the curve or right line AQ. And conſequently that, in any curve, the ſquare of the fluxion of the curve, is

equal

equal to the ſum of the ſquares of the fluxions of the abſciſs and ordinate, when theſe two are at right angles to each other.

11. From the premiſes it alſo appears, that contemporaneous fluents, or quantities that flow or increaſe together, which are always in a conſtant ratio to each other, have their fluxions alſo in the ſame conſtant ratio, at every poſition. For, let AP and BQ be two contemporaneous fluents, deſcribed in the ſame time by the motion of the points P and Q, the contemporaneous poſitions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, conſtantly in the ratio of 1 to n. Then is $n \times AP = BQ$,



and $n \times Ap = Bq$;
therefore, by ſubtraction, $n \times Pp = Qq$;
that is, the fluxion of Pp : fluxion of Qq :: 1 : n,
the ſame as the fluent AP : fluent BQ :: 1 : n;
or, the fluxions and fluents are in the ſame conſtant ratio.

But if the ratio of the fluents be variable, ſo will that of the fluxions be alſo, though not in the ſame variable ratio with the former, at every poſition.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraical quantities, by means of which thoſe of all other kinds are determined, it will be neceſſary firſt to premiſe the notation commonly uſed in this ſcience, with ſome obſervations. As, firſt, that the final letters of the alphabet z, y, x, w, &c, are uſed to denote variable or flowing quantities; and the initial letters a, b, c, d, &c, to denote conſtant or invariable ones: Thus, the variable baſe AP of the flowing rectangular figure ABQP, in art. 6, may be repreſented by x; and the invariable altitude PQ, by a: alſo, the variable baſe or abſciſs AP, of the figures in art. 8, may be repreſented by x; the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the fluxion of a quantity denoted by a ſingle letter, is repreſented by the ſame letter with a point over it: Thus, the fluxion of x is expreſſed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. As to the fluxions of conſtant or invariable quantities, as of a, b, c, &c, they are equal to nothing, becauſe they do not flow or change their magnitude.

Thirdly, that the increments of variable or flowing quantities, are alſo denoted by the ſame letters with a ſmall (′) over them: Thus, the increments of x, y, z, are $\dot{x}$, $\dot{y}$, $\dot{z}$.

13. From theſe notations, and the foregoing principles, the quantities, and their fluxions, there conſidered, will be denoted as below. Thus, in all the foregoing figures put

the variable or flowing line - - AP $= x$,
in art. 6, the conſtant line - - PQ $= a$,
in art. 8, the variable ordinate - PQ $= y$,
alſo, the variable line or curve - AQ $= z$:

Then ſhall the ſeveral fluxions be thus repreſented, namely,

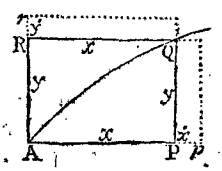
$\dot{x}$ = Pp the fluxion of the line AP,
$a\dot{x}$ = PQqp the fluxion of ABQP in art. 6,
$y\dot{x}$ = PQrp the fluxion of APQ in art. 8,
$\dot{z}$ = Qq $= \sqrt{\dot{x}^2 + \dot{y}^2}$ the fluxion of AQ; and
$a\dot{x}$ = Pr the fluxion of the ſolid in art. 7, if a denote the conſtant generating plane PQR. Alſo,
nx = BQ in the figure to art. 11, and
$n\dot{x}$ = Qq the fluxion of the ſame.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine, which conſiſts of two principal parts, called the Direct and Inverſe Method of Fluxions; namely, the direct method, which conſiſts in finding the fluxion of any propoſed fluent or flowing quantity; and the inverſe method, which conſiſts in finding the fluent of any propoſed fluxion. As to the former of theſe two problems, it can always be determined, and that in finite algebraical terms; but the latter, or finding of fluents, can only be effected in ſome certain caſes, except by means of infinite ſeries.—Firſt then of

The DIRECT METHOD of FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.

15. Let ARQP, $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ moving always perpendicular to each other, from the poſitions AR and AP; denoting the one by x, and the other by y; and ſuppoſe x and y to be ſo related, that the curve line AQ may always paſs through the interſection Q of thoſe lines, or the oppoſite angle of the rectangle.



Now,

Now, the rectangle conſiſts of the two trilinear ſpaces APQ, ARQ, of which the

fluxion of the former is PQ × Pp, or $y\dot{x}$, and

that of the latter is - RQ × Rr, or $x\dot{y}$, by art. 8;

therefore the ſum of the two, $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwiſe.

16. Let the ſides of the rectangle, x and y, by flowing, become $x + \dot{x}$ and $y + \dot{y}$: then the product of theſe two, or $xy + x\dot{y} + y\dot{x} + \dot{x}\dot{y}$ will be the new or contemporaneous value of the flowing rectangle PR or xy; ſubtract the one value from the other, and the remainder, $x\dot{y} + y\dot{x} + \dot{x}\dot{y}$, will be the increment generated in the ſame time as $\dot{x}$ or $\dot{y}$; of which the laſt term $\dot{x}\dot{y}$ is nothing, or indefinitely ſmall, in reſpect of the other two terms, becauſe $\dot{x}$ and $\dot{y}$ are indefinitely ſmall in reſpect of x and y; which term being therefore omitted, there remains $x\dot{y} + y\dot{x}$ for the value of the increment: and hence, by ſubſtituting $\dot{x}$ and $\dot{y}$ for $\dot{x}$ and $\dot{y}$, to which they are proportional, there ariſes $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the ſame as before.

17. Hence may be eaſily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $wxyz$, or $vwxyz$, &c. And, firſt, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the laſt article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and ſo $\dot{p} = \dot{x}y + x\dot{y}$ by the ſame article; ſubſtituting therefore theſe values of p and $\dot{p}$ inſtead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, the fluxion of xyz required; which is therefore equal to the ſum of the products, ariſing from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $wxyz$, the continual product of four variable quantities; put this product, namely $wxyz$ or $qw = r$, where $q = xyz$ as above. Then, taking the fluxions by the laſt article, $\dot{r} = q\dot{w} + \dot{q}w$; which, by ſubſtituting for q and $\dot{q}$ their values as above, becomes - - - $\dot{r} = \dot{w}xyz + w\dot{x}yz + wx\dot{y}z + wxy\dot{z}$, the fluxion of $wxyz$ as required; conſiſting of the fluxion of each quantity, drawn into the products of the other three.

In the very same manner, it is found that the fluxion of $vwxyz$ is $\dot{v}wxyz + v\dot{w}xyz + vw\dot{x}yz + vwx\dot{y}z + vwxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. From hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that
the fluxion of x^4 is $= 4x^3\dot{x}$,
the fluxion of x^5 is $= 5x^4\dot{x}$, and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$; where n is any positive whole number whatever.

That is, the fluxion of any positive integral power, is equal to the fluxion of the root $(\dot{x})$, multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1}).

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of $\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then, multiplying by the denominator, $x = qy$; and, taking the fluxions,
$\dot{x} = q\dot{y} + \dot{q}y$, or $\dot{q}y = \dot{x} - q\dot{y}$; and, by division,
$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} =$ (by substituting the value of q, or $\frac{x}{y}$),
$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}$, the fluxion of $\frac{x}{y}$, as required.

That

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the ſquare of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence, too, is eaſily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the ſame thing. For here the numerator of the fraction is 1, whoſe fluxion is nothing; and therefore, by the laſt article, the fluxion of ſuch a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the ſquare of the ſaid denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$, is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power leſs by 1.

The ſame thing is otherwiſe obtained thus: Put the propoſed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$;

and, taking the fluxions,

$\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$;

and (dividing by x^n),

$\dot{q} = -\frac{qn\dot{x}}{x} =$ (by ſubſtituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; the ſame as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$ or $-\frac{\dot{x}}{x^2}$,

that of x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $-\frac{\dot{x}}{(a+x)^2}$,

and that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x} \times (a+3x^2)^{-3}$,

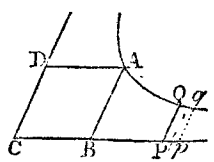
or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each to the n power, $x^m = q^n$;
take the fluxions, so shall $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$;
divide by nq^{n-1}, $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{mx^{m-1}\dot{x}}{nx^{m-\frac{m}{n}}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$: Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$, that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$, and that of $\sqrt{a^2-x^2}$ or $(a^2-x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{a^2-x^2}}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities, it remains to determine those of logarithmic expressions, and also of exponential ones, or such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.

23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c, parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known that any space ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put 1 = CB or BA the side of the square or rhombus DB; m = the modulus, or CB × BA; or area of DB, or sine of the angle C to the radius 1; also the abscis CP = x, and the ordinate PQ = y. Then, by the nature of the hyperbola, CP × PQ is always equal to DB, that is, $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the space, or $\dot{x}y$ is $\frac{m\dot{x}}{x}$ = PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, therefore $\frac{\dot{x}}{x}$ is



$\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itſelf.

Hence the fluxion of the hyp. log.

of $1 + x$ is $\frac{\dot{x}}{1+x}$,

of $1 - x$ is $\frac{-\dot{x}}{1-x}$,

of $x + z$ is $\frac{\dot{x}+\dot{z}}{x+z}$,

of $\frac{a+x}{a-x}$ is $\frac{\dot{x}(a-x)+\dot{x}(a+x)}{(a-x)^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{a^2-x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are uſually determined thoſe of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. Theſe exponentials are of two kinds, namely, when the root is a conſtant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the firſt caſe, put the exponential, whoſe fluxion is to be found, equal to a ſingle variable quantity z, namely, $z = e^x$; then take the logarithm of each, ſo ſhall $\log. z = x \times \log. e$; take the fluxions of theſe, ſo ſhall $\frac{\dot{z}}{z} = \dot{x} \times \log. e$ by the laſt article; hence $\dot{z} = z\dot{x} \times \log. e = e^x\dot{x} \times \log. e$, which is the fluxion of the propoſed quantity e^x or z, and which therefore is equal to the ſaid given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence alſo, the fluxion of $(a + c)^{nx}$ is $(a + c)^{nx} \times n\dot{x} \times \log. (a + c)$.

26. In like manner, in the ſecond caſe, put the given quantity $y^x = z$; then the logarithms give $\log. z = x \times \log. y$, and the fluxions give $\frac{\dot{z}}{z} = \dot{x} \times \log. y + x \times \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x} \times \log. y + \frac{zx\dot{y}}{y}$ = (by ſubſtituting y^x for z) $y^x\dot{x} \times \log. y + xy^{x-1}\dot{y}$, which is the fluxion of the propoſed quantity y^x; and which therefore conſiſts of two terms, of which the

the one is the fluxion of the given quantity conſidering the exponent as conſtant, and the other the fluxion of the ſame quantity conſidering the root as conſtant.

Of SECOND, THIRD, &c, FLUXIONS.

HAVING explained the manner of conſidering and determining the firſt fluxions of flowing or variable quantities; it remains now to conſider thoſe of the higher orders, as ſecond, third, fourth, &c, fluxions.

27. If the rate or celerity with which any flowing quantity changes its magnitude, be conſtant, or the ſame, at every poſition; then is the fluxion of it alſo conſtantly the ſame. But if the variation of magnitude be continually changing, either increaſing or decreaſing; then will there be a certain degree of fluxion peculiar to every point or poſition; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this ſecond fluxion, is called the Third Fluxion of the firſt propoſed fluent quantity; and ſo on.

Theſe orders of fluxions are denoted by the ſame fluent letter, with the correſponding number of points over it; namely, two points for the ſecond fluxion, three points for the third fluxion, four points for the fourth fluxion, and ſo on. So, the different orders of the fluxions of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c; where each is the fluxion of the one next before it.

28. This deſcription of the higher orders of fluxions may be illuſtrated by the figures exhibited in page 274; where, if x denote the abſciſs AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the abſciſs AP or x, with an uniform motion; then the fluxion of x, namely, $\dot{x} = Pp$ or Qr is a conſtant quantity, or $\ddot{x} = 0$, in all the figures. Alſo, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a conſtant quantity, or $\ddot{y} = 0$; for, the angle Q = the angle A, being conſtant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a conſtant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increaſes more and more; and

in

in fig. 3 it continually decreaſes more and more, and therefore in both theſe caſes y has a ſecond fluxion, being poſitive in fig. 2, but negative in fig. 3. And ſo on, for the other orders of fluxions.

Thus, if, for inſtance, the nature of the curve be ſuch, that x^3 is every where equal a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, conſidering $\dot{x}$ always as a conſtant quantity, and taking always the fluxions, the equations of the ſeveral orders of fluxions will be as below, viz.

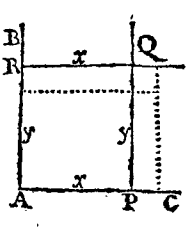
the 1ſt fluxions $a^2\dot{y} = 3x^2\dot{x}$,
the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,
the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,
the 4th fluxions $a^2\ddddot{y} = 0$,

and all the higher fluxions alſo $= 0$, or nothing.

Alſo, the higher orders of fluxions are found in the ſame manner as the lower ones. Thus,

the firſt fluxion of y^3 is - - - - - $3y^2\dot{y}$;
its 2d flux. or the flux. of $3y^2\dot{y}$, conſidered as the rectangle of $3y^2$ and $\dot{y}$, is $- 3y^2\ddot{y} + 6y\dot{y}^2$;
and the flux. of this again, or the 3d flux. of y^3, is - - - - - - $- 3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

29. In the foregoing articles, it has been ſuppoſed that the fluents increaſe, or that their fluxions are poſitive; but it often happens that ſome fluents decreaſe, and that therefore their fluxions are negative: and whenever this is the caſe, the ſign of the fluxion muſt be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increaſe together, the fluxion is $\dot{x}y + x\dot{y}$; but if one of them, as y, decreaſe, while the other, x, increaſes; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that caſe be $\dot{x}y - x\dot{y}$. This may be illuſtrated by the annexed rectangle $APQR = xy$, ſuppoſed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increaſed, and its fluxion is $+ \dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreaſed, and the fluxion of the decreaſe is $x\dot{y}$; therefore, taking



taking the fluxion of the decreaſe from that of the increaſe, the fluxion of the rectangle xy, when x increaſes and y decreaſes, is $\dot{x}y - x\dot{y}$.

30. We may now collect all the rules together, demonſtrated in the foregoing articles, for finding the fluxions of all ſorts of quantities. And,

1ſt, *For the fluxion of Any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next leſs by 1 of the ſame root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper ſigns.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper ſigns.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, ſubtract the fluxion of the denominator drawn into the numerator, and divide the reſult by the ſquare of the denominator.

5th, *Or, the 2d, 3d, and 4th caſes may be all included under one, and performed thus.*—Take the fluxion of the given expreſſion as often as there are variable quantities in it, ſuppoſing firſt only one of them variable, and the reſt conſtant; then another variable, and the reſt conſtant; and ſo on, till they have all in their turns been ſingly ſuppoſed variable: and connect all theſe fluxions together with their own ſigns.

6th, *For the fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itſelf, and multiply the reſult by the modulus of the ſyſtem of logarithms.

Note, The modulus of the hyp. logs. is 1, ,
and the modulus of the com. logs. is - 0·43429448.

7th, *For the fluxion of an Exponential quantity, having the root Conſtant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity having the root Variable.*—To the fluxion of the given quantity, found by the 1ſt rule, as if the root only were variable, add the fluxion of the ſame quantity, found by the 7th rule, as if the exponent only were variable; and the ſum will be the fluxion for both of them variable.

Note, When the given quantity conſiſts of ſeveral terms, find the fluxion of each term ſeparately, and connect them all together with their proper ſigns.

31. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4y$ is
10. The fluxion of $4x^{\frac{2}{3}}y^4$ is
11. The fluxion of $ax^2y - x^{\frac{1}{2}}y^3$ is
12. The fluxion of $4x^4 - x^3y + 3byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is
14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is
15. The fluxion of $\frac{1}{\sqrt[n]{x^m}}$ or $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is
16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is
17. The fluxion of $\sqrt[3]{}$ or $x^{\frac{1}{3}}$ is
18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is
19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is
20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is
21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is
22. The fluxion of $\sqrt{a^2 + x^2}$ or $\overline{a^2 + x^2}^{\frac{1}{2}}$ is
23. The fluxion of $\sqrt{a^2 - x^2}$ or $\overline{a^2 - x^2}^{\frac{1}{2}}$ is
24. The fluxion of $\sqrt{2rx - xx}$ or $\overline{2rx - xx}^{\frac{1}{2}}$ is
25. The fluxion of $\frac{1}{\sqrt{a^2 - x^2}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is
26. The fluxion of $\overline{ax - xx}^{\frac{1}{3}}$ is

27. The

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $\overline{a^2 - x^2}^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$ or $\overline{xz}^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $\overline{xz - zz}^{\frac{1}{2}}$ is

31. The fluxion of $-\frac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{-\frac{1}{2}}$ is

32. The fluxion of $\frac{ax3}{a+x}$ is

33. The fluxion of $\frac{x^m}{y^n}$ is

34. The fluxion of $\frac{xy}{z}$ is

35. The fluxion of $\frac{c}{xx}$ is

36. The fluxion of $\frac{3x}{a-x}$ is

37. The fluxion of $\frac{z}{x+z}$ is

38. The fluxion of $\frac{x^2}{z^2}$ is

39. The fluxion of $\frac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}}$ is

40. The fluxion of $\frac{axy^2}{z}$ is.

41. The fluxion of $\frac{3}{\sqrt{x^2 - y^2}}$ is

42. The fluxion of the hyp. log. of ax is

43. The fluxion of the hyp. log. of $1 + x$ is

44. The fluxion of the hyp. log. of $1 - x$ is

45. The fluxion of the hyp. log. of x^2 is

46. The fluxion of the hyp. log. of $\sqrt{z}$ is

47. The fluxion of the hyp. log. of x^m is

48. The

48. The fluxion of the hyp. log. of $\frac{2}{x^2}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is

52. The fluxion of 10^x is

53. The fluxion of $\overline{a+c}^x$ is

54. The fluxion of 100^{xy} is

55. The fluxion of x^x is

56. The fluxion of y^{10^x} is

57. The fluxion of x^x is

58. The fluxion of $\overline{xy}^{xz}$ is

59. The fluxion of $x\dot{y}$ is

60. The fluxion of $\dot{x}\dot{y}^2$ is

61. The second fluxion of xy is

62. The second fluxion of xy, when $\dot{x}$ is constant, is

63. The second fluxion of x^n is

64. The third fluxion of x^n, when $\dot{x}$ is constant, is

65. The third fluxion of xy is

The INVERSE METHOD, or the FINDING of FLUENTS.

32. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

33. It may farther be observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

34. It

34. It is eaſy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberleſs caſes in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite ſeries. For, it is only in certain particular forms and caſes that the fluents of given fluxions can be found; there being no method of performing this univerſally, *à priori*, by a direct inveſtigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for ſuch forms of fluxions as we know, from the direct method, belong to ſuch and ſuch kinds of flowing quantities: and theſe rules, it is evident, muſt chiefly conſiſt in performing ſuch operations as are the reverſe of thoſe by which the fluxions are found of given fluent quantities. The principal caſes of which are as follow:

35. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely ſubſtituting the variable or flowing quantity inſtead of its fluxion; being the reſult or reverſe of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.
The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.
The fluent of $\sqrt{a^2 + x^2}$ is $\sqrt{a^2 + x^2}$.

36. *When any power of a flowing quantity is multiplied by the fluxion of the root.*

Then, having ſubſtituted, as before, the flowing quantity for its fluxion, divide the reſult by the new index of the power. Or, which is the ſame thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index ſo increaſed. Which is the reverſe of the 1ſt rule for finding fluxions.

So, if the fluxion propoſed be - $3x^5\dot{x}$.
Leave out, or divide by, $\dot{x}$, then it is $3x^5$;
add 1 to the index, and it is - $3x^6$;
divide by the index 6, and it is - $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the propoſed fluxion $3x^5\dot{x}$.

In like manner,
The fluent of $2ax\dot{x}$ is
The fluent of $3x^2\dot{x}$ is

The

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is
The fluent of $2y^{\frac{3}{2}}\dot{y}$ is
The fluent of $az^{\frac{3}{2}}\dot{z}$ is
The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is
The fluent of $x^{n-1}\dot{x}$ is
The fluent of $ny^{n-1}\dot{y}$ is
The fluent of $\frac{\dot{z}}{z^2}$ or $z^{-2}\dot{z}$ is
The fluent of $\frac{a\dot{y}}{y^n}$ is
The fluent of $(a + x)^4\dot{x}$ is
The fluent of $(a^4 + y^4)y^3\dot{y}$ is
The fluent of $(a^3 + z^3)^4z^2\dot{z}$ is
The fluent of $(a^n + x^n)^m x^{n-1}\dot{x}$ is
The fluent of $(a^2 + y^2)^3 y\dot{y}$ is
The fluent of $\frac{z\dot{z}}{\sqrt{a^2 + z^2}}$ is
The fluent of $\frac{\dot{x}}{\sqrt{a - x}}$ is

37. *When the root under a vinculum is a compound quantity; and the index of the part or factor without the vinculum, increased by* 1, *is some multiple of that under the vinculum.*

Put a single variable letter for the compound root; and substitute its powers and fluxion instead of those, of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot{F} = (a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum: therefore, putting $z = a^2 + x^2$, hence $x^2 = z - a^2$, the fluxion of which is $2x\dot{x} = \dot{z}$; hence then $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z - a^2)$, and the given fluxion $\dot{F}$, or $(a^2+x^2)^{\frac{2}{3}}x^3\dot{x}$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z - a^2)$ or $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$; and hence the fluent F is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{1}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2 + x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16} \cdot \sqrt[3]{a^2+x^2}^5 \cdot x^2 - \frac{3}{5}a^2$.

In like manner,

To find the fluent of $\sqrt{a+cx} \times x^3\dot{x}$.

To find the fluent of $\overline{a+cx}^{\frac{3}{4}} x^2\dot{x}$.

To find the fluent of $(a+cx^2)^{\frac{1}{3}} \times dx^3\dot{x}$.

To find the fluent of $\frac{cz\dot{z}}{\sqrt{a+z}}$ or $(a+z)^{-\frac{1}{2}}cz\dot{z}$.

To find the fluent of $\frac{cz^{3n-1}\dot{z}}{\sqrt{a+z^n}}$ or $(a+z^n)^{-\frac{1}{2}}cz^{3n-1}\dot{z}$.

To find the fluent of $\frac{\dot{z}\sqrt{a^2+z^2}}{z^6}$ or $(a^2+z^2)^{\frac{1}{2}}z^{-6}\dot{z}$.

To find the fluent of $\frac{\dot{x}\sqrt{a-x^n}}{x^{\frac{7}{2}n-1}}$ or $(a-x^n)^{\frac{1}{2}}x^{-\frac{7}{2}n-1}\dot{x}$.

38. *When there are ſeveral terms, involving two or more variable quantities, having the fluxion of each multiplied by the other quantity or quantities:*

Take the fluent of each term, as if there was only one variable quantity in it, namely, that whoſe fluxion is contained in it, ſuppoſing all the others to be conſtant in that term; then, if the fluents of all the terms, ſo found, be the very ſame quantity, that quantity will be the fluent of the whole. Which is the reverſe of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, ſuppoſing y conſtant; and the fluent of $x\dot{y}$ is alſo xy, ſuppoſing x conſtant; therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is

The fluent of $2xy\dot{x} + x^2\dot{y}$ is

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$ is

The fluent of $\frac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\frac{2ax\dot{x}}{\sqrt{y}} - \frac{ax^2\dot{y}}{2y\sqrt{y}}$ is

39. *When the given fluxional expreſſion is in this form* $\frac{\dot{x}y - x\dot{y}}{y^2}$, *namely, a fraction including two quantities, being the fluxion of the former of them drawn into the latter, minus the fluxion of the latter drawn into the former, and divided by the ſquare of the latter.*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity divided by the latter. That is,

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this caſe may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which,

the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ when y is conſtant; and

the fluent of $-x\dot{y}y^{-2}$ is alſo xy^{-1} or $\frac{x}{y}$ when x is conſtant;

theref. by that caſe, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

40. *When the Fluxion of a quantity is divided by the quantity itſelf.*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the ſame thing, the fluent is equal to 2·302585 09 multiplied by the common log. of the ſame quantity.

So, the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$ is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is 2 × the hyp. log. of x, or equal to the hyp. log. of x^2.

The fluent of $\frac{a\dot{x}}{x}$ is a × hyp. log. x or = hyp. log. x^a.

The fluent of $\frac{\dot{x}}{a+x}$ is

The fluent of $\frac{3x^2\dot{x}}{a+x^3}$ is

41. *Many fluents may be found by the direct method thus:*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent ſought; into which ſubſtitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c; that is, make x, $\dot{x}$, $\ddot{x}$, as alſo, y, $\dot{y}$, $\ddot{y}$, &c, to be in continual proportion, or ſo that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c; then divide the ſquare of the given fluxional expreſſion by the ſecond fluxion, juſt found, and the quotient will be the fluent required in many caſes.

Or the ſame rule may be otherwiſe delivered thus:

In the given fluxion $\dot{F}$, write x for $\dot{x}$, y for $\dot{y}$, &c, and call the reſult G, taking alſo the fluxion of this quantity, $\dot{G}$; then make $\dot{G} : \dot{F} :: G : F$; ſo ſhall the fourth proportional F be the fluent ſought in many caſes.

It may be proved if this be the true fluent, by taking the fluxion of it again, which, if it agree with the propoſed fluxion, will ſhew that the fluent is right; otherwiſe, it is wrong.

EXAMPLES.

EXAM. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{F} = nx^{n-1}\dot{x}$, write x for $\dot{x}$, and it is $nx^{n-1}x$ or - - $nx^n = G$;

the fluxion of this is $\dot{G} = n^2x^{n-1}\dot{x}$; therefore $\dot{G} : \dot{F} :: G : F$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = F$, the fluent ſought.

EXAM. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{F} = \dot{x}y + x\dot{y}$; then, writing x for $\dot{x}$, and y for $\dot{y}$, it is $xy + xy$ or $2xy = G$; hence $\dot{G} = 2\dot{x}y + 2x\dot{y}$; then $\dot{G} : \dot{F} :: G : F$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$, the fluent ſought.

42. To

42. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the moſt uſual forms of fluxions that occur in the practical ſolution of problems, with their correſponding fluents ſet oppoſite to them; by means of which, namely, by comparing any propoſed fluxion with the correſponding form in the table, the fluent of it will be found.

Forms.	*Fluxions.*	*Fluents.*
I	$x^{n-1}\dot{x}$	$\frac{x^n}{n}$, or $\frac{1}{n}x^n$
II	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm\frac{1}{mn}(a\pm x^n)^m$
III	$\frac{x^{mn-1}\dot{x}}{(a\pm x^n)^{m+1}}$	$\frac{1}{mna}\times\frac{x^{mn}}{(a\pm x^n)^m}$
IV	$\frac{(a\pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna}\times\frac{(a\pm x^n)^m}{x^{mn}}$
V	$(my\dot{x}+nx\dot{y})\times x^{m-1}y^{n-1}$, or $(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y})\,x^m y^n$	$x^m y^n$
VI	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$, or $(m\dot{x}yz+nx\dot{y}z+rxy\dot{z})\,x^{m-1}y^{n-1}z^{r-1}$, or $(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y}+\frac{r\dot{z}}{z})\,x^m y^n z^r$,	$x^m y^n z^r$
VII	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x
VIII	$\frac{x^{n-1}\dot{x}}{a\pm x^n}$	$\pm\frac{1}{n}$ log. of $a\pm x^n$
IX	$\frac{x^{-1}\dot{x}}{a\pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a\pm x^n}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a-x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a}+\sqrt{x^n}}{\sqrt{a}-\sqrt{x^n}}$
XI	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$	$\frac{2}{n\sqrt{a}}\times$ arc to tan. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}}\times$ arc to cofine $\frac{a-x^n}{a+x^n}$
XII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{\pm a+x^n}}$	$\frac{2}{n}$ log. of $\sqrt{x^n}+\sqrt{\pm a+x^n}$

Forms.	*Fluxions.*	*Fluents.*
XIII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a-x^n}}$	$\frac{2}{n}\times$ arc to fin. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n}\times$ arc to verf. $\frac{2x^n}{a}$
XIV	$\frac{x^{-1}\dot{x}}{\sqrt{a\pm x^n}}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\pm\sqrt{a\pm x^n}\mp\sqrt{a}}{\sqrt{a\pm x^n}+\sqrt{a}}$
XV	$\frac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\frac{2}{n\sqrt{a}}\times$ arc to fecant $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}}\times$ arc to cofin. $\frac{2a-x^n}{x^n}$
XVI	$\dot{x}\sqrt{dx-x^2}$	$\frac{1}{2}$ circ. feg. to diam. d & verf. x
XVII	$c^{nx}\dot{x}$	$\frac{c^{nx}}{n\log. c}$
XVIII	$\dot{x}y^x\log. y+xy^{x-1}\dot{y}$	y^x

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whofe fine, or tangent, &c, are mentioned, have the radius 1, and are thofe in the common tables of fines, tangents, and fecants. Alfo, the numbers m, n, &c, are to be fome real quantities, as the forms fail when $m = 0$, or $n = 0$, &c.

The Ufe of the foregoing Table of Forms of Fluxions and Fluents.

43. In ufing the foregoing table, it is to be obferved, that the firft column ferves only to fhew the number of the form; in the fecond column are the feveral forms of fluxions, which are of different kinds or claffes; and in the third or laft column, are the correfponding fluents.

The method of ufing the table, is this. Having any fluxion given, to find its fluent: Firft, Compare the given fluxion with the feveral forms of fluxions in the fecond column of the table, till one of the forms be found that agrees

with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correſpond, will give the particular values of the general quantities in the tabular form: then ſubſtitute theſe particular values in the general or tabular form of the fluent, and the reſult will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the propoſed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the firſt form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being ſubſtituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent ſought.

EXAM. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3 - x^3}$, or $5x^2\dot{x}\,(c^3 - x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, alſo $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent $n-1$ of x^{n-1} without the vinculum, by uſing 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: ſo that all the parts of the form are found to correſpond. Then, ſubſtituting theſe values into the general fluent, $-\frac{1}{mn}(a - x^n)^m$, it becomes $-\frac{1}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

EXAM. 3. To find the fluent of $\frac{x^2\dot{x}}{1 + x^3}$.

This is found to agree with the 8th form: where $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; alſo $a = 1$. Hence then, by ſubſtituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a + x^n$, it becomes $\frac{1}{3}$ log. of $1 + x^3$.

EXAM. 4. To find the fluent of $ax^4\dot{x}$.

EXAM. 5. To find the fluent of $2\,(10 + x^2)^{\frac{2}{3}}x\dot{x}$.

EXAM. 6. To find the fluent of $\frac{a\dot{x}}{\sqrt{c^2 + x^2}^3}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{\sqrt{a - x}^4}$.

EXAM. 8.

EXAM. 8. To find the fluent of $\frac{c^2-x^2}{x^5}\dot{x}$.

EXAM. 9. To find the fluent of $\frac{1+3x}{2x^4}\dot{x}$.

EXAM. 10. To find the fluent of $(\frac{3\dot{x}}{x}+\frac{2\dot{y}}{y})x^3y^2$.

EXAM. 11. To find the fluent of $(\frac{\dot{x}}{x}+\frac{\dot{y}}{3y})xy^{\frac{1}{3}}$.

EXAM. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

EXAM. 13. To find the fluent of $\frac{a\dot{x}}{3-2x}$.

EXAM. 14. To find the fluent of $\frac{3\dot{x}}{2x-x^2}$ or $\frac{3x^{-1}\dot{x}}{2-x}$.

EXAM. 15. To find the fluent of $\frac{2\dot{x}}{x-3x^3}$ or $\frac{2x^{-1}\dot{x}}{1-3x^2}$.

EXAM. 16. To find the fluent of $\frac{3x\dot{x}}{1-x^4}$.

EXAM. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2-x^5}$.

EXAM. 18. To find the fluent of $\frac{2x\dot{x}}{1+x^4}$.

EXAM. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2+x^5}$.

EXAM. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1+x^4}}$.

EXAM. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2-4}}$.

EXAM. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1-x^4}}$.

EXAM. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4-x^2}}$.

EXAM. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1-x^2}}$.

EXAM. 25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2+x^{\frac{1}{2}}}}$.

EXAM. 26. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{x^2-1}}$.

EXAM. 27.

EXAM. 27. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}}-ax^2}}$.

EXAM. 28. To find the fluent of $2\dot{x}\sqrt{2x-x^2}$.

EXAM. 29. To find the fluent of $a^x\dot{x}$.

EXAM. 30. To find the fluent of $3a^{2x}\dot{x}$.

EXAM. 31. To find the fluent of $3z^x\dot{x}\log. z + 3xz^{x-1}\dot{z}$.

EXAM. 32. To find the fluent of $(1+x^3)x\dot{x}$.

EXAM. 33. To find the fluent of $(2+x^4)x^{\frac{3}{2}}\dot{x}$.

EXAM. 34. To find the fluent of $x^2\dot{x}\sqrt{a^2+x^2}$.

To find Fluents by Infinite Series.

44. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse is then to be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra; viz. either by division, or extraction of roots, or by the binomial theorem, &c; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after multiplied by any constant factor or co-efficient which is contained in the given fluxional expression.

45. It is to be noted, however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging; and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend, or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quantity will decrease always more and more in the succeeding terms, or increase in the denominators of them.

For

For example, to find the fluent of $\frac{1-x}{1+x-x^2}$.

Here, by dividing the numerator by the denominator, the propoſed fluxion becomes $\dot{x}-2x\dot{x}+3x^2\dot{x}-5x^3\dot{x}+8x^4\dot{x}-$&c; then the fluents of all the terms, being taken, give - - - $x-x^2+x^3-\frac{5}{4}x^4+\frac{8}{5}x^5-$ &c, for the fluent ſought.

Again, to find the fluent of $\dot{x}\sqrt{1-x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1-x^2}$, the given fluxion becomes - - - - $\dot{x}-\frac{1}{2}x^2\dot{x}-\frac{1}{8}x^4\dot{x}-\frac{1}{16}x^6\dot{x}-$ &c. Then, the fluents of all the terms, being taken, give $x-\frac{1}{6}x^3-\frac{1}{40}x^5-\frac{1}{112}x^7-$ &c, for the fluent ſought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a-x}$ both in an aſcending and deſcending ſeries.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a+x}$ in both ſeries.

EXAM. 3. To find the fluent of $\frac{3\dot{x}}{(a+x)^2}$.

EXAM. 4. To find the fluent of $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$.

EXAM. 5. Given $\dot{z}=\frac{b\dot{x}}{a^2+x^2}$, to find z.

EXAM. 6. Given $\dot{z}=\frac{a^2+x^2}{a+x}\dot{x}$, to find z.

EXAM. 7. Given $\dot{z}=3\dot{x}\sqrt{a+x}$, to find z.

EXAM. 8. Given $\dot{z}=2\dot{x}\sqrt{a^2+x^2}$, to find z.

EXAM. 9. Given $\dot{z}=4\dot{x}\sqrt{a^2-x^2}$, to find z.

EXAM. 10. Given $\dot{z}=\frac{5a\dot{x}}{\sqrt{x^2-a^2}}$, to find z.

EXAM. 11. Given $\dot{z}=2\dot{x}\sqrt[3]{a^3-x^3}$, to find z.

EXAM. 12. Given $\dot{z}=\frac{3a\dot{x}}{\sqrt{ax-xx}}$, to find z.

EXAM. 13. Given $\dot{z}=2\dot{x}\sqrt[3]{x^3+x^4+x^5}$, to find z.

EXAM. 14. Given $\dot{z}=5\dot{x}\sqrt{ax-xx}$, to find z.

To Correct the Fluent of any Given Fluxion.

46. The fluxion found from a given fluent, is always perfect and complete: but the fluent found from a given fluxion, is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c: for, the fluent of any given fluxion, as $\dot{x}$, may be either x, which is found by the rule, or it may be $x \pm c$, that is, x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion, $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, already found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

47. EXAM. 1. To find the correct fluent of $\dot{z} = a\dot{x}^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $0 = c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and from hence we find $c = -ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or,

Or, if it be known that z is = ſome quantity d, when x is = ſome other quantity as b; then ſubſtitute d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; conſequently, writing this value for c in the general equation, it becomes - - - $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this caſe.

48. And hence ariſes another eaſy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or poſition; then ſubtract the ſides of the reſulting particular equation from the correſponding ſides of the general one, and the remainders will give the correct equation of the fluents ſought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by ſubtraction, - $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents, as before.

EXAM. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being = 0 when x is = a.

EXAM. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being = 0 at the ſame time.

EXAM. 4. To find the correct fluents of $\dot{z} = \frac{3a\dot{x}}{a + x}$; ſuppoſing z and x to begin to flow together, or to be each = 0 at the ſame time.

EXAM. 5. To find the correct fluents of $\dot{z} = \frac{2\dot{x}}{a^2 + x^2}$; ſuppoſing z and x to begin together.

OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

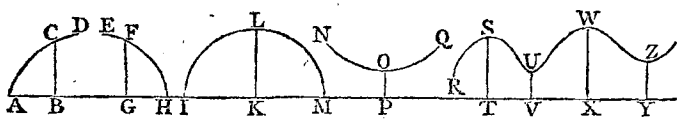
49. MAXIMUM, denotes the greateſt ſtate or quantity attainable in any given caſe, or the greateſt value of a variable quantity: by which it ſtands oppoſed to Minimum, which is the leaſt poſſible quantity in any caſe.

As

As the expreſſion or ſum $a^2 + bx$, evidently increaſes as x, or the term bx, increaſes; therefore the given expreſſion will be the greateſt, or a maximum, when x is the greateſt, or infinite; and the ſame expreſſion will be a minimum, or the leaſt, when x is the leaſt, or nothing.

Again, in the algebraic expreſſion $a^2 - bx$, where a and b denote conſtant or invariable quantities, and x a flowing or variable one. Now, it is evident that the value of this remainder or difference, $a^2 - bx$, will increaſe, as the term bx, or as x, decreaſes; therefore the former will be the greateſt, when the latter is the ſmalleſt; that is, $a^2 - bx$ is a maximum, when x is the leaſt, or nothing at all; and the difference is the leaſt, when x is the greateſt.

50. Some variable quantities increaſe continually; and ſo have no maximum, but what is infinite. Others again decreaſe continually; and ſo have no minimum, but what is of no magnitude, or nothing. But, on the other hand, ſome variable quantities increaſe only to a certain finite magnitude, called their Maximum, or greateſt ſtate, and after that they decreaſe again. While others decreaſe to a certain finite magnitude, called their Minimum, or leaſt ſtate, and afterwards increaſe again. And laſtly, ſome quantities have ſeveral maxima and minima.



Thus, for example, the ordinate BC of the parabola, or ſuch-like curve, flowing along the axis AB from the vertex A, continually increaſes, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreaſes to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increaſes to a certain magnitude, namely the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreaſes again to nothing, at the point M. And in the curve NOQ, the ordinate decreaſes only to the poſition OP, where it is leaſt, or a minimum; and after that it continually increaſes towards Q. But in the curve RSU, &c, the ordinates have ſeveral maxima, as ST, WX, and ſeveral minima, as VU, YZ, &c.

51. Now,

51. Now, becauſe the fluxion of a variable quantity, is the rate of its increaſe or decreaſe; and becauſe the maximum or minimum of a quantity neither increaſes nor decreaſes, at thoſe points or ſtates; therefore ſuch maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

52. From the nature of the queſtion or problem, find an algebraical expreſſion for the value, or general ſtate, of the quantity whoſe maximum or minimum is required; then take the fluxion of that expreſſion, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum ſtate of the compound expreſſion $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expreſſion is $100\dot{x} - 10x\dot{x} = 0$; which being made $= 0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expreſſion $100x - 5x^2 \pm c$ is the greateſt. As is eaſily tried: for if 10 be ſubſtituted for x in that expreſſion, it becomes $\pm c + 500$: but if, for x, there be ſubſtituted any other number, whether greater or leſs than 10; that expreſſion, for the maximum, will always be found to be leſs than $\pm c + 500$, which is its greateſt poſſible value.

53. It is evident that, if a maximum or minimum be any way compounded with, or operated upon, by a given conſtant quantity, the reſult will ſtill be a maximum or minimum. That is, if a maximum or minimum be increaſed, or decreaſed, or multiplied, or divided, by a given quantity, or any given power or root of it be taken; the reſult will ſtill be a maximum or minimum. Thus, if x be a maximum or minimum, then alſo is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or $\dot{x}^n$, or $\sqrt[n]{x}$, ſtill a maximum or minimum. Alſo, the logarithm of the ſame will be a maximum or a minimum. And therefore, if any propoſed maximum or minimum can be made ſimpler by performing any of theſe operations, it is better to do ſo, before the expreſſion is put into fluxions.

54. When

54. When the expreſſion for a maximum or minimum contains ſeveral variable letters or quantities; take the fluxion of it as often as there are variable letters; ſuppoſing firſt one of them only to flow, and the reſt to be conſtant; then another only to flow, and the reſt conſtant; and ſo on for all of them: then putting each of theſe fluxions = o, there will be as many equations as unknown letters, from which theſe may be all determined. For the fluxion of the expreſſion muſt be equal to nothing in each of theſe caſes; otherwiſe the expreſſion might become greater or leſs, without altering the values of the other letters, which are conſidered as conſtant.

So, if it be required to find the values of x and y when $4x^2 - xy + 2y$ is a minimum. Then we have,

Firſt, $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.

Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.

And hence y or $8x = 16$.

55. *To find whether a propoſed quantity admits of a Maximum or a Minimum.*

Every algebraic expreſſion does not admit of a maximum or minimum, properly ſo called; for it may either increaſe continually to infinity, or decreaſe continually to nothing; and, in both theſe caſes, there is neither a proper maximum nor minimum: for the true maximum is that finite value to which an expreſſion increaſes, after which it decreaſes again; and the minimum is that finite value to which the expreſſion decreaſes, and after that it increaſes again. Therefore, when the expreſſion admits of a maximum, its fluxion is poſitive before that point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and poſitive after it. Hence, taking the fluxion of the expreſſion a little before the fluxion is equal to nothing, and again a little after the ſame; if the former fluxion be poſitive, and the latter negative, the middle ſtate is a maximum; but if the former fluxion be negative, and the latter poſitive, the middle ſtate is a minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that ſtate. Now, if in the fluxion $(a - 2x)\,\dot{x}$, the value of x be taken rather leſs than its true value, $\frac{1}{2}a$, that fluxion will evidently be poſitive: but if x be taken ſomewhat greater than $\frac{1}{2}a$, the value of $a - 2x$, and conſequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being poſitive before, and negative

gative after the ſtate when its fluxion is = 0, it follows that at this ſtate the expreſſion is not a minimum, but a maximum.

Again, taking the expreſſion $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expreſſion be a maximum or a minimum at that time, take x a little leſs than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)x\dot{x}$, and this will evidently be negative; and again, take x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently poſitive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is = 0, and poſitive after it, it follows that in this ſtate the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

56. Some Examples for Practice.

Exam. 1. To divide a line, or any other given quantity a, into two parts, ſo that their rectangle or product may be the greateſt poſſible.

Exam. 2. To divide the given quantity a into two parts ſuch, that the product of the m power of one, by the n power of the other, may be a maximum.

Exam. 3. To divide the given quantity a into three parts ſuch, that the continual product of them all may be a maximum.

Exam. 4. To divide the given quantity a into three parts ſuch, that the continual product of the 1ſt, the ſquare of the 2d, and the cube of the 3d, may be a maximum.

Exam. 5. To determine a fraction ſuch, that the difference between its m power and n power ſhall be the greateſt poſſible.

Exam. 6. To divide the number 80 into two ſuch parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

Exam. 7. To find the greateſt rectangle that can be inſcribed in a given right-angled triangle.

Exam. 8. To find the greateſt rectangle that can be inſcribed in the quadrant of a given circle.

Exam. 9. To find the leaſt right-angled triangle that can circumſcribe the quadrant of a given circle.

Exam. 10. To find the greateſt rectangle inſcribed in, and the leaſt iſoſceles triangle circumſcribed about, a given ſemi-ellipſe.

EXAM. 11. To determine the ſame for a given parabola.

EXAM. 12. To determine the ſame for a given hyperbola.

EXAM. 13. To inſcribe the greateſt cylinder in a given cone; or to cut the greateſt cylinder out of a given cone.

EXAM. 14. To determine the dimenſions of a rectangular ciſtern, capable of containing a given quantity a of water, ſo as to be lined with lead at the leaſt poſſible expence.

EXAM. 15. Required the dimenſions of a cylindrical tankard, to hold one quart of ale meaſure, that can be made of the leaſt poſſible quantity of ſilver, of a given thickneſs.

EXAM. 16. To cut the greateſt parabola from a given cone.

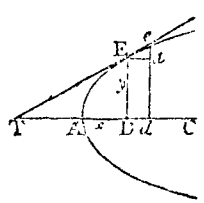
EXAM. 17. To cut the greateſt ellipſe from a given cone.

EXAM. 18. To find the value of x when x^x is a minimum.

THE METHOD OF TANGENTS; OR, TO DRAW TANGENTS TO CURVES.

THE Method of Tangents, is a method of determining the quantity of the tangent and ſubtangent of any algebraic curve; the equation of the curve being given. Or, *vice verſa*, the nature of the curve, from the tangent given.

57. If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the ſubtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent ſought.



58. Let *dae* be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced, in e; and let E*a* be parallel to the axis AD. Then is the elementary triangle E*ea* ſimilar to the triangle TDE;

and

and therefore - $ea : aE :: ED : DT$.
But - - $ea : aE :: \text{flux. } ED : \text{flux. } AD$.
Therefore - $\text{flux. } ED : \text{flux. } AD :: DE : DT$.
That is, - $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = DT$,

which is therefore the value of the ſubtangent ſought; where x is the abſciſs AD, and y the ordinate DE.

Hence we have this general rule.

GENERAL RULE.

59. By means of the given equation of the curve, put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$, or of $\frac{\dot{x}}{\dot{y}}$; which value ſubſtitute for it in the expreſſion $DT = \frac{y\dot{x}}{\dot{y}}$, and, when reduced to its ſimpleſt terms, it will be the value of the ſubtangent ſought.

EXAMPLES.

EXAM. 1. Let the propoſed curve be that which is defined, or expreſſed, by the equation $ax^2 + xy^2 - y^3 = 0$.

Here the fluxion of the equation of the curve is $2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by tranſpoſition, $2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$; and hence, by diviſion, $\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2}$; conſeq. $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2}$, which is the value of the ſubtangent TD ſought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the abſciſs, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $ax = y^2$; where a denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipſe; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

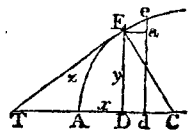
EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x^2) = a^2y^2$; where a and c are the two axes.

EXAM. 6. To draw a tangent to the hyperbola referred to the aſymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the abſciſs and ordinate anſwering to the vertex of the curve.

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

RECTIFICATION, is the finding the length of a curve line, or to find a right line equal to a propoſed curve.

60. By art. 10 it appears, that the elementary triangle E*ae*, formed by the increments of the abſciſs, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuſe; and therefore the ſquare of the latter is equal to the ſum of the ſquares of the two former; that is, $Ee^2 = Ea^2 + ae^2$. Or, ſubſtituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its abſciſs AD, and y its ordinate DE. Hence this rule.



RULE.

61. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value ſubſtitute inſtead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the abſciſs or ordinate.

EXAMPLES.

EXAM. 1. To find the length of the arc of a circle, in terms both of the ſine, verſed ſine, tangent, and ſecant.

The equation of the circle may be expreſſed in terms of the radius, and either the ſine, or the verſed ſine, or tangent, or ſecant, &c, of an arc. Let, therefore, the radius of the circle be CA or CE $= r$, the verſed ſine AD (of the arc AE) $= x$, the right ſine DE $= y$, the tangent TE $= t$, and the ſecant CT $= s$; then, by the nature of the circle, there ariſe theſe equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2+t^2} = \frac{s^2 - r^2}{s^2}r^2.$$

Then, by means of the fluxions of theſe equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itſelf; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - x^2}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}}.$$

Hence

Hence the value of the curve, from the fluent of each of theſe, gives the four following forms, in ſeries, viz. putting $d = 2r$ the diameter, the curve is z

$$= (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c)\sqrt{dx},$$

$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c)\, y,$$

$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c)\, t,$$

$$= (\frac{s - r}{s} + \frac{s3 - r3}{2.3s^3} + \frac{3(s5 - r^5)}{2.4.5s^5} + \&c)\, r.$$

Now, it is evident that the ſimpleſt of theſe ſeries, is the third in order, or that which is expreſſed in terms of the tangent. That form will therefore be the fitteſt to calculate an example by in numbers. And, for this purpoſe, it will be convenient to aſſume ſome arc whoſe tangent, or at leaſt the ſquare of it, is known to be ſome ſmall ſimple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and conſequently the tangent of 45°, or t, $= 1$ alſo, in this caſe the arc of 45° to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite ſeries $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c$.

But as this ſeries converges very ſlowly, it will be proper to take ſome ſmaller arc, that the ſeries may converge faſter; ſuch as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its ſquare $t^2 = \frac{1}{3}$; which being ſubſtituted in the ſeries, the length of the arc of 30° comes out - - - -

$(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c)\sqrt{\frac{1}{3}}$. Hence, to compute theſe terms in decimal numbers, after the firſt, the ſucceeding terms will be found by dividing always by 3, and theſe quotients again by the abſolute numbers 3, 5, 7, 9, &c; and laſtly, adding every other term together, into two ſums, the one the ſum of the poſitive terms, and the other the ſum of the negative ones; then laſtly, the one ſum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, conſequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the firſt term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3.4641016$; and hence the operation will be conveniently made as follows:

			+ Terms.	− Terms.
1)	3·4641016	(	3·4641016	
3)	1·1547005	(		0·3849002
5)	384900 2	(	769800	
7)	128300 1	(		183286
9)	427667	(	47519	
11)	14 556	(		12960
13)	47519	(	3655	
15)	15840	(		1056
17)	5280	(	311	
19)	1760	(		93
21)	587	(	28	
23)	196	(		8
25)	65	(	3	
27)	22	(		1
			+ 3 54 2332	−0·4046406
			− 0·4046406	

So that at laſt 3·1415926 is the whole circumference, to the diameter 1.

EXAM. 2. To find the length of a parabola.

EXAM. 3. To find the length of the ſemicubical parabola, whoſe equation is $ax^2 = y^3$.

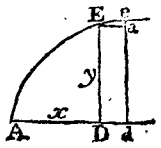
EXAM. 4. To find the length of an elliptical curve.

EXAM. 5. To find the length of an hyperbolic curve.

OF QUADRATURES; OR, FINDING THE AREAS OF CURVES.

62. THE Quadrature of Curves, is the meaſuring their areas, or finding a ſquare, or other right-lined ſpace, equal to a propoſed curvilineal one.

By art. 9 it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the



the flowing. That is, $Dd \times DE$ or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

63. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value ſubſtitute inſtead of it in the expreſſion $y\dot{x}$; then the fluent of that expreſſion, being taken, will be the area of the curve ſought.

EXAMPLES.

EXAM. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the abſciſs AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This ſubſtituted in the general fluxion of the area $y\dot{x}$, gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ is the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, and which is therefore equal to $\frac{2}{3}$ of its circumſcribing rectangle.

EXAM. 2. To ſquare the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by ſubſtitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a ſeries, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times \left(1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c\right);$$

then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times \left(\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \&c\right)$$

for the general expreſſion of the ſemiſegment ADE.

When the point D arrives at the extremity of the diameter, then the ſpace becomes a ſemicircle, and $x = a$; and then the ſeries above becomes barely

$$a^2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \&c\right)$$

for the area of the ſemicircle whoſe diameter is a.

EXAM. 3. To find the area of any parabola, whoſe equation is $a^m x^n = y^{m+n}$.

EXAM. 4. To find the area of an ellipſe.

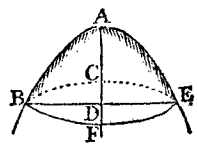
EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and aſymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola, whoſe general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS.

64. IN the ſolid formed by the rotation of any curve about its axis, the ſurface may be conſidered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the ſolid. Therefore, as the fluxion of any generated quantity, is produced, by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the ſurface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the ſurface BAE, is equal to AE drawn into the circumference BCEF, whoſe radius is the ordinate DE.



65. But, if c be $= 3{\cdot}1416$, the circumference of a circle whoſe diameter is 1, $x = $ AD the abſciſs, $y = $ DE the ordinate, and $z = $ AE the curve; then $2y = $ the diameter BE, and $2cy = $ the circumference BCEF; alſo, $\text{AE} = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2cy\dot{z}$ or $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the ſurface. And conſequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and ſubſtituted in this expreſſion $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expreſſion, being then taken, will be the ſurface of the ſolid required.

EXAMPLES.

EXAM. 1. To find the ſurface of a ſphere, or of any ſegment.

In

In this caſe, AE is a circular arc, whoſe equation is $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$.

The fluxion of this gives $\dot{y} = \frac{a - 2x}{2\sqrt{ax - x^2}}\dot{x} = \frac{a - 2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \frac{a^2 - 4y^2}{4y^2}\dot{x}^2$;

conſeq. $\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more eaſily thus: In the fig. to art. 60, the two triangles EDC, E*ae* are equiangular, being each of them equiangular to the triangle ETC; conſeq. ED : EC :: E*a* : E*e*, that is, $y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$, the ſame as before.

The value of $\dot{z}$ being found, by ſubſtitution is obtained $2cy\dot{z} = ac\dot{x}$ for the fluxion of the ſpherical ſurface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives acx for the ſaid ſurface of the ſpherical ſegment BAE.

But ac is equal to the whole circumference of the generating circle; and therefore it follows, that the ſurface of any ſpherical ſegment, is equal to the ſaid circumference of the generating circle, drawn into x or AD, the height of the ſegment.

Alſo, when x or AD becomes equal to the whole diameter a, the expreſſion acx becomes aca or ca^2, or 4 times the arc of the generating circle, for the ſurface of the whole ſphere.

And theſe agree with the rules before found in Menſuration of Solids.

EXAM. 2. To find the ſurface of a ſpheroid.

EXAM. 3. To find the ſurface of a paraboloid.

EXAM. 4. To find the ſurface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS.

66. ANY ſolid which is formed by the revolution of a curve about its axis (ſee laſt fig.), may alſo be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore

fore the area of that circle being drawn into the fluxion of the axis, will produce the fluxion of the ſolid. That is, AD × area of the circle BCF, whoſe radius is DE, or diameter BE, is the fluxion of the ſolid, by art. 9.

67. Hence, if AD = $\dot{x}$, DE = y, c = 3·1416; becauſe cy^2 is equal to the area of the circle BCF; therefore $cy^2\dot{x}$ is the fluxion of the ſolid. And conſequently if, from the given equation of the curve, the value of either y^2 or $\dot{x}$ be found, and that value ſubſtituted for it in the expreſſion $cy^2\dot{x}$, the fluent of the reſulting quantity, being taken, will be the ſolidity of the figure propoſed.

EXAMPLES.

EXAM. 1. To find the ſolidity of a ſphere, or any ſegment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by ſubſtitution, the general fluxion of the ſolid, $cy^2\dot{x}$, becomes $ca\cdot\dot{x} - cx\dot{x}$, the fluent of which gives $\frac{1}{2}cax^2 - \frac{1}{3}cx^3$, or $\frac{1}{6}cx^2(3a - 2x)$, for the ſolid content of the ſpherical ſegment BAE, whoſe height AD is x.

When the ſegment becomes equal to the whole ſphere, then $x = a$, and the above expreſſion for the ſolidity, becomes $\frac{1}{6}ca^3$ for the ſolid content of the whole ſphere.

And theſe deductions agree with the rules before given and demonſtrated in the Menſuration of Solids.

EXAM. 2. To find the ſolidity of a ſpheroid.

EXAM. 3. To find the ſolidity of a paraboloid.

EXAM. 4. To find the ſolidity of an hyperboloid.

TO FIND LOGARITHMS

68. IT has been proved, art. 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the ſame quantity. Therefore, when any quantity is propoſed, to find its logarithm; take the fluxion of that quantity, and divide it by the ſame quantity; then take the fluent of the quotient, either in a ſeries or otherwiſe, and it will be the logarithm ſought; when corrected as uſual, if need be; that is, the hyperbolic logarithm.

69. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the ſyſtem, for the logarithm ſought.

Note,

Note, The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·434294481 90, &c; and, in general, the modulus of any ſyſtem, is equal to the logarithm of 10 in that ſyſtem divided by the number 2·302585092994 0, &c, which is the hyp. log. of 10.

EXAM. 1. To find the log. of $\frac{a+x}{a}$

Denoting any propoſed number z, whoſe logarithm is required to be found, by the compound expreſſion $\frac{a+x}{a}$, the fluxion of the number z, is $\frac{\dot{x}}{a}$, and the fluxion of the log. $\frac{\dot{z}}{z} = \frac{\dot{x}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} +$ &c.

Then the fluents of theſe terms give the logarithm of z or logarithm of - - - - $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

writing $-x$ for x, gives log. $\frac{a-x}{a} = -\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c

Mult. theſe numbs. and adding their logs. gives } log. $\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5}$ &c.

Alſo, becauſe $\frac{a}{a\pm x} = 1 \div \frac{a\pm x}{a}$, or log. $\frac{a}{a\pm x} = 0 -$ log. $\frac{a\pm x}{a}$.

therefore log. of $\frac{a}{a+x}$ is $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

and the log. of $\frac{a}{a-x}$ is $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c.

the prod. gives log. $\frac{a^2}{a^2-x^2} = \frac{2x^2}{2a^2} + \frac{2x^4}{4a^4} + \frac{2x^6}{6a^6} +$ &c.

Now, for an example in numbers, ſuppoſe it were required to compute the common logarithm of the number 2. This will be beſt done by the ſeries,

$$\text{log. of } \frac{a+x}{a-x} = 2m \times \left(\frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \frac{x^7}{7a^7} \text{ &c}\right).$$

Now, making $\frac{a+x}{a-x} = 2$, gives $a = 3x$; conſequently $\frac{x}{a} = \frac{1}{3}$, and $\frac{x^2}{a^2} = \frac{1}{9}$, which is the conſtant factor for every ſucceeding term; alſo, $2m = 2 \times$ ·434294481 90 = ·868588964; therefore the calculation will be conveniently made, by firſt dividing this number by 3, then the quotients ſucceſſively

ſucceſſively by 9, and laſtly theſe quotients in order by the reſpective numbers 1, 3, 5, 7, 9, &c, and after that, adding the terms all together, as follows:

3)	·868588964			
9)	289529654	1) ·289529654	(·289529654	
9)	32169962	3) 32169962	(10723321	
9)	3574440	5) 3574440	(714888	
9)	397160	7) 397160	(56737	
9)	44129	9) 44129	(4903	
9)	4903	11) 4903	(446	
9)	545	13) 545	(42	
9)	61	15) 61	(4	

Sum of the terms gives log. 2 = ·301029995

EXAM. 2. To find the log. of $\frac{a+x}{b}$.

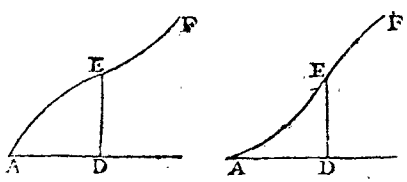
EXAM. 3. To find the log. of $a - x$.

EXAM. 4. To find the log. of $\frac{a+x}{a-x}$.

EXAM. 5. To find the log. of $\frac{a-x}{a+x}$.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE, IN CURVES.

70. THE Point of Inflexion in a curve, is that point of it which ſeparates the concave from the convex part, lying between the two; or where the curve changes from concave to convex, or from convex to concave, on the ſame ſide of the curve. Such as the point E in the annexed figures; where the former of the two is concave towards the axis AD, from A to E, but convex from E to F; and, on the contrary, the latter figure is convex from A to E, and concave from E to F.



71. From

71. From the nature of curvature, as has been remarked before at art. 28, it is evident that, when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the abscis; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the abscis, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a constant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a constant quantity. But constant quantities have no fluxion, or their fluxion is equal to nothing; so that, in this case, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

72. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal o or nothing; and from this last equation find also the value of the same $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which will form an equation from whence, and the first given equation, of the curve, x and y will be determined, being the abscis and ordinate answering to the point of inflexion in the curve, as required.

EXAMPLES.

EXAM. 1. To find the point of inflexion in the curve whose equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$. Then the fluxion of this quantity made = o, gives $2x\dot{x}(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$.

Lastly, this value of $\frac{\dot{x}}{\dot{y}}$, being put equal the former, gives $\frac{a^2 + x^2}{a^2 - x^2} \cdot \frac{x}{a - y} = \frac{a^2 + x^2}{2x} \cdot \frac{1}{a - y}$; and hence $2x^2 = a^2 - x^2$, or $3x^2 = a^2$, and $x = a\sqrt{\frac{1}{3}}$, the abscis.

Hence

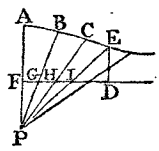
Hence alſo, from the original equation,

$y = \frac{ax^2}{a^2 + x^2} = \frac{\frac{1}{5}a^3}{\frac{4}{5}a^2} = \frac{1}{4}a$, the ordinate to the point of inflexion ſought.

EXAM. 2. To find the point of inflexion in á curve defined by the equation $ay = a\sqrt{ax} + xx$.

EXAM. 3. To find the point of inflexion in a curve defined by the equation $ay^2 = a^2x + x^3$.

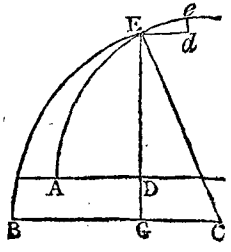
EXAM. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or conſtructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c, cutting the given line FD in the points F, G, H, I, &c: then make the diſtances FA, GB, HC, IE, &c, equal to each other, and equal to a given line; and the curve line ABCE, &c, will be the conchoid; a curve ſo called by its inventor Nicomedes.



TO FIND THE RADIUS OF CURVATURE OF CURVES.

73. THE Curvature of a Circle is conſtant, or the ſame in every point of it, and its radius is the radius of curvature. But the caſe is different in other curves, every one of which has its curvature continually varying, either increaſing or decreaſing, and every point having a degree of curvature peculiar to itſelf; and the radius of a circle which has the ſame curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the buſineſs of this chapter to find.

74. Let AEe be any curve, concave toward its axis AD; draw an ordinate DE to the point E, where the curvature is required to be found; and ſuppoſe EC perpendicular to the curve, and equal to the radius of curvature ſought, or equal to the radius of a circle having the ſame curvature there, and with that radius deſcribe the ſaid equally-curved circle BEe;



BE*e*; laſtly, draw E*d* parallel to AD, and *de* parallel and indefinitely near to DE; thereby making E*d* the fluxion, or increment of the abſciſs AD, alſo *de* the fluxion of the ordinate DE, and E*e* that of the curve AE. Then put x = AD, y = DE, z = AE, and r = CE the radius of curvature; then is E*d* = $\dot{x}$, *de* = $\dot{y}$, and E*e* = $\dot{z}$.

Now, by ſim. triangles, the three lines E*d*, *de*, E*e*,
or $\dot{x}$, $\dot{y}$, $\dot{z}$,
are reſpectively as the three - - GE, GC, CE;
therefore - - - - - - GC . $\dot{x}$ = GE . $\dot{y}$;
and the flux. of this eq. is GC . $\ddot{x}$ + GC . $\dot{x}$ = GE . $\ddot{y}$ + GE . $\dot{y}$,
or, becauſe GC = − BG, it is, GC . $\ddot{x}$ − BG . $\dot{x}$ = GE . $\ddot{y}$ + GE . $\dot{y}$.

But, ſince the two curves AE and BE have the ſame curvature at the point E, their abſciſſes and ordinates have the ſame fluxions at that point, that is, E*d* or $\dot{x}$ is the fluxion both of AD and BG, and *de* or $\dot{y}$ is the fluxion both of DE and GE. In the equation above, therefore, ſubſtitute $\dot{x}$ for BG, and $\dot{y}$ for GE, and it becomes

$$\text{GC}\ddot{x} - \dot{x}\dot{x} = \text{GE}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or} \quad \text{GC}\ddot{x} - \text{GE}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now, multiply the three terms of this equation reſpectively by theſe three quantities, $\frac{\dot{y}}{\text{GC}}$, $\frac{\dot{x}}{\text{GE}}$, $\frac{\dot{z}}{\text{CE}}$, which are equal, and it becomes - - $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{\text{CE}}$ or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the abſciſs and ordinate.

75. Farther, as in any caſe either x or y may be ſuppoſed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ conſtant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this ſuppoſition, either of the terms in the denominator, of the value of r, may be made to vaniſh. Thus, when $\dot{x}$ is ſuppoſed conſtant, $\ddot{x}$ being then = 0, the value of r is barely $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is = $\frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is conſtant.

EXAMPLES.

EXAM. 1. To find the radius of curvature to any point of

of a parabola, whose equation is $ax = y^2$; its vertex being A, and axis AD.

Now, the equation to the curve being $ax = y^2$; the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, supposing $\dot{y}$ constant; hence then r or

$$\frac{\dot{x}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2+4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a+4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the abscis AD $= x$.

Hence, when the abscis x is nothing, the last expression becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis.

EXAM. 2. To find the radius of curvature of an ellipse, whose equation is $a^2y^2 = c^2 \cdot ax - x^2$.

$$\text{Ans. } r = \frac{(a^2c^2 + 4(a^2-c^2) \times (ax-x^2))^{\frac{3}{2}}}{2a^4c}.$$

EXAM. 3. To find the radius of curvature of an hyperbola, whose equation is $a^2y^2 = c^2 \cdot ax + x^2$.

EXAM. 4. To find the radius of curvature of the cycloid.

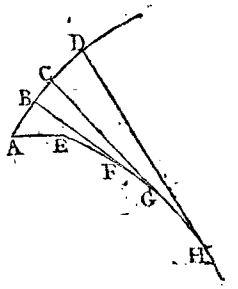
Ans. $r = 2\sqrt{aa - ax}$, where x is the abscis, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

76. AN Evolute is any curve supposed to be evolved or opened, by having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the same time always stretched.

77. Thus,

77. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plyed close to the curve, &c, from H to A, keeping the thread always ſtretched tight; the other end of the thread will deſcribe a certain curve ABCD, called an Involute; the firſt curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will deſcribe the ſame involute ABCD.



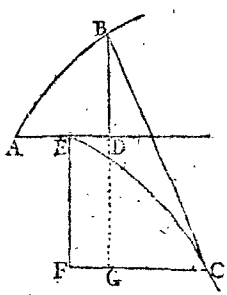
78. If AE, BF, CG, DH, &c, be any poſitions of the thread, in evolving or unwinding; it follows, that theſe parts of the thread are always the radii of curvature, at the correſponding points, A, B, C, D; and alſo equal to the correſponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE = AE is the radius of curvature to the point A,
BF = AF is the radius of curvature to the point B,
CG = AG is the radius of curvature to the point C,
DH = AH is the radius of curvature to the point D.

79. It alſo follows, from the premiſes, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Alſo, that the evolute is the locus of the centre of curvature of the involute curve.

80. Hence, and from art. 74, it will be eaſy to find one of theſe curves, when the other is given. To this purpoſe, put

x = AD, the abſciſs of the involute,
y = DB, an ordinate to the ſame,
z = AB, the involute curve,
r = BC, the radius of curvature,
v = EF, the abſciſs of the evolute EC,
u = FC, the ordinate of the ſame, and
a = AE, a certain given line.



Then, by the nature of the radius of curvature, it is

$$r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \mathrm{BC} = \mathrm{AE} + \mathrm{EC};$$ alſo, by ſim. triangles,

$$\dot{z} : \dot{x} :: r : \mathrm{GB} = \frac{r\dot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}};$$

$$\dot{z} : \dot{y} :: r : \mathrm{GC} = \frac{r\dot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}.$$

Hence $\mathrm{EF} = \mathrm{GB} - \mathrm{DB} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} - y = v;$

and $\mathrm{FC} = \mathrm{AD} - \mathrm{AE} + \mathrm{GC} = x - a + \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = u;$

which are the values of the abſciſs and ordinate of the evolute curve EC; from which, therefore, theſe may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve $\mathrm{EC} = s$; ſince $\mathrm{CB} = \mathrm{AE} + \mathrm{EC}$, or $r = a + s$, this gives r the radius of curvature. Alſo, by ſimilar triangles, there ariſe theſe proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{\dot{s}} = \frac{a + s}{\dot{s}}\dot{v} = \mathrm{GB},$$

and $\dot{s} : \dot{u} :: r : \frac{r\dot{u}}{\dot{s}} = \frac{a + s}{\dot{s}}\dot{u} = \mathrm{GC};$

therefore $\mathrm{AD} = \mathrm{AE} + \mathrm{FC} - \mathrm{GC} = a + u - \frac{a + s}{\dot{s}}\dot{u} = x,$

and $\mathrm{DB} = \mathrm{GB} - \mathrm{GD} = \mathrm{GB} - \mathrm{EF} = \frac{a + s}{\dot{s}}\dot{v} - v = y;$

which are the abſciſs and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$. Alſo, either of the quantities x, y, may be ſuppoſed to flow equably, in which caſe the reſpective ſecond fluxion, $\ddot{x}$ or $\ddot{y}$, will be nothing, and the correſponding term in the denominator $\dot{y}\ddot{x} - \dot{x}\ddot{y}$ will vaniſh, leaving only the other term in it; which will have the effect of rendering the whole operation ſimpler.

81. EXAMPLES.

EXAM. 1. To determine the nature of the curve by whoſe evolution the common parabola AB is deſcribed.

Here

Here the equation of the given involute AB, is $cx = y^2$, where c is the parameter of the axis AD. Hence then $y = \sqrt{cx}$, and $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$, alſo $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$ by making $\dot{x}$ conſtant. Conſequently the general values of v and u, or of the abſciſs and ordinate, EF and FC, above given, become, in that caſe,

$$EF = v = \frac{\dot{x}^2}{-\ddot{y}} - y = \frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$FC = u = x - a + \frac{\dot{y}\dot{x}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

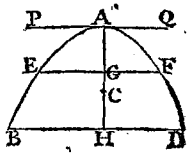
But the value of the quantity a or AE, by exam. 1 to art. 75, was found to be $\frac{1}{2}c$; conſequently the laſt quantity, FC or u, is barely $= 3x$.

Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the abſciſs and ordinate of the evolute curve EC, ſhewing it to be the ſemicubical parabola.

EXAM. 2. To determine the evolute of the common cycloid. Anſ. another cycloid, equal to the former.

TO FIND THE CENTRE OF GRAVITY.

82. By referring to prop. 41, &c, in Mechanics, it is ſeen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expreſſed. It there appears, that if PAQ be a line, or plane, drawn through any point, as ſuppoſe the vertex of any body, or figure, ABD, and if s denote any ſection EF of the figure, $d = AG$, its diſtance below PQ, and $b =$ the whole body or figure ABD; then the diſtance AC, of the centre of gravity below PQ, is univerſally denoted by $\frac{\text{ſum of all the } ds}{b}$;



whether ABD be a line, or a plane ſurface, or a curve ſuperficies, or a ſolid.

But the sum of all the ds, is the same as the fluent of $d\dot{s}$, and b is the same as the fluent of $\dot{b}$; therefore the general expression for the distance of the centre of gravity, is AC $=$ $\frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable distance AG. Which will divide into the following four cases.

83. CASE 1. When AE is some line, as a curve supposed. In this case, $\dot{b}$ is $= \dot{z}$ or $\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2+\dot{y}^2}}{z}$ is the distance of the centre of gravity in a curve.

84. CASE 2. When the figure ABD is a plane; then $\dot{b} = y\dot{x}$; therefore the general expression becomes AC $= \frac{\text{fluent of } yx\dot{x}}{\text{fluent of } y\dot{x}}$ for the distance of the centre of gravity in a plane.

85. CASE 3. When the figure is the superficies of a body generated by the rotation of a line AEB, about the axis AH.

Then, putting $c = 3{\cdot}14159$ &c, $2cy$ will denote the circumference of the generating circle, and $2cy\dot{z}$ the fluxion of the surface; therefore AC $= \frac{\text{fluent of } 2cyx\dot{z}}{\text{fluent of } 2cy\dot{z}} = \frac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$ will be the distance of the centre of gravity for a surface generated by the rotation of a curve line z.

86. CASE 4. When the figure is a solid generated by the rotation of a plane ABH, about the axis AH.

Then, putting $c = 3{\cdot}14159$ &c, it is cy^2 = the area of the circle whose radius is y, and $cy^2\dot{x} = \dot{b}$, the fluxion of the solid; therefore

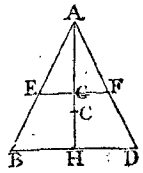
$$\text{AC} = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } cy^2x\dot{x}}{\text{fluent of } cy^2\dot{x}} = \frac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$$

is the distance of the centre of gravity below the vertex in a solid.

87. EXAMPLES.

EXAM. 1. Let the figure proposed be the isosceles triangle ABD.

It is evident that the centre of gravity C, will be somewhere



where in the perpendicular AH. Now, if a denote AH, c = BD, x = AG, and y = EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d case, $AC = \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}}$ $= \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2}{3}x = \frac{2}{3}AH$, when x becomes = AH: consequently $CH = \frac{1}{3}AH$.

In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{1}{3}$ of the altitude of the triangle; the same as it was found in prop. 42, Mechanics.

EXAM. 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

EXAM. 3. In a circular arc; the distance from the centre of the circle, is $\frac{ac}{r}$; where a denotes the arc, c its chord, and r the radius.

EXAM. 4. In a circular sector; the distance from the centre of the circle, is $\frac{2cr}{3a}$: where a, c, r, are the same as in exam. 3.

EXAM. 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

EXAM. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

EXAM. 7. In the semisphere, or semispheroid; the distance from the centre is $\frac{3}{8}r$, or $\frac{3}{8}$ of the radius.

EXAM. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis.

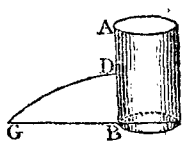
EXAM. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a - x}{6a - 4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

EXAM. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a + x}{6a + 4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

PRACTICAL QUESTIONS.

QUESTION I.

A LARGE veſſel, of 10 feet, or any other given depth, and of any ſhape, being kept conſtantly full of water, by means of a ſupplying cock at the top; it is propoſed to aſſign the place where a ſmall hole muſt be made in the ſide of it, ſo that the water may ſpout through it to the greateſt diſtance on the plane of the baſe.

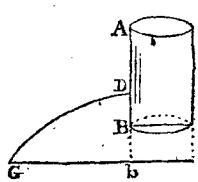


Let AB denote the height or ſide of the veſſel; D the required hole in the ſide, from whence the water ſpouts, in the parabolic curve DG, to the greateſt diſtance BG, on the horizontal plane.

By the Scholium to prop. 61, Hydraulics, the diſtance BG is always equal to $2\sqrt{AD \cdot DB}$, which is equal to $2\sqrt{x(a-x)}$ or $2\sqrt{ax-x^2}$, if a be put to denote the whole height AB of the veſſel, and $x = AD$ the depth of the hole. Hence $2\sqrt{ax-x^2}$, or $ax-x^2$, muſt be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or $x = \frac{1}{2}a$. So that the hole D muſt be in the middle between the top and bottom; the ſame as before found at the end of the ſcholium above quoted.

QUESTION II.

If the ſame veſſel, as in Queſt. 1, ſtand on high, with its bottom a given height above a horizontal plane below; it is propoſed to determine where the ſmall hole muſt be made, ſo as to ſpout fartheſt on the ſaid plane.



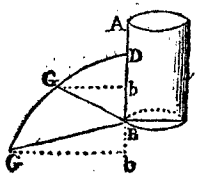
Let the annexed figure repreſent the veſſel as before, and bG the greateſt diſtance ſpouted by the fluid DG, on the plane bG.

Here, as before, $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(c-x)} = 2\sqrt{cx-x^2}$, by putting $Ab = c$, and $AD = x$. So that $2\sqrt{cx-x^2}$ or $cx-x^2$ muſt be a maximum. And hence, like as in the former queſtion, $x = \frac{1}{2}c = \frac{1}{2}AB$. So that, the hole D muſt be made in the middle

middle between the top of the veſſel, and the given plane, that the water may ſpout fartheſt.

QUESTION III.

But if the ſame veſſel as before, ſtand on the top of an inclined plane, making a given angle, as ſuppoſe of 30 degrees, with the horizon; it is propoſed to determine the place of the ſmall hole, ſo as the water may ſpout the fartheſt on the ſaid inclined plane.



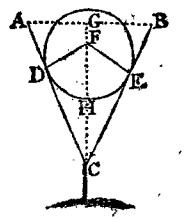
Here again (D being the place of the hole, and BG the given inclined plane), $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(a - x \pm z)}$, putting $z = Bb$, and, as before, $a = AB$, and $x = AD$. Then bG muſt ſtill be a maximum, as alſo Bb, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm xz$, as well as z, is a maximum. Hence, by art. 54 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conſeq. $\pm z = 2x - a$; and hence $bG = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But, as the given angle GBb is $= 30°$, the ſine of which is $\frac{1}{2}$; therefore $BG = 2Bb$ or $2z$, and $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x - a)^2$, or $bG = \pm(2x - a)\sqrt{3}$.

Putting now theſe two values of bG equal to each other, gives the equation $2x = \pm(2x - a)\sqrt{3}$, from which is found $x = \frac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \pm 1} = \frac{3 \mp \sqrt{3}}{4}a$, the value of AD required.

Note. In the Select Exerciſes, page 165, this anſwer is brought out $\frac{6 + \sqrt{6}}{10}a$, by taking the velocity as the root of *half* the altitude.

QUESTION IV.

It is required to determine the ſize of a ball, which, being let fall into a conical glaſs full of water, ſhall expel the moſt water poſſible from the glaſs; its depth being 6, and diameter 5 inches.



Let ABC repreſent the cone of the glaſs, and DHE the ball, touching the ſides in the points D and E, the centre of the ball being at ſome point F in the axis GC of the cone.

Put AG = GB = $2\frac{1}{2} = a$,
GC = 6 = b,
AC = $\sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$,
FD = FE = FH = x the radius of the ball.

The two triangles ACG and DCF are equiangular; theref. AG : AC :: DF : FC, that is, $a : c :: x : \frac{cx}{a}$ = FC; hence GF = GC − FC = $b - \frac{cx}{a}$, and GH = GF + FH = $b + x - \frac{cx}{a}$, the height of the ſegment immerſed in the water. Then (by rule 1 for the ſpherical ſegment, page 48), the content of the ſaid immerſed ſegment will be $(6DF - 2GH) \times GH^2 \times .5236 = (2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1.0472$, which muſt be a maximum by the queſtion: the fluxion of this made = 0, and divided by $2\dot{x}$ and the common factors, gives $\frac{2a+c}{a} \times (b - \frac{c-a}{a}x) - (\frac{2a+c}{a}x - b) \times \frac{c-a}{a} \times 2 = 0$; this reduced gives $x = \frac{abc}{(c-a) \times (c+2a)} = 2\frac{11}{92}$, the radius of the ball. Conſequently its diameter is $4\frac{11}{46}$ inches, as required.

PRACTICAL EXERCISES CONCERNING FORCES, WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

BEFORE entering on the following problems, it will be convenient here, to lay down a ſynopſis of the theorems which expreſs the ſeveral relations between any forces, and their correſponding times, velocities, and ſpaces deſcribed; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F be any two conſtant accelerative forces, acting on any body, during the reſpective times t, T, at the end of which are generated the velocities v, V, and deſcribed the ſpaces s, S. Then, becauſe the ſpaces are as the times and velocities conjointly, and the velocities as the forces and times; we ſhall have,

I. *In*

I. *In Conſtant Forces.*

1. $\frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}$.

2. $\frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}$.

3. $\frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}$.

4. $\frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}$.

And if one of the forces, as F, be the force of gravity at the ſurface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the correſponding ſpace S is $= 16\frac{1}{12}$ feet, and conſequently its velocity $V = 2S = 32\frac{1}{6}$, which call $2g$, namely, $g = 16\frac{1}{12}$ feet or 193 inches. Then the above four theorems, in this caſe, become as here below:

5. $s = \frac{1}{2}tv = gft^2 = \frac{v^2}{4gf}$.

6. $v = \frac{2s}{t} = 2gft = \sqrt{4gfs}$.

7. $t = \frac{2s}{v} = \frac{v}{2gf} = \sqrt{\frac{s}{gf}}$.

8. $f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}$.

And from theſe are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

9. $\dot{s} = v\dot{t} = \frac{v\dot{v}}{2gf}$.

10. $\dot{v} = 2gf\dot{t} = \frac{2gf\dot{s}}{v}$.

11. $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}$.

12. $f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}$.

In

In theſe laſt four theorems, the force f, though variable, is ſuppoſed to be conſtant for the indefinitely ſmall time t, and they are to be uſed in all caſes of variable forces, as the former ones in conſtant forces; namely, from the circumſtances of the problem under conſideration, an expreſſion is deduced for the value of the force f, which being ſubſtituted in one of theſe theorems, that may be proper to the caſe in hand; the equation thence reſulting will determine the correſponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the queſtion, it may be proper to obſerve, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between theſe three quantities being univerſally expreſſed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or elſe brought into it.

Alſo, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is alſo to be obſerved, that the theorems equally hold good for the deſtruction of motion and velocity, by means of retarding forces, as for the generation of the ſame, by means of accelerating forces.

And to the following problems, which are all reſolved by the application of theſe theorems, it has been thought proper to ſubjoin their ſolutions, for the better information and convenience of the ſtudent.

PROBLEM I.

To determine the time and velocity of a body deſcending, by the force of gravity, down an inclined plane; the length of the plane being 20 *feet, and its height* 1 *foot.*

HERE, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as 20 : 1 :: 1 (the force of gravity) : $\frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4} \times 16\frac{1}{12} \times \frac{1}{20} \times 20 = \sqrt{4} \times 16\frac{1}{12} = 2 \times 4\frac{1}{96} = 8\frac{1}{48}$ feet, the laſt velocity per ſecond.

And, by theor. 7, t or $\sqrt{\frac{s}{gf}}$, is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}} = 4\frac{76}{77}$ ſeconds, the time of deſcending.

PRO-

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 *feet per ſecond, up a ſmooth inclined plane, which riſes* 1 *foot in* 20: *it is propoſed to aſſign the length which it will aſcend up the plane, before it ſtops and begins to return down again, and the time of its aſcent.*

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the diſtance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193} =$ $621''\frac{147}{193} = 10' 21''\frac{147}{193}$, the time of aſcent.

PROBLEM III.

If a ball be projected up a ſmooth inclined plane, which riſes 1 *foot in* 10, *and aſcend* 100 *feet before it ſtop: required the time of aſcent, and the velocity of projection.*

FIRST, by theor. 6, $v = \sqrt{4gfs} = \sqrt{4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100} = 8\frac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per ſecond, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} =$ $\frac{192}{77}\sqrt{10} = 7{\cdot}88516$ ſeconds, the time in motion.

PROBLEM IV.

If a ball be obſerved to aſcend up a ſmooth inclined plane 100 *feet in ten ſeconds, before it ſtop, to return back again: required the velocity of projection, and the angle of the plane's inclination.*

FIRST, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per ſecond, the velocity.

And, by theor. 8, $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore, 193 : 12 :: 100 : 6·2176 the height of the plane, or the ſine of elevation to radius 100, which anſwers to 3° 34′, the angle of elevation of the plane.

PRO-

PROBLEM V.

By a mean of ſeveral experiments, I have found, that a caſt iron ball, of 2 inches diameter, impinging perpendicularly on the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per ſecond, penetrated 13 inches deep into its ſubſtance. It is propoſed then to determine the time of the penetration, and the reſiſting force of the wood, as compared to the force of gravity, ſuppoſing that force to be a conſtant quantity.

FIRST, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a ſecond, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is, the reſiſting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its denſity or ſpecific gravity. For, ſince f is as $\frac{v^2}{s}$ by theor. 4, the denſity and ſize of the ball remaining the ſame; if the denſity, or ſpecific gravity, n, vary, and all the reſt be conſtant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the ſize of the ball only is conſtant. But when only the diameter d varies, all the reſt being conſtant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the reſiſting ſurface, or force of reſiſtance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the reſt are conſtant. Conſequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And ſo $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$, and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$; where f denotes the ſtrength or firmneſs of the ſubſtance penetrated, and is here ſuppoſed to be the ſame, for all balls and velocities, in the ſame ſubſtance, which it is either accurately or nearly ſo. See page 264, &c. of my Tracts, vol. i.

Hence, taking the numbers in the problem, it is - - -

$f = \frac{dnv^2}{s} = \frac{2 \times 7\frac{1}{2} \times 1500^2}{1\frac{1}{2}} = \frac{44 \times 1500^2}{39} = 2538462$

the value of f for elm wood. Where the ſpecific gravity of the

the ball is taken $7\frac{1}{3}$, which is a little leſs than that of ſolid caſt iron, as it ought, on account of the air bubble which is found in all caſt balls.

PROBLEM VI.

To find how far a 24lb *ball of caſt iron will penetrate into a block of ſound elm, when fired with a velocity of* 1600 *feet per ſecond.*

HERE, becauſe the ſubſtance is the ſame as in the laſt problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5{\cdot}55 \times 1600^2 \times 13}{2 \times 1500^2} =$ $41\frac{2}{45}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18 *pounder ball, fired with a velocity of* 1200 *feet per ſecond, penetrated* 34 *inches into ſound dry oak. It is required to aſcertain the comparative ſtrength or firmneſs of oak and elm.*

THE diameter of an 18lb ball is 5·04 inches $= D$. Then, by the numbers given in this problem for oak, and in prob. 5 for elm, we have $\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5{\cdot}04 \times 1200^2 \times 13} =$ $\frac{100 \times 17}{5{\cdot}04 \times 16 \times 13} = \frac{1700}{1048}$ or $= \frac{8}{5}$ nearly.

From which it would ſeem, that elm timber reſiſts more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be ſuſpected that Mr. Robins's great penetration was owing to the ſplitting of his timber in ſome degree.

PROBLEM VIII.

A 24 *pounder ball being fired into a bank of firm earth, with a velocity of* 1300 *feet per ſecond, penetrated* 15 *feet: It is required then to aſcertain the comparative reſiſtances of elm and earth.*

COMPARING the numbers here with thoſe in prob. 5, it is

is $\frac{f}{F} = \frac{dv^2s}{Dv^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5{\cdot}55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0{\cdot}37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm timber reſiſts about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; ſuppoſing it fired with a velocity of 1700 feet per ſecond, and that the lead does not change its figure by the ſtroke againſt the wood.

HERE $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $S =$ - - - - - - - -

$\frac{DNV^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600}$ $= 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber muſt have reſiſted about twice as much; or elſe, which is much more probable, that the defect in his penetration aroſe from the change of figure in the leaden ball, from the blow againſt the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per ſecond, having been found to penetrate 13 inches deep into dry elm: It is required to aſcertain the time of paſſing through every ſingle inch of the 13, and the velocity loſt at each of them; ſuppoſing the reſiſtance of the wood conſtant or uniform.

THE velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the ſpaces, in conſtant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a ſecond, the whole time of paſſing through the 13 inches; therefore as

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. loſt			Time in the
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}v$	$= 58{\cdot}9 :: t :$	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}t$	$= {\cdot}00005$ 1ſt
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}v$	$= 61{\cdot}4 :: t :$	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}t$	$= {\cdot}00006$ 2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}v$	$= 64{\cdot}2$ &c	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}t$	$= {\cdot}00006$ 3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}v$	$= 67{\cdot}5$	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}t$	$= {\cdot}00007$ 4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}v$	$= 71{\cdot}4$	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}t$	$= {\cdot}00007$ 5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}v$	$= 76{\cdot}0$	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}t$	$= {\cdot}00007$ 6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}v$	$= 81{\cdot}7$	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}t$	$= {\cdot}00008$ 7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}v$	$= 88{\cdot}8$	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}t$	$= {\cdot}00008$ 8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}v$	$= 98{\cdot}2$	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}t$	$= {\cdot}00009$ 9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}v$	$= 111{\cdot}4$	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}t$	$= {\cdot}00011$ 10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}v$	$= 132{\cdot}2$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}t$	$= {\cdot}00013$ 11th.
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}v$	$= 172{\cdot}3$	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}t$	$= {\cdot}00017$ 12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}v$	$= 416{\cdot}0$	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}t$	$= {\cdot}00040$ 13th
Sum	$1500{\cdot}0$	Sum	$\frac{1}{692}$ or ${\cdot}00144$ inch.

Hence, as the motion loſt at the beginning is very ſmall; and conſequently the motion communicated to any body, as an inch plank, in paſſing through it, is very ſmall alſo; we can conceive how ſuch a plank may be ſhot through, without overſetting it.

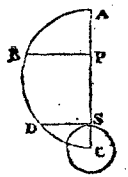
PRO-

PROBLEM XI.

The force of attraction, above the earth, being inverſely as the ſquare of the diſtance from the centre; it is propoſed to determine the time, velocity, and other circumſtances, attending a heavy body falling from any given height; the deſcent at the earth's ſurface being $16\frac{1}{12}$ *feet, or* 193 *inches, in the firſt ſecond of time.*

Put

r = CS the radius of the earth,
a = CA the diſt. fallen from,
x = CP any variable diſtance,
v = the velocity at P,
t = time of falling there, and
g = $16\frac{1}{12}$, half the veloc. or force at S,
f = the force at the point P.



Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is conſidered as 1;

$\dot{v} = -\dot{x}$, becauſe x decreaſes; and

$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the laſt equation give $v^2 = \frac{4gr^2}{x}$.

But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{\frac{4gr^2}{a} \times \frac{a-x}{x}}$, a general expreſſion for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{4gr \times \frac{a-r}{a}}$ for the greateſt velocity, or the velocity when the body ſtrikes the earth.

When a is very great in reſpect of r, the laſt velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greateſt velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per ſecond. Alſo, the velocity acquired in falling from the

the diſtance of the ſun, or 12000 diameters of the earth, is 6·9505 miles per ſecond. And the velocity acquired in falling from the diſtance of the moon, or 30 diameters, is 6·8927 miles per ſecond.

Again, to find the time; ſince $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} +$ arc to diameter a and verſ. $a - x$); or the time of falling to any point $P = \frac{1}{2r}\sqrt{\frac{a}{g}} \times (AB + BP)$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{AD + DS}{SC}$ for the whole time of falling to the ſurface at S; which is evidently infinite when a or AC is infinite, although the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's ſurface is given $= g$; becauſe r is then nearly $= a$, and AD nearly $=$ DS, the time t for the diſtance g will be nearly $\sqrt{\frac{r}{4gr^2}} \times 2DS = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

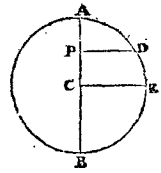
If a body at the diſtance of the moon at A fall to the earth's ſurface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$da. 19h. $46'46''$, which is the time of falling from the moon to the earth.

When the attracting body is conſidered as a point C; the whole time of deſcending to C will be - - - - - $\frac{1}{2r}\sqrt{\frac{a}{g}} \times ABDC = \frac{1{\cdot}7854a}{r}\sqrt{\frac{a}{g}}$.

PROBLEM XII.

The force of attraction below the earth's ſurface being directly as the diſtance from the centre: it is propoſed to determine the circumſtances of velocity, time, and ſpace fallen by a heavy body from the ſurface, through a perforation made ſtraight to the centre of the earth; abſtracting from the effect of the earth's rotation, and ſuppoſing it to be a homogeneous ſphere of 3965 miles radius.

Put r = AC the radius of the earth,
x = CP the dist. from the centre,
v = the velocity at P,
t = the time there,
$g = 16\frac{1}{12}$, half the force at A,
f = the force at P.



Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -2gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{2g \times \frac{r^2 - x^2}{r}} = \mathrm{PD}\sqrt{\frac{2g}{r}} = \mathrm{PD}\sqrt{\frac{2g}{\mathrm{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or fine PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{2g \cdot \mathrm{AC}}$ = 25950 feet or 4·9148 miles per second, which is the greateſt velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to coſine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of deſcent to any point P, is as the correſponding arc AD.

When P arrives at C, the above becomes $t = \sqrt{\frac{1}{2gr}} \times$ quadrant AE $= \frac{\mathrm{AE}}{\mathrm{AC}}\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ ſeconds = $21' 7''\frac{1}{4}$, for the time of falling to the centre C.

The time of falling to the centre, is the ſame quantity $1{\cdot}5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given diſtance from C at which the motion commences: then, by correction, $v = \sqrt{\frac{2g}{r}(n^2 - x^2)}$; and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to coſine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1{\cdot}5708\sqrt{\frac{r}{2g}}$, for the time of deſcent to the centre C, the ſame as before.

As

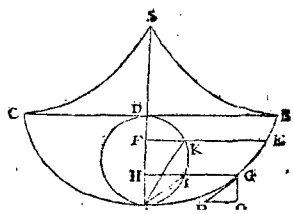
As an equal force, acting in contrary directions, generates or deſtroys an equal quantity of motion, in the ſame time; it follows that, after paſſing the centre, the body will juſt aſcend to the oppoſite ſurface at B, in the ſame time in which it fell to the centre from A. Then from B it will return again in the ſame manner, through C to A; and ſo vibrate continually between A and B, the velocity being always equal at equal diſtances from C on both ſides; and the whole time of a double oſcillation, or of paſſing from A and arriving at A again, will be quadruple the time of paſſing over the radius AC, or $= 2 \times 3{\cdot}1416\sqrt{\frac{r}{2g}} = 1\text{h. } 24' 29''$.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.

Let

S be the point of ſuſpenſion;
SA, the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG are perpendiculars.
SC, SB two other equal ſemi-cycloids, upon which the thread wrapping, the end A is made to deſcribe the cycloid BAC.



Now, by the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = SA = AB. Alſo, if at any point G be drawn the tangent GP; alſo GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP :: GP : GQ :: AI : AH :: AD : AI; in like manner, the force of gravity : force in the curve at E :: AD : AK; that is, the accelerative force in the curve, is every where as the correſponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, ſince AG is always = 2AI, by the nature of the curve. So that the proceſs and concluſions, for the velocity and time of deſcribing any arc in this caſe, will be the very ſame as in the laſt problem, the nature of the forces being the ſame, viz. as the diſtance to be paſſed over to the loweſt point A.

From whence it follows, that the time of a ſemi-vibration, in all arcs, AG, AE, &c, is the ſame conſtant quantity $1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{AS}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$; and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{2g}}$; where $l = AS = AB$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3·1416 the circumference of a circle whoſe diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, is $\sqrt{\frac{l}{2g}}$ by the nature of deſcents, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

If the time of the whole vibration be 1 ſecond, this equation ariſes, viz. $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$; and hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of theſe, g or l, be given by experiment, theſe equations will give the other. When g, for inſtance, is ſuppoſed to be given $= 16\frac{1}{12}$ feet, or 193 inches; then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$, the length of a pendulum to vibrate ſeconds. Or if $l = 39\frac{1}{8}$, the length of the ſeconds pendulum for the latitude of London, by experiment; then is $g = 4{\cdot}9348l = 193{\cdot}07$ inches $= 16\frac{1}{12}\frac{07}{00}$ feet, or nearly $16\frac{1}{12}$ feet, for the ſpace deſcended by gravity in the firſt ſecond of time in the latitude of London; alſo agreeing with experiment.

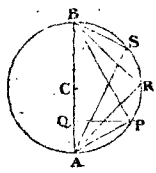
Hence the times of vibration of pendulums, are as the ſquare roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the ſquare roots of the lengths. And hence alſo, the length of a pendulum vibrating n times in a minute, or $60''$, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc; as the length of the ſtring is conſtantly the ſame, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is ſmaller; ſo that when

when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To determine the Time of a Body descending down the Chord of a Circle.



LET C be the centre; AB the vertical diameter; AP any chord, down which a body is to descend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the time in PA : time in QA :: PA : QA :: BA : PA; but time in BA : time in QA :: $\sqrt{BA} : \sqrt{QA}$ :: BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 131, Mechanics. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and r = the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

THE capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the sluice, the area of which is 4 square feet;

therefore $8\sqrt{10g}$ is the quantity per ſecond running through it; and conſequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31' 3''$ nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Veſſel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom; the Height of the Aperture being very ſmall in reſpect of the Altitude of the Fluid.

PUT a = the area of the aperture or ſluice;
$2g = 32\frac{1}{6}$ feet, the force of gravity;
d = the whole depth of water;
x = the variable altitude of the ſurface above the aperture;
A = the area of the ſurface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will iſſue at the ſluice; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$ the velocity with which the ſurface of the water will deſcend at the altitude x, or the ſpace it would deſcend in 1 ſecond with the velocity there. Now, in deſcending the ſpace $\dot{x}$, the velocity may be conſidered as uniform; and uniform deſcents are as their times; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of deſcending $\dot{x}$ ſpace, or the fluxion of the time of exhauſting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}$; which is made negative, becauſe x is a decreaſing quantity, or its fluxion negative.

Then, when the nature or figure of the veſſel is given, the area A will be given in terms of x; which value of A being ſubſtituted into this fluxion of the time, the fluent of the reſult will be the time of exhauſting ſought.

So if, for example, the veſſel be any priſm, or every where of the ſame breadth; then A is a conſtant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and ſhould be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the ſurface deſcending

ſcending till the depth of the water be x. And when $x = 0$, the whole time of exhauſting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

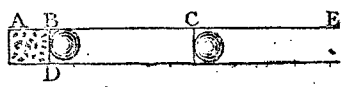
Hence, if A be 10000 ſquare feet, $a = 1$ ſquare foot, and $d = 10$ feet; the time is $7885\frac{1}{5}$ ſeconds, or 2h. 11′ $25''\frac{1}{5}$.

Again, if the veſſel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90+x}{9} \times 2$ the breadth of the ſurface of the water when its depth in the canal is x; and therefore $A = \frac{90+x}{9} \times 2000$ is the ſurface at that time. Conſequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{gx}} = 1100 \times \frac{90+x}{9} \times \frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180+\frac{2}{3}d}{9a} \times \sqrt{\frac{d}{g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{2}\sqrt{g}} =$ $15459''\frac{2}{3}$ nearly, or 4h. 17′ $39''\frac{2}{3}$, being the whole time of exhauſting by a ſluice of 1 foot ſquare.

PROBLEM XVII.

To determine the Velocity with which a Ball is diſcharged from a Given Piece of Ordnance, with a Given Charge of Gunpowder.

LET the annexed figure repreſent the bore of the gun; AD being the part filled with gunpowder. And put



a = AB, the part at firſt filled with powder;
b = AE, the whole length of the gun bore;
c = ·7854, the area of a circle whoſe diameter is 1;
d = BD, the diameter of the bore;
e = the ſpecific gravity of the ball, or weight of 1 cubic foot;
g = $16\frac{1}{12}$ feet, deſcended by a body in 1 ſecond;
m = 230 ounces, the preſſure of the atmoſphere on a ſq. inch;
n to 1 the ratio of the firſt force of the fired powder, to the preſſure of the atmoſphere;
w = the weight of the ball. Alſo, let
x = AC, be any variable diſtance of the ball from A, in moving along the gun barrel.

First, cd^2 is = the area of the circle BD of the bore; therefore mcd^2 is the preſſure of the atmoſphere on BD; conſeq. $mncd^2$ is the firſt force of the powder on BD.

But the force of the inflamed powder is proportional to its denſity, and the denſity is inverſely as the ſpace it fills; therefore the force of the powder on the ball at B, is to its force on the ſame at C, as AC is to AB; that is,

$x : a :: mncd^2 : \frac{mnacd^2}{x} = F$, the motive force at C;

conſeq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, $x = a$; therefore, by correction,

$v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent;

conſeq. $v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the vel. of ball at C,

and $v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball iſſues from the muzzle at E.

Or, by ſubſtituting the numbers for g and m, and changing the hyperbolic logarithms for the common ones, then

$v = \sqrt{(\frac{223onad^2}{w} \times \text{com. log. } \frac{b}{a})}$ the veloc. at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being ſubſtituted for w, in the value of v, it becomes

$v = 2713\sqrt{(\frac{na}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of caſt iron; taking $e = 7333$, the rule becomes

$v = 100\sqrt{(\frac{na}{10d} \times \text{log. } \frac{b}{a})}$ for the veloc. of the caſt-iron ball.

Or, when the ball is of lead; then

$v = 80\frac{3}{5}\sqrt{\frac{na}{10d} \times \text{log. } \frac{b}{a}}$ for the veloc. of the leaden ball.

Corol. From the general expreſſion for the velocity v, above given, may be derived what muſt be the length of the charge of powder a, in the gun barrel, ſo as to produce the greateſt

greateſt poſſible velocity in the ball; namely, by making the value of v a maximum, or, by ſquaring and omitting the conſtant quantities, the expreſſion $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is, $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whoſe hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 7 : 19 very nearly; that is, the length of the charge, to produce the greateſt velocity, is the $\frac{7}{19}$th part of the length of the bore.

By actual experiment it is found, that the charge for the greateſt velocity, ſomewhat exceeds that which is here computed from theory; as may be ſeen by turning to page 269 of my volume of Tracts, where the correſponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the conſtant quantity n remains to be determined. It denotes the firſt ſtrength or force of the fired gunpowder, juſt before the ball is moved out of its place. This value is aſſumed, by Mr. Robins, equal to 1000, that is, 1000 times the preſſure of the atmoſphere, on any equal ſpaces.

But the value of the quantity n, may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as above computed by theory, viz. $v = 1000\sqrt{(\frac{na}{10d} \times \text{log. of } \frac{b}{a})}$. Now, ſuppoſing that v is a given quantity, as well as all the other quantities, excepting only the number n, by reducing this equation, the value of the letter n is found to be as follows, viz. $n = \frac{dvv}{1000a} \div$ com. log. of $\frac{b}{a}$.

Now, to apply this to the experiments. By page 257 of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 110 feet per ſecond; and, by page 109, the length of the gun was 38·2; alſo, by page 237, the length of the charge was 3·12 inches:

3·12 inches: so that the values of the quantities in the rule, are thus: $a = 3{\cdot}12$; $b = 28{\cdot}2$; $d = 1{\cdot}96$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div$ com. log. of $\frac{b}{a}$, it comes out $n = 893$. And so on, for the other experiments there treated of.

But it will be best to arrange the results in a table, with the several dimensions, from which they are computed, as here below.

Table of Velocities of Balls, and First Force of Powder, &c.

Gun.		Charge.		Velocity, or value of v.	First force, or value of n.
No.	Length, or value of b.	Weight in ounces.	Length, or value of a.		
	inches.	4	2·54	1100	893
1	28·2	8	5·08	1430	1060
		16	10·16	1430	890
		4	2·54	1180	914
2	38·1	8	5·08	1580	1101
		16	10·16	1660	926
		4	2·54	1300	963
3	57·37	8	5·08	1790	1174
		16	10·16	2000	1026
		4	2·54	1370	967
4	79·9	8	5·08	1940	1213
		16	10·16	2200	1043

where the numbers in the column of velocities, 1420 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be.

Note, The diameter of the ball, or value of the letter d, is 1·96 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not greatly different from the number 1000, used by Mr. Robins, is rather various, both for the different

different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing to the omission of a material datum in the calculation of the last problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.

Beside the notation used in the last problem, let $2p$ denote the weight of the powder in the charge.

Now, because the inflamed powder occupies all the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force, as half the weight of the powder, moved with the same velocity as the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p + w$; and when that is done, the last velocity will come out, $v = \sqrt{(\frac{2230nad^2}{p+w} \times \text{com. log. } \frac{b}{a})}$.

Now, to find the value of $p + w$ in terms of ad, the dimensions of the ball and powder; it appears, in the first place, from the calculations in the last scholium, that about 5 inches in length of a gun, of near 2 inches diameter, contains just 8 ounces of powder; and as the contents of cylinders, or the weight of powder they will contain, are in the compound ratio of their length and square of the diameter; therefore as $5 \times 2^2 : ad^2 :: 8 : \frac{2ad^2}{5}$, which will be the weight of powder contained in the bore, or charge, whose length is a, and diameter d; that is, $2p = \frac{2ad^2}{5}$, or $p = \frac{ad^2}{5}$, the value of the quantity p, in terms of the dimensions of the charge.

But

But the value of the quantity w, or weight of the ball, as found in the laſt problem, was $w = \frac{ed^3}{3300}$. Conſequently $p + w = \frac{ad^2}{5} + \frac{ed^3}{3300} = \frac{660ad^2 + ed^3}{3300}$.

Then, ſubſtituting this value of $p + w$, in the expreſſion above found for the velocity, gives

$v = 2713\sqrt{\frac{na}{660a + de}} \times \log. \frac{b}{a}$, for the laſt veloc. of ball.

And, when the ball is of caſt iron, uſing 7333 for n, the ſame theorem becomes

$v = 1100\sqrt{\frac{na}{9a + 10d}} \times \log. \frac{b}{a}$, for the laſt velocity of the ball, when of caſt iron.

Corol. To determine the maximum velocity, from the foregoing expreſſion, by ſquaring, and expelling the conſtant quantities, there is obtained at length $\frac{a}{9a + 10d} \times$ hyp. log. $\frac{b}{a} =$ a maximum. Then, by making the fluxion of this quantity $= 0$, there is produced hyp. log. $\frac{b}{a} = 1 + \frac{9a}{100d}$. From which a may eaſily be found, in any particular caſe, by the method of trial-and-error, and a table of hyperbolic logarithms.

Thus, taking the firſt caſe in the corollary to the laſt problem, in which $b = 28{\cdot}2$, and $d = 2$ nearly; then, by double poſition, is ſoon found $a = 7{\cdot}4$; which is $\frac{3}{11}$ of b, being very near $\frac{3}{10}b$, as found by experiment.

Again, taking the ſecond caſe there mentioned, in which $b = 38{\cdot}1$, and $d = 2$, as before. Then is eaſily found $a = 9{\cdot}3$; which is $\frac{1}{4}$ or $\frac{3}{12}$ of b, the very ſame as found by experiment.

Thirdly, taking the third caſe, in which $b = 57{\cdot}37$, and $d = 2$. Then a comes out 12; which is between $\frac{1}{14}$ and $\frac{1}{15}$ of b; which was found $\frac{1}{16}b$ by experiment.

Laſtly, taking the fourth caſe, in which $b = 79{\cdot}9$, and $d = 2$. Then a comes out 15, which is $\frac{3}{16}$ of b, but was found near $\frac{3}{20}$ of b by experiment.

The near agreement of theſe calculations, from theory, with the experiments, is a confirmation of the truth, both of the one and the other. And this gives a reaſonable ground

ground to hope, that the value of the quantity n, or the firſt force of the fired gunpowder, when calculated from the laſt theorem for the velocity, may come nearly the ſame, when fired in the ſame quantity, by uſing different experiments. Which may now be tried in the following ſcholium.

SCHOLIUM.

Proceeding here as in the ſcholium to the laſt problem, to find the value of the letter n, in terms of v and the known quantities, from the theorem for the velocity laſt found, viz. $v = 100\sqrt{\frac{na}{\cdot 9a + 10d}} \times \frac{b}{a}$, by reduction it comes out $n = \frac{(10d + \cdot 9a)v^2}{10000a} \div \log. \frac{b}{a}$.

Then, calculating the value of n, by this theorem, from the ſame data as in the 12 experiments made uſe of in the ſcholium to the laſt problem, the values of n come out as they are here placed in the margin, which ſtand in the ſame order as thoſe in the laſt column of the table of data and reſults in the ſcholium before mentioned. Alſo, the correſponding values of the ſame force n, from the former problem and ſcholium, are ſet after theſe, that their differences may be ſeen at one view.

Values of n from this prob.	Values of n from the former prob.
998	893
1307	1060
1305	890
1020	914
1358	1101
1358	926
1076	963
1448	1174
1505	1026
1080	967
1496	1213
1529	1043

From a view of theſe reſults, it clearly appears, that the theorems, for the values of v and n, found and employed in this problem, and the corollary and ſcholium to the ſame, muſt be very near the truth, as the values of n, or firſt force of the powder, come out very uniform, for all the guns, when the ſame quantity of powder is fired. To make this uniformity more obvious, they are here arranged, according to the like weight of powder in each column, and for each gun on the ſame line. From which it appears, that

Length of gun.	Value of n, with powder		
	4 oz.	8 oz.	16 oz.
inc.			
28·2	998	1307	1305
38·1	1020	1358	1358
57·37	1076	1448	1505
79·9	1080	1496	1529

that there is very little difference on account of the length of the gun; but a conſiderable one for the different quantities of powder, being, on an average, nearly, as follows, viz.

That the firſt force of fired gunpowder,
when 4oz are fired, is about 1050 times
when 8oz are fired, is about 1400 times
when 16oz are fired, is about 1500 times
} the ſtrength of the atmoſphere.

And this increaſe of ſtrength is probably owing to the greater heat of the elaſtic fluid, ariſing from the greater quantity of powder that is fired in the gun.

ON THE MOTION OF BODIES IN FLUIDS.

PROBLEM XIX.

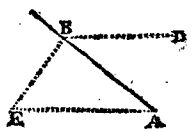
To determine the Force of Fluids in Motion; and the Circumſtances attending Bodies moving in Fluids.

1. IT is evident that the reſiſtance to a plane, moving perpendicularly through an infinite fluid, at reſt, is equal to the preſſure or force of the fluid on the plane at reſt, and the fluid moving with the ſame velocity, and in the contrary direction, to that of the plane in the former caſe. But the force of the fluid in motion, muſt be equal to the weight or preſſure which generates that motion; and which, it is known, is equal to the weight or preſſure of a column of the fluid, whoſe baſe is equal to the plane, and its altitude equal to the height through which a body muſt fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the ſake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the ſpecific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole reſiſtance, or motive force m, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{12}$ feet. And hence, *cæteris paribus*, the reſiſtance is as the ſquare of the velocity.

2. This ratio, of the ſquare of the velocity, may be otherwiſe derived thus. The force of the fluid in motion, muſt be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number

number of them ſtriking the plane in a given time, is alſo as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the ſquare of the velocity.

3. If the direction of motion, inſtead of being perpendicular to the plane, as above ſuppoſed, be inclined to it in any angle, the ſine of that angle being s, to the radius 1; then the reſiſtance to the plane, or the force of the fluid againſt the plane, in the direction of the motion, as aſſigned above, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination, or in the ratio of 1 to s^3. For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whoſe ſine is s; the number of particles, or quantity of the fluid, ſtriking the plane, will be diminiſhed in the ratio of 1 to s, or of radius to the ſine of the angle B of inclination; and the force of each particle will alſo be diminiſhed in the ſame ratio of 1 to s: ſo that, on both theſe accounts, the whole reſiſtance will be diminiſhed in the ratio of 1 to s^2, or in the duplicate ratio of radius to the ſine of the ſaid angle. But again, it is to be conſidered that this whole reſiſtance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all theſe accounts, the reſiſtance in the direction of motion, is diminiſhed in the ratio of 1 to s^3, or in the triplicate ratio of radius to the ſine of inclination. Hence, comparing this with article 1, the whole reſiſtance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{4g}$.



4. Alſo, if w denote the weight of the body, whoſe plane face a is reſiſted by the abſolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

5. And if the body be a cylinder, whoſe face or end is a, and diameter d, or radius r, moving in the direction of its axis; becauſe then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the reſiſting force m will be $\frac{npd^2v^2}{16g} = \frac{npr^2v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw} = \frac{npr^2v^2}{4gw}$.

6. This

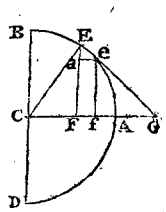
6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{npd^2v^2s^2}{16g} = \frac{npr^2v^2s^2}{4g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a farther investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder with a Hemispherical End, &c.

1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG.



Putting CF $= x$, EF $= y$, BE $= z$, s = sine $\angle$G to radius 1, and $p = 3.1416$: then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{4g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{2g} \times y\dot{z}$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form; putting the radius CA

or

or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{EF}{EG} = \frac{CF}{CE} = \frac{x}{r}$, and $y\dot{z}$ or EF $\times$ Ee $=$ CE $\times$ ae $= r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3 y\dot{z}$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{2gr^2} \times x^3\dot{x}$; the fluent of which, or $\frac{pnv^2}{8gr^2}x^4$, is the reſiſtance to the ſpherical ſurface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the reſiſtance on the whole hemiſphere; which is alſo equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular reſiſtance to the circle of the ſame diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, ſhews that the reſiſtance to the ſphere, is juſt equal to half the direct reſiſtance to a great circle of it, or to a cylinder of the ſame diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its denſity or ſpecific gravity, its weight w will be $= \frac{1}{6}pd^3\mathrm{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{p\mathrm{N}d^3} = \frac{3nv^2}{16g\mathrm{N}d}$; which is alſo $= \frac{v^2}{4gs}$ by art. 8 of the general theorems in page 329; hence then $\frac{3n}{4\mathrm{N}d} = \frac{1}{s}$, and $s = \frac{\mathrm{N}}{n} \times \frac{4}{3}d$; which is the ſpace that would be deſcribed by the globe, while its whole motion is generated or deſtroyed by a conſtant force which is equal to the force of reſiſtance, if no other force acted on the globe to continue its motion. And if the denſity of the fluid were equal to that of the globe, the reſiſting force is ſuch, as, acting conſtantly on the globe without any other force, would generate or deſtroy its motion in deſcribing the ſpace $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greateſt velocity that a globe will acquire by deſcending in a fluid, by means of its relative weight in the fluid, will be found by making the reſiſting force equal to that weight. For, after the velocity is arrived at ſuch a degree, that the reſiſting force is equal to the weight that

urges it, it will increaſe no longer, and the globe will afterwards continue to deſcend with that velocity uniformly. Now, N and n being the ſeparate ſpecific gravities of the globe and fluid, $N - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3(N - n)$ is the weight by which it is urged; alſo, $m = \frac{pnv^2d^2}{32g}$ is the reſiſtance; conſequently $\frac{pnv^2d^2}{32g} = \frac{1}{6}pd^3(N - n)$ when the velocity becomes uniform; from which equation is found $v = \sqrt{4g \cdot \frac{4}{3}d \cdot \frac{N - n}{n}}$, for the ſaid uniform or greateſt velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 329, it will appear that its greateſt velocity, is equal to the velocity generated by the accelerating force $\frac{N - n}{n}$, in deſcribing the ſpace $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely deſcribing the ſpace $\frac{N - n}{n} \times \frac{4}{3}d$.—If $N = 2n$, or the ſpecific gravity of the globe be double that of the fluid, then $\frac{N - n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greateſt velocity in deſcribing $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter.—It is farther evident, that if the body be very ſmall, it will very ſoon acquire its greateſt velocity, whatever its denſity may be.

EXAM. If a leaden ball, of 1 inch diameter, deſcend in water, and in air of the ſame denſity as at the earth's ſurface, the three ſpecific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{4 \cdot 16\frac{1}{12} \cdot \frac{4}{3 \cdot 6} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8{\cdot}5944$ feet, is the greateſt velocity per ſecond the ball can acquire by deſcending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{3 \cdot 6} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$ is the greateſt velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greateſt velocities it could acquire, would be only $\frac{1}{10}$ of theſe, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were ſtill farther diminiſhed, the greateſt velocity would alſo be diminiſhed, and that in the ſubduplicate ratio of the diameter of the ball.

PRO-

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a Given Velocity.

1. LET a = the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{16gNd}$, where N is the density of the ball, n that of the fluid, and d the diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8Nd}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8Nd}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$. The correct fluent of this, is log. a — log. v or log. $\frac{a}{v} = bx$. Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

EXAM. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and the velocity lost is $\frac{c^{bx} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$,

$x = 500$, $N = 7\frac{1}{3}$, $n = \cdot 0012$; and therefore $bx = \frac{3nx}{8Nd} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second: having loſt 202 feet, or nearly $\frac{1}{6}$ of its firſt velocity.

EXAM. 3. If the earth revolved about the ſun, in a medium as denſe as the atmoſphere near the earth's ſurface; and it were required to find the quantity of motion loſt in a year. Then, ſince the earth's mean denſity is about $4\frac{1}{2}$, and its diſtance from the ſun 12000 of its diameters, we have $24000 \times 3 \cdot 1416 = 75398$ diameters $= x$, and $bx = \frac{3 \cdot 75398 \cdot 12 \cdot 2}{8 \cdot 10000 \cdot 9} = 7 \cdot 5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are loſt of the firſt motion in the ſpace of a year, and only the $\frac{1}{1881}$ part remains.

EXAM. 4. If it be required to determine the diſtance moved, x, when the globe has loſt any part of its motion, as ſuppoſe $\frac{1}{2}$, and the denſity of the globe and fluid equal; The general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1 \cdot 8483925d$. So that the globe loſes half its motion before it has deſcribed twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conſequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vaniſh together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time ſought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

EXAM. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per ſecond; and it were required to determine in what time it would paſs over 500 yards

500 yards or 1500 feet, and what would be its velocity at the end of that time: We ſhould have, as in exam. 2 above, $b = \frac{3 \cdot 12 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1{\cdot}7372}{1200} = \frac{1}{690}$ nearly. Conſequently $v = 690$ is the velocity; and $t = \frac{1}{b}(\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{40}$ ſeconds, is the time required, or $1''$ and $\frac{2}{3}$ nearly.

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe deſcends, by its own Weight, in an infinite Fluid.

THE foregoing notation remaining, viz. d = diameter, N and n the denſity of the ball and fluid, and v, s, t, the velocity, ſpace, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3(N-n)$ = its weight in the fluid, alſo $m = \frac{pnd^2v^2}{32g}$ = its reſiſtance from the fluid; conſequently $\frac{1}{6}pd^3(N-n) - \frac{pnd^2v^2}{32g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}pNd^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{16gNd}$ for the accelerative force.

2. Hence $v\dot{v} = 2gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{Nv\dot{v}}{2g(N-n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8Nd}$, and $\frac{1}{a} = \frac{3n}{2g \cdot 8d(N-n)}$, or $ab = 2g$ nearly; the fluent of which is $s = \frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expreſſion for the ſpace s, in terms of the velocity v.

3. But now, to determine v in terms of s, put $c = 2{\cdot}71828182\,8$; then, ſince the log. of $\frac{a}{a - v^2} = 2bs$, there-

fore $\frac{a}{a-v^2} = c^{2bs}$, or $\frac{a-v^2}{a} = c^{-2bs}$; and hence $v = \sqrt{a - ac^{-2bs}}$, the velocity fought.

4. The greateſt velocity is to be found, as in art. 5 of prob. 20, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$, which gives $v = \sqrt{2g \cdot 8d \cdot \frac{N-n}{3n}} = \sqrt{a}$. The ſame value is alſo obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$. And the ſame value of v is obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident, however, that it will approximate towards it the faſter, the greater b is, or the leſs d is; and that, the diameters being very ſmall, the bodies deſcend by nearly uniform velocities, which are directly in the ſubduplicate ratio of the diameters. See alſo art. 5, prob. 20, for other obſervations on this head.

5. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then, to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1-z^2}$; conſequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1-z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a}+v}{\sqrt{a}-v}$, which is the general expreſſion for the time.

EXAM. If it were required to determine the time and velocity, by deſcending in air 1000 feet; the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{1} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} =$

$\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{1\frac{1}{2}}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68.50^2}$; consequently $v = \sqrt{a} \times \sqrt{1 - c^{-2bs}} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times \sqrt{1 - c^{-\frac{81}{85}}} = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \sqrt{\frac{34 \cdot 2500}{27 \cdot 193}} \times \log. \frac{1{\cdot}78383}{0{\cdot}21617} = 8{\cdot}5236''$, the time.

Note. If the globe be ſo light as to aſcend in the fluid; it is only neceſſary to change the ſigns of the firſt two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all reſpects as before.

SCHOLIUM.

To compare this theory, contained in the laſt four problems, with experiment, the few following numbers are here extracted from extenſive tables of velocities and reſiſtances, reſulting from a courſe of many hundred very accurate experiments, made in the courſe of the year 1786.

In the firſt column are contained the mean uniform or greateſt velocities acquired in air, by globes, hemiſpheres, cylinders, and cones, all of the ſame diameter, and the altitude of the cone nearly equal to the diameter alſo, when urged by the ſeveral weights, expreſſed in averdupois ounces, and ſtanding on the ſame line with the velocities, each in their proper column. So, in the firſt line, the numbers ſhew that, when the greateſt or uniform velocity was accurately 3 feet per ſecond, the bodies were urged by theſe weights, according as their different ends went foremoſt; namely, by ·028oz. when the vertex of the cone went foremoſt; by ·064oz. when the baſe of the cone went foremoſt; by ·027 oz. for a whole ſphere; by ·050oz. for a cylinder; by ·051 oz. for the flat ſide of the hemiſphere; and by ·100z. for the round or convex ſide of the hemiſphere. Alſo, at the bottom of all, are placed the mean proportions of the reſiſtances of theſe figures, in the neareſt whole numbers. Note, the common diameter of all the figures, was 6·375 or $6\frac{3}{8}$ inches; ſo that the area of the circle of that diameter is juſt 32 ſquare inches, or $\frac{2}{9}$ of a ſquare foot; and the alti-

tude of the cone was $6\frac{5}{8}$ inches. Alſo, the diameter of the ſmall hemiſphere was $4\frac{3}{4}$ inches, and conſequently the area of its baſe is $17\frac{3}{4}$ ſquare inches, or $\frac{1}{8}$ of a ſquare foot nearly.

From the given dimenſions of the cone, it appears, that the angle made by its ſide and axis, or direction of the path, is 26 degrees, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; conſequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz. nearly, in thoſe circumſtances.

Veloc. per ſec.	Cone		Whole globe.	Cylinder.	Hemiſphere.		Small Hemiſ.
	vertex	baſe			flat	round	flat
feet	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·292	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of reſiſtances, ſeveral practical inferences may be drawn. As,

1. That the reſiſtance is nearly as the ſurface; the reſiſtance increaſing but a very little above that proportion in the greater

greater ſurfaces. Thus, by comparing together the numbers in the 6th and laſt columns, for the baſes of the two hemiſpheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in thoſe two columns, expreſſing the reſiſtances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the reſiſtances on the greater ſurface increaſe gradually more and more above that proportion. And the mean reſiſtances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumſtance therefore agrees nearly with the theory.

2. The reſiſtance to the ſame ſurface, is nearly as the ſquare of the velocity; but gradually increaſes more and more above that proportion as the velocity increaſes. This is manifeſt from all the columns. And therefore this circumſtance alſo differs but little from the theory, in ſmall velocities.

3. When the hinder parts of bodies are of different forms, the reſiſtances are different, though the fore parts be alike; owing to the different preſſures of the air on the hinder parts. Thus, the reſiſtance to the fore part of the cylinder, is leſs than that on the flat baſe of the hemiſphere, or of the cone; becauſe the hinder part of the cylinder is more preſſed or puſhed, by the following air, than thoſe of the other two figures.

4. The reſiſtance on the baſe of the hemiſphere, is to that on the convex ſide, nearly as $2\frac{2}{5}$ to 1, inſtead of 2 to 1; as the theory aſſigns the proportion. And the experimented reſiſtance, in each of theſe, is nearly $\frac{1}{4}$ part more than that which is aſſigned by the theory.

5. The reſiſtance on the baſe of the cone, is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the ſame ratio is radius to the ſine of the angle of the inclination of the ſide of the cone, to its path or axis. So that, in this inſtance, the reſiſtance is directly as the ſine of the angle of incidence, the tranſverſe ſection being the ſame, inſtead of the ſquare of the ſine.

6. Hence we can find the altitude of a column of air, whoſe preſſure ſhall be equal to the reſiſtance of a body, moving through it with any velocity. Thus,

Let

Let a = the area of the ſection of the body, ſimilar to any of thoſe in the table, perpendicular to the direction of motion;

r = the reſiſtance to the velocity, in the table; and

x = the altitude ſought, of a column of air, whoſe baſe is a, and its preſſure r.

Then ax = the content of the column in feet, and $1\frac{1}{3}ax$ or $\frac{6}{5}ax$ its weight in ounces;

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude ſought in feet, namely, $\frac{5}{6}$ of the quotient of the reſiſtance of any body divided by its tranſverſe ſection; which is a conſtant quantity for all ſimilar bodies, however different in magnitude, ſince the reſiſtance r is as the ſection a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the ſmall hemiſphere; then, $x = \frac{5}{6} \times \frac{r}{a}$, becomes $x = \frac{15}{4}r$, where r is the reſiſtance in the table, to the ſimilar body.

If, for example, we take the convex ſide of the large hemiſphere, whoſe reſiſtance is ·634 oz, to a velocity of 16 feet per ſecond; then $r = ·634$, and $x = \frac{15}{4}r = 2·3775$ feet, is the altitude of the column of air whoſe preſſure is equal to the reſiſtance on a ſpherical ſurface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the preſſure. And as the altitude is proportional to the ſquare of the velocity, therefore, in ſmall velocities, the reſiſtance to any ſpherical ſurface, is equal to the preſſure of a column of air on its great circle, whoſe altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whoſe reſiſtance $r = 1·526$: then $x = \frac{15}{4}r = 5·72$; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be ſtill greater, if the body were larger; and alſo if the velocity were more.

7. Alſo, if it be required to find with what velocity any flat ſurface muſt be moved, ſo as to ſuffer a reſiſtance juſt equal to the whole preſſure of the atmoſphere.

The

The reſiſtance on the whole circle whoſe area is $\frac{2}{9}$ of a foot, is ·051 oz. with the velocity of 3 feet per ſecond, it is $\frac{1}{9}$ of ·051, or ·0056 oz. only with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole preſſure of the atmoſphere. Therefore, as $\sqrt{\cdot 0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity ſought. Being almoſt equal to the velocity with which air ruſhes into a vacuum.

8. Hence may be inferred the great reſiſtance ſuffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal to the ſize of an iron ball weighing 36 lb, moving with a velocity of only 16 feet per ſecond, meets with a reſiſtance equal to the preſſure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the ſquare of the velocity, the leaſt reſiſtance that ſuch a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the preſſure of 417 lb, and that independent of the preſſure of the atmoſphere itſelf on the fore part of the ball, which would be 487 lb more, as there would be no preſſure from the atmoſphere on the hinder part, in the caſe of ſo great a velocity as 1600 feet per ſecond. So that the whole reſiſtance would be more than 900 lb to ſuch a velocity.

9. Having ſaid, in the laſt article, that the preſſure of the atmoſphere is taken entirely off the hinder part of the ball, moving with a velocity of 1600 feet per ſecond; which muſt happen when the ball moves faſter than the particles of air can follow by ruſhing into the place quitted and left void by the ball, or when the ball moves faſter than the air ruſhes into a vacuum from the preſſure of the incumbent air: let us therefore inquire what this velocity is. Now, the velocity with which any fluid iſſues, depends upon its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, ſuppoſing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of an uniform atmoſphere, all of the ſame denſity as at the earth's ſurface, would be $2\frac{1}{2} \times 13{\cdot}6 \times 833\frac{1}{3}$ or 28333 feet; therefore $\sqrt{16} : \sqrt{28333} :: 32 : 8\sqrt{28333} = 1346$ feet, which is the velocity ſought. And therefore, with a velocity of 1600 feet per ſecond, or any velocity above 1346 feet, the ball muſt continually leave a vacuum behind it, and ſo muſt ſuſtain the whole preſſure of the atmoſphere on its fore part, as well as the reſiſtance ariſing from the vis inertia of the particles of air ſtruck by the ball.

10. Upon

10. Upon the whole, we find, that the reſiſtance of the air, as determined by the experiments, differs very widely, both in reſpect to the quantity of it on all figures, and in reſpect to the proportions of it on oblique ſurfaces, from the ſame as determined by the preceding theory; which is the ſame as that of Sir Iſaac Newton, and moſt modern philoſophers. Neither ſhould we ſucceed better if we have recourſe to the theory given by Profeſſor Graveſande, or others, as ſimilar differences and inconſiſtencies ſtill occur.

We conclude, therefore, that all the theories of the reſiſtance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important ſubject, ſhall enable us to deduce from them, another, that ſhall be more conſonant to the true phenomena of nature.

FINIS.

ERRATA IN VOL. II.

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for ·9999999153, read ·9999999577

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